Mond - question on 6 + 7

Wed - put 6 on board
+ questions on 7

Friday - 6 due

Monday - 7 due - questions on 8 (?)
Def: External Direct Product

If $G_i$ is a group for all $i$

The external direct product is

$$\bigoplus_{i=1}^{n} G_i = G_1 \oplus G_2 \oplus \cdots \oplus G_n$$

$$= \{ (g_1, \ldots, g_n) \mid g_i \in G_i \text{ and } i \in \mathbb{Z}\}$$

with operation

$$(g_1, \ldots, g_n) \times (h_1, \ldots, h_n) =$$

$$(g_1h_1, g_2h_2, \ldots, g_nh_n)$$

where $g_i h_i$ is the product of $g_i$ and $h_i$ under the operation of the group $G_i$.

Note: $|\bigoplus_{i=1}^{n} G_i| = \prod_{i=1}^{n} |G_i| = |\mathbb{Z}_2 \oplus \mathbb{Z}_3| = 6$
Theorem

\[ |(g_1, \ldots, g_n)| = \text{lcm}(|g_1|, \ldots, |g_n|) \]

Proof: Let \( t = \text{lcm}(|g_1|, \ldots, |g_n|) \)

Then \((g_1, \ldots, g_n)^{-1} = (g_1^t, \ldots, g_n^t) = (e, \ldots, e_n)\)
Thus \(|(g_1, \ldots, g_n)| \leq t\).

So \(|(g_1, \ldots, g_n)| \leq t\).

Now let \(S = |(g_1, \ldots, g_n)|\).

Then \((g_1, \ldots, g_n)^S = (e_1, \ldots, e_n)\).

\[\Rightarrow (g_1^S, \ldots, g_n^S) = (e_1, \ldots, e_n)\]

\[\Rightarrow g_i^S = e_i \quad \forall i \Rightarrow |g_i|^S \quad \forall i\]

\[\Rightarrow S \text{ is a common multiple of the } |g_i|s,\]

but \(t\) was the least common multiple, so \(t \leq S\).

But we noted above that \(t = S\).

\[\therefore t = S\]

So \(\text{lcm}(|g_1|, \ldots, |g_n|) = |(g_1, \ldots, g_n)|\).

\[\text{ and } S\]
Example:

How many elements of order 7 are there in $\mathbb{Z}_{49} \oplus \mathbb{Z}_7$?

By previous theorem, $(a, b) \in \mathbb{Z}_{49} \oplus \mathbb{Z}_7$ has order 7 if $\text{lcm}(|a|, |b|) = 7$.
The possible orders for $a \in \mathbb{Z}_9$ are $|a| \leq 3, 7, 49$.

For $b \in \mathbb{Z}_7$, orders are $|b| \leq 3, 7$.

What are the possibilities so that $|\text{lcm}(|a|, |b|)| = 7$?

<table>
<thead>
<tr>
<th>Possibilities</th>
<th>How many?</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>a</td>
<td>= 1,</td>
</tr>
<tr>
<td>$</td>
<td>a</td>
<td>= 7,</td>
</tr>
<tr>
<td>$</td>
<td>a</td>
<td>= 7,</td>
</tr>
</tbody>
</table>

Total: 48 elements of order 7 in $\mathbb{Z}_9 \times \mathbb{Z}_7$. 

Title: Oct 29 - 9:36 AM (6 of 6)