Theorem: Let $G_i$ be cyclic for all $i$. Then

$\bigoplus_{i=1}^{n} G_i$ is cyclic $\iff$

$\gcd(|G_i|,|G_j|)=1$ whenever $i \neq j$
Proof: “⇒” Suppose $\prod_{i=1}^{n} G_i = G$ is cyclic. Recall that for each divisor $d$ of $|G|$, $E!$ subgroup of order $d$.

Now B.W.O.C. suppose $i \neq j$ with $\gcd(|G_i|, |G_j|) = s + 1$. 
Now, if \( g_i \in G_i \) such that \( \langle g_i \rangle = G_i \) since \( G_i \) is cyclic.

Similarly, if \( g_j \in G_j \) such that \( \langle g_j \rangle = G_j \)

Also, \( |G_i| = s \cdot m \) for some \( m \)
\( |G_j| = s \cdot k \) for some \( k \)
Thus
\[ |g_{m}^{i}| = S, \text{ and } |g_{j}^{k}| = S \]

Thus
\[ |\langle e_{i}, \ldots, e_{i-1}, g_{m}^{i}, e_{i+1}, \ldots, e_{n} \rangle| = S \text{ in } G \]
and
\[ |\langle e_{1}, \ldots, g_{j}^{k}, \ldots, e_{n} \rangle| = S \text{ in } G \]

Thus $G$ has more than one subgroup of orders $S$ \( \Rightarrow \), \( \ldots \) \( \gcd(|g_{i}|, |g_{j}|) = 1 \) for \( i \neq j \)
if \( \gcd(|G_i|, |G_j|) = 1 \) \( \forall i \neq j \),

and \( G_i \) is cyclic \( \forall i \),

let \( g_i \) be a generator of \( G_i \), so

\[ |g_i| = |G_i| \]

Recall that \( \gcd(m, n) = 1 \) \( \Rightarrow \text{LCM of } m = mn \)
Thus
\[ |(g_1, g_2, \ldots, g_n)| = \text{cm}(|g_1|, |g_2|, \ldots, |g_n|) \]
\[ = \text{cm}(|g_1|, |g_2|, \ldots, |g_n|) = \prod_{i=1}^{n} |G_i| = |G| \]  
\[ \Rightarrow \langle (g_1, \ldots, g_n) \rangle = G \Rightarrow G \text{ is cyclic.} \]
Cor \[ Z_m \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \]

\[ \iff m = \prod_{i=1}^{k} n_i \quad \text{and} \quad \gcd(n_i, n_j) = 1 \quad \forall i \neq j \]
eg. \( \sqrt{100} \sim 225 \oplus Z_4 \) can't split up

#2 combine

\[
\begin{align*}
Z & \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_5 \\
& \approx \ \\
& \sim \ \\
& \approx \ \\
& \sim \ \\
& \approx \ \\
& \sim \ \\
Z_2 \oplus Z_6 \oplus Z_15 \sim & Z_6 \oplus Z_{30}
\end{align*}
\]
\[\mathbb{Z}_{10} \oplus \mathbb{Z}_{22} \]

\[\mathbb{Z}_{32.5} \oplus \mathbb{Z}_{11.2} \]

\[\mathbb{Z}_{32} \oplus \mathbb{Z}_{5} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{2} \]