$|G| = 300 = 2^2 \cdot 3 \cdot 5^2$

**Elementary divisor form:**

$G \cong \mathbb{Z}_{25} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$, or $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$

or $\mathbb{Z}_{25} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$ or $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$

**Invariant factor form:**

$G \cong \mathbb{Z}_{30} \oplus \mathbb{Z}_{10}$ or $\mathbb{Z}_{10} \oplus \mathbb{Z}_5$ or $\mathbb{Z}_{30} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{300}$

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$p_k$

$5^5$

$\begin{align*}
5^5 & \equiv 2 \pmod{3} \\
5 & \equiv 2 \pmod{3}
\end{align*}$

$|G| = 625$
Bigger example:

\[ |G| = 2^3 \cdot 3^2 \cdot 7 = 504 \]

\[ \{ 2^3 \} \times \{ 3^2 \} \times \{ 7 \} \]

good elem. divisor

\[ \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_7 \]
Outline of Fund Thm of Fin. Gen. Abelian Groups

Suppose $G$ is Abelian in all cases.

(Lemma 1) Suppose $|G| = p^m$ where $p^m$ a prime.

Then $G = H \times K$ where $H = \{ x \in G | x^{p^n} = e \}$

and $K = \{ x \in G | x^m = e \}$.

(recall that we showed that if $G$ is Abelian, $n \in \mathbb{Z}$ then $\{ x \in G | x^n = e \} \leq H$).

What this does is let us "pick off" subgroups of prime power orders.
So if $|G| = \prod_{i=1}^{k} P_i^{n_i}$ w/ $(P_i, P_j) = 1$

Then $G_l = H_1 \times \ldots \times H_k^{n_i}$

where $H_i = \{ x \in G | P_i^n = e \}$.

II. Show that if $H = \{ x \in G | x^{p^n} = e \}$

then $|H| = p^n$.

So now all we have to do is analyze Abelian groups with order $p^n$ for some prime $p$ and $n \in \mathbb{Z}$. 
Lemma 2

If $|H| = p^n$, and $|a| \geq |x|$ $\forall x \in H$, then $H = \langle a \rangle \times K$.

Suppose $|a| = p^r$. Then $\langle a \rangle \cong \mathbb{Z} p^r$.

(note $|a| \mid |H| \Rightarrow |a| \mid p^n \Rightarrow |a| = p^r$ for some $r$ since $p$ is prime.)

Since $|H| = |\langle a \rangle| \cdot |K| \Rightarrow |K| = p^{n-r}$

$\Rightarrow K = \langle b \rangle \times L$ (i.e. the lemma applies to $K$ as well)

Thus, by induction

$|H| \cong \mathbb{Z} p^{r_1} \times \mathbb{Z} p^{r_2} \times \ldots \times \mathbb{Z} p^{r_s}$
Thus

$G \cong H_1 \times \ldots \times H_k$ where $|H_i| = p_i^{n_i}$

where $p_i \neq p_j$ for $i \neq j$

$\cong \prod_{i} Z_{p_i}^{r_i} \times Z_{p_i}^{r_i} \times \ldots \times Z_{p_i}^{r_i}$

$H_i$

$\ldots Z_{p_k}^{r_k} \times Z_{p_k}^{r_k} \times \ldots \times Z_{p_k}^{r_k}$

$H_k$

And this is the elementary divisor form of the Theorem.
So now we just need to prove the lemmas.

Gabelian,

Lemma I: \(|G| = p^m\) where \(p \nmid m\), \(p\) is prime.

Show \(G = H \times K\) where \(H = \{ x \in G \mid x^p = e^3 \}\)

and \(K = \{ x \in G \mid x^m = e^3 \}\)

Proof: Recall (pg. 18) that if \(H \triangleleft G, K \triangleleft G\),

then \(G = H \times K\) if \(G = HK\) and \(H \cap K = e\).