Theorem: \( f(x), g(x) \) continuous at \( a \), \( c \) a constant

then \( f \pm g, \ f \cdot g, \ c \cdot f \), and \( \frac{f}{g} \) if \( g(a) \neq 0 \)

are all cont at \( a \).
Most common application.

1. Polynomials are continuous everywhere.

2. Rational functions are continuous everywhere except where the denominator is zero.
Then:

if \( f(x) \) is cont. at \( b \),

and \( \lim_{x \to a} g(x) = b \), then

\[
\lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) = f(b)
\]

can push limits inside cont. functs.
Thm
Intermediate value theorem

If $f(x)$ is cont. on $[a, b]$, $f(a) \neq f(b)$, and $N$ is between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ s.th. $f(c) = N$. 
Example 48: Show the equation has a solution.

\[ 3 \sqrt{x} = 1- x \text{ on } (0, 1) \]

i.e. \[ 3 \sqrt{x} - 1 + x = 0 \]

We want \( f(x) = 3 \sqrt{x} - 1 + x \) to equal 0.

We have \( f(x) \) cont on \([0, 1]\)

\( a = 0, \ b = 1 \)

\( f(a) = f(0) = -1, \ f(b) = f(1) = 1 \)

\( N = 0 \) is between -1 and 1,

So by the Intermediate Value Theorem \( \exists \ c \in (0, 1) \)

so that \( f(c) = N \) i.e.

\[ 3 \sqrt{c} - 1 + c = 0 \]

so \( c \) is a sol. to

\[ 3 \sqrt{c} = 1 - c \]
Example 10.
\[ f(x) = x^2 + \sqrt{7} - x \] at \( a = 4 \)

\[ f(4) = 16 + \sqrt{3} \]

\[ \lim_{x \to 4} x^2 + \sqrt{7} - x = 16 + \sqrt{3} \]

and yes \( f(4) = \lim_{x \to 4} x^2 + \sqrt{7} - x \)

so yes \( \frac{\text{cont.}}{} \).
\[ f(x) = \begin{cases} 
\frac{1}{x-1} & x \neq 1 \\
2 & x = 1 
\end{cases} \quad \text{at } a = 1 \]

\[ f(1) = 2 \]

\[ \lim_{x \to 1} f(x) = \text{DNE} \]

\[ \therefore f(x) \text{ is not cont.} \]
\[ f(x) = \sum_{x=1}^{\infty} \left\{ \frac{1}{x-1} \quad \begin{cases} x+1 \end{cases} \right. \]
\text{Eq. #34}
\theta(x) = \begin{cases} 
\sin x & x < \frac{\pi}{4} \\
\cos x & x \geq \frac{\pi}{4}
\end{cases} \quad a = \frac{\pi}{4}

\theta \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}

\lim_{x \to \frac{\pi}{4}} \theta(x)

\lim_{x \to \frac{\pi}{4}^-} \theta(x) = \lim_{x \to \frac{\pi}{4}^-} \sin x = \frac{\sqrt{2}}{2}

\lim_{x \to \frac{\pi}{4}^+} \theta(x) = \lim_{x \to \frac{\pi}{4}^+} \cos x = \frac{\sqrt{2}}{2}

\text{so } \lim_{x \to \frac{\pi}{4}} \theta(x) = \frac{\sqrt{2}}{2}

\text{and } \lim_{x \to \frac{\pi}{4}} \theta(x) = \frac{\sqrt{2}}{2} = \theta(x)

\text{so yes cont.}
Find $c$ so that
\[ g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases} \]
is continuous at 4.
\[ g(4) = 4c + 20 \]
\[ \lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x^2 - c^2) = 16 - c^2 \]
\[ \lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} (cx + 20) = 4c + 20 \]
We need $16 - c^2 = 4c + 20$
\[ 0 = c^2 + 4c + 4 \]
\[ c = -2 \]
If $c = -2$, then $g(x) = \begin{cases} x^2 - (-2)^2 & x < 4 \\ -2x + 20 & 4 \leq x \end{cases}$
\[ g(4) = -2(4) + 20 = 12 \]
\[ \lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x^2 - 4) = 12 \\ \lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} (-2x + 20) = 12 \]
So $\lim_{x \to 4} g(x) = 12$
and yes $g(4) = 12 = \lim_{x \to 4} g(x)$
so $c = -2$ makes $g(x)$ cont at 4.