4.2

18. Show $2x - 1 - \sin x = 0$ has exactly one root.

At least one: Use Intermediate Value Theorem

Let $f(x) = 2x - 1 - \sin x$

$f(0) = 0 - 1 - 0 = -1 < 0$ and

$f(\pi) = 2\pi - 1 - \sin \pi = 2\pi - 1 > 0$

$\therefore \exists c \in [0, \pi]$ s.t.

$f(c) = 0$, so at least one root.
No more than one root: B WOC

Assume there are two roots, i.e. \( f(a) = 0 \) and \( f(b) = 0 \)

Now apply Rolle's:

- \( f(x) \) is cont on \([a, b]\)
- \( f'(x) = 2 - \cos x \) exists on \((a, b)\)
- \( f(a) = f(b) = 0 \), so criteria are satisfied
So by Rolle's, \( \exists \ c \in (a, b) \) \n\[ \sin f'(c) = 0 \]

but \( f'(c) = 2 - \cos(c) \)

but \( 2 - \cos(c) \geq 1 > 0 \)

\[
\Rightarrow
\]
\[ f(x) = x^3 - 15x + C \quad \text{on } [-2, 2] \]

\[ f(-2) = -8 + 30 + C = 22 + C \]

\[ f(2) = 8 - 30 + C = -22 + C \]

Rolles & assume BWOC that there are two roots in \([-2, 2]\)
so \( a, b \in [-2, 2] \)

3. \( f(a) = 0 = f(b) \)

1. \( f(x) \) is cont on \([a, b]\)

2. \( f'(x) = 3x^2 - 15 \) exists on \((a, b)\)

so Rolle says \( \exists k \in (a, b) \subset [-2, 2] \)

so \( f'(k) = 0 \)
1 \times

3k^2 - 15 = 0

3k^2 = 15

k^2 = 5

k = \pm \sqrt{5}

but \sqrt{5}, -\sqrt{5} \notin [-2, 2] \implies \iff
23. \( f(1) = 10, \ f'(x) = 2 \)

for \( x \in [1, 4] \) how small can \( f(4) \) be? (Assuming \( f(x) \) is cont. on \([1, 4]\) and \( f'(x) \) exists on \((1, 4)\))
MVT says

If \( c \in (1, 4) \) s.t.

\[
f'(c) = \frac{f(4) - f(1)}{4 - 1}
\]

then \( f'(c) \geq 2 \), so . . .
\[ \frac{f(4) - f(1)}{4 - 1} \geq 2 \]

\[ f(4) - 10 \geq 6 \]

\[ f(4) \geq 16 \]
4.3.

Theorem: on an interval $I$

$f'(x) > 0 \Rightarrow f(x)$ is increasing

$f'(x) < 0 \Rightarrow f(x)$ is decreasing
Proof: Let \( x_1, x_2 \in I \)

with \( x_1 < x_2 \)

MVT \( \Rightarrow \exists c \in (x_1, x_2) \)

so \( f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \)

if \( f(x_2) - f(x_1) \)

so \( f(x_2) > f(x_1) \)

\( f(x) \) is increasing

\( f(x) \) is always pos
$1^{st}$ derivative test:

If $f(x)$ is cont. and $c$ is a crit. number then

$f' \quad + + + + c \quad + + +$

$f(c)$ is a rel. min

$f' \quad + + + + - c \quad + + +$

$f(c)$ is a rel. max
The second derivative, $f''$, tells the concavity of the function:

- $f'' > 0$: concave up
- $f'' < 0$: concave down

The first derivative, $f'$, indicates the increasing or decreasing nature of the function:

- $f' > 0$: increasing
- $f' < 0$: decreasing

Combining these, we have:

- $f' > 0$, then $f'' > 0$: increasing and concave up
- $f' > 0$, then $f'' < 0$: increasing and concave down
- $f' < 0$, then $f'' > 0$: decreasing and concave up
- $f' < 0$, then $f'' < 0$: decreasing and concave down
2nd derivative test

if \( f'(c) = 0 \) (\( c \) is a crit #)

and \( f''(c) > 0 \) then \( f(c) \) is local min

and \( f''(c) < 0 \) then \( f(c) \) is local max

\( \frac{d}{dx} \)