\[ f'(x) = \frac{4}{5} x^{-\frac{1}{5}} (x-4)^2 + x^{\frac{4}{5}} \cdot 2(x-4) \]

\[ = \frac{-1}{5} x^{-\frac{1}{5}} (x-4) \left[ \frac{4}{5} (x-4) + 2x \right] \]

\[ = \frac{-1}{5} x^{-\frac{1}{5}} (x-4) \cdot \frac{2}{5} (3x-8) \]

\[ = \frac{2}{5} \cdot \frac{1}{x^{\frac{1}{5}}} (x-4) (3x-8) \]

\[ f'(x) \text{ does not exist at } x = 0 \]

\[ f'(x) = 0 \text{ when } x = 4, \frac{8}{7} \]

Critical numbers are 0, 4, \frac{8}{7}
Theorem (Fermat)

if \( f(x) \) has a local max at \( c \), then either \( f'(c) \) DNE or \( f'(c) = 0 \)
Proof:

Need to start with a more careful definition of local max:

\[ f(c) \text{ is a local max if } \exists \delta \text{ such that } |x-c| < \delta \Rightarrow f(x) \leq f(c). \]
Back to the proof:

If $f'(c)$ DNE, we're done, so assume $f'(c)$ exists.

Thus

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

and this limit exists.
Since \( h \to 0 \), we can assume \( |h| < \delta \) so \( f(c+h) - f(c) \leq 0 \) because \( f(c) \) is a local max.
Look at
\[
\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} < 0 \\
\text{This is } \leq 0.
\]
look at
\[
\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \not\to < 0
\]
so this is \geq 0
But \( f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} \) exists.

So,
\[
\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}
\]
not positive
not negative

The only way this can happen is if both • the limit from the right and the limit from the left are zero.

Thus \( f'(c) = 0 \).
4.2. Rolle's Theorem: If

a. \( f \) is continuous on \([a, b]\)

b. \( f' \) exists on \((a, b)\)

c. \( f(a) = f(b) \)

Then \( \exists \ c \in (a, b) \) such that \( f'(c) = 0 \).
in this case, \( f(x) = \text{constant} \)

\[
f'(c) = 0 \quad \forall \ c \in (a, b)
\]