Drawings of Graphs in Two and Three Dimensions

David Wood
McGill University
Montréal, Canada
graphs

A graph $G$ consists of

- a set $V(G)$ of vertices
- a set $E(G)$ of unordered pairs of vertices called *edges*

Graph $G$

$V(G) = \{1, 2, 3, 4, 5\}$

$E(G) = \{12, 13, 14, 15, 23, 24, 25, 34\}$

Drawing of $G$
graphs

graphs are ubiquitous objects that model:

- transportation networks
- communication networks
- information flows
- spread of disease
- social networks, etc.

and

- arise in other area of mathematics
  (e.g. probability, geometry, number theory)
**topological origins of graph theory**

Euler’s *Seven bridges of Königsberg* problem (1735)
topological origins of graph theory

4-colouring the regions of a map (Guthrie 1852)
topological origins of graph theory

Tremaux’s algorithm for finding your way through a maze (1882)
• abstract graph theory has flourished in the past 40 years

• recent (renewed) interest in graphs in topological and geometric contexts
  – topological graph theory
    e.g. crossing-free embeddings of graphs on surfaces
  – geometric graph theory
    e.g. straight-line drawings of graphs in the plane
  – computational geometry— algorithmics of geometric objects
    e.g. how fast can one compute the convex hull of a set of points in the plane?
  – graph drawing—automatic production of ‘nice’ drawings of graphs
applications of graph drawing

drawings of information flowcharts
applications of graph drawing

drawings of metro maps
applications of graph drawing

drawings of biochemical pathways

Boehringer
graph drawing

fundamental problem: automatically produce a nice drawing of a given graph $G$

what makes a drawing nice?

- edges are straight line-segments
- few edge crossings
- small area
- symmetry, etc.

(aesthetic criteria)
2D straight-line grid drawings

- **vertices** $\rightarrow$ grid-points in $\mathbb{Z}^2$
- **edges** $\rightarrow$ straight line segments

[de Fraysseix–Pach–Pollack–90; Schnyder–89]

Every planar graph has a $O(n) \times O(n)$ straight-line grid drawing with no crossings.
3D straight-line grid drawings

- vertices $\rightarrow$ grid-points in $\mathbb{Z}^3$
- edges $\rightarrow$ straight line segments

such that

- no edge crossings

e.g. $K_6$

main aesthetic criterion: small volume (of bounding box)
measuring the volume of a box

3 \times 3 \times 3 \text{ box with volume } 27

hence 2D drawings have positive volume
folklore theorem: every graph $G$ has a 3D drawing
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- let $V(G) = (1, 2, \ldots, n)$; place vertex $i$ at $(i, i^2, i^3)$
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why no crossings? if the edges $ij$ and $kl$ cross, then the vertices $i, j, k$ and $l$ are coplanar, and the determinant

$$\det \begin{pmatrix}
1 & i & i^2 & i^3 \\
1 & j & j^2 & j^3 \\
1 & k & k^2 & k^3 \\
1 & l & l^2 & l^3 \\
\end{pmatrix} = 0$$

but this is a Vandermonde matrix which has nonzero determinant
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\[
\begin{vmatrix}
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\]

but this is a Vandermonde matrix which has nonzero determinant

- vertex $i$ at $(i, i^2 \mod p, i^3 \mod p)$ for prime $p \geq n$

$\Rightarrow O(n^3)$ volume [Eades–Cohen–Lin–Ruskey–96]

- Erdös discovered this trick in 2D in 1951
volume lower bounds

what is a lower bound on the volume of a 3D drawing of a given graph?

what is the maximum number of edges in a 3D drawing with given bounding box volume?
**volume lower bounds**

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**theorem [Bose–Czyzowicz–Morin–W.]**

the maximum number of edges in an $X \times Y \times Z$ drawing is exactly

$$(2X - 1)(2Y - 1)(2Z - 1) - XYZ,$$

which is $\lt 7XYZ$

**corollary:** every 3D drawing of a graph $G$ has volume $\gt \frac{1}{8}(|V(G)| + |E(G)|)$
proof of lower bound theorem

given an $X \times Y \times Z$ box $B$

let $G$ be a graph with 3D drawing in $B$ with maximum number of edges
proof of lower bound theorem

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define ‘half-grid’ $P \overset{\text{def}}{=} \{(x, y, z) \in B : 2x, 2y, 2z \in \mathbb{Z}\}$

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\[ |P| = (2X - 1)(2Y - 1)(2Z - 1) \]

the midpoint of every edge is in $P$, and no two edges share a common midpoint

\[ |E(G)| \leq |P| \]
**proof of lower bound theorem**

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$\Rightarrow |E(G)| \leq |P|$

the midpoint of an edge does not intersect a vertex

$\Rightarrow |E(G)| \leq |P| - |V(G)|$

no edge passes through a grid-point, otherwise subdivide the edge
proof of lower bound theorem

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no edge passes through a grid-point, otherwise subdivide the edge

$\Rightarrow |V(G)| = XYZ$

$\Rightarrow |E(G)| \leq |P| - XYZ$
given a $X \times Y \times Z$ box, have one vertex per grid-point

every point in half-grid $P$ is a vertex or the midpoint of an edge

$\Rightarrow$ # edges is $|P| - XYZ$
upper bounds on 3D drawings

- we know every graph has a 3D drawing with $O(n^3)$ volume, which is best possible for the complete graph
upper bounds on 3D drawings

- we know every graph has a 3D drawing with $O(n^3)$ volume, which is best possible for the complete graph
- which graphs have 3D drawings with $O(n)$ volume? they certainly have $O(n)$ edges
- this question leads to the definition of a ‘track layout’
**track layouts**

- **t-track layout** of graph $G \equiv$
  
  - vertex colouring $\{V_i : 1 \leq i \leq t\}$
  
  - total ordering $<_i$ of each $V_i$ (a track)
  
  - no X-crossing $\equiv$ two edges $vw$ and $xy$ such that $v <_i x$ and $y <_j w$
**track layouts**

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The **track-number** $\text{tn}(G) \overset{\text{def}}{=} \text{minimum } t \text{ such that } G \text{ has a } t\text{-track layout}$

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**Lemma:** $G$ has an $A \times B \times C$ drawing $\implies$ $G$ has a $2AB$-track layout
$j$-th vertex in $i$-th track at \{$(i, i^2, jp + (i^3 \mod p))$\} . . . no crossings

\[
y = x^2
\]

volume $\leq \mathcal{O}(tn(G)^4 \cdot n)$
track layout $\rightarrow$ 3D drawing

$j$-th vertex in $i$-th track at $(i, i^2 \mod p, jp + i^3 \mod p)$ \ldots no crossings

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\[
\text{volume} \leq \mathcal{O}(tn(G)^3 \cdot n)
\]

by balancing the number of vertices in each track \ldots \) volume \( \leq \mathcal{O}(tn(G)^2 \cdot n) \)
example 1: trees

$G$ has a 2-track layout iff $G$ is a caterpillar forest \[\text{[Harary–Schwenk–72]}\]
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$G$ has a \textbf{2}-track layout iff $G$ is a caterpillar forest [Harary–Schwenk–72]

every tree has a \textbf{3}-track layout [Felsner–Liotta–Wismath–01]
example 2: square grid has a 3-track layout

wrap modulo 3
**example 3: interval graphs**

- *interval graph* is the intersection graph of a set of intervals in \( \mathbb{R} \).
- Well known that every interval graph with maximum clique size \( k \) is \( k \)-colourable.
example 3: interval graphs

- interval graph is the intersection graph of a set of intervals in $\mathbb{R}^2$
- well known that every interval graph with maximum clique size $k$ is $k$-colourable

theorem: every such interval graph has a $k$-track layout

i.e. track-number is at most pathwidth $+ 1$
upper bounds on volume of 3D drawings

[Eades–Cohen–Lin–Ruskey–96]
every graph has a 3D drawing with $O(n^3)$ volume,
and $\Theta(n^3)$ volume is necessary for $K_n$
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every $O(1)$-colourable graph has a 3D drawing with $O(n^2)$ volume,
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[Felsner–Liotta–Wismath–01]
every outerplanar graph has a 3D drawing with \( O(n) \) volume.
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graphs with bounded treewidth have bounded track-number,
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[Dujmović–W.–04]  
for fixed $H$, every graph with no $H$-minor (e.g. planar graphs)  
has a 3D drawing with $O(n^{3/2})$ volume.
**open problem: planar graphs**

Does every planar graph have $O(1)$ queue-number?

[Heath–Leighton–Rosenberg–92]  
equivalently: does every planar graph have $O(1)$ track-number?
[Dujić–Morin–Pór–W.–04]
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  and thus have bounded queue/track-number
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** best known lower bound on track-number is 7
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* is track/queue-number bounded for any proper minor-closed family?
**Queue layouts**

A **$k$-queue layout** of graph $G$ is:

- a linear order of vertices
- at most $k$ pairwise nested edges

**Queue-number** $qn(G)$ is defined as the minimum $k$ such that there is a $k$-queue layout of $G$.


Theorem:

$$qn(G) \cdot t_n(G) \leq 1$$

(Dujmović–Morin–W.–03)

And:

$$qn(G) \cdot k = t_n(G) \cdot 4k \cdot (2k - 1) \cdot (4k - 1)$$

(Dujmović, Pór–W.–04)
queue layouts

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queue layouts

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queue-number \(qn(G)\) \(\overset{\text{def}}{=} \) minimum \(k\) such that there is a \(k\)-queue layout of \(G\)


theorem: \(qn(G) \leq tn(G) - 1\) [Dujmović–Morin–W.–03]
and \(qn(G) \leq k \implies tn(G) \leq 4k \cdot 4^{k(2k-1)(4k-1)}\) [Dujmović, Pór–W.–04]