NONLINEAR GYROKINETIC
TOKAMAK PHYSICS

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Abstract

The gyrokinetic reduced description of low-frequency and small-perpendicular-wavelength nonlinear tokamak dynamics is presented in three different versions: the reduced dynamical description of test particles moving in electromagnetic fields; the reduced gyrokinetic description of the self-consistent interaction of particles and fields through the Maxwell-Vlasov equations; and the reduced description of nonlinear fluid motion.

The unperturbed tokamak plasma is described in terms of a noncanonical Hamiltonian guiding-center theory. The unperturbed guiding-center tokamak plasma is then perturbed by gyrokinetic electromagnetic fields and consequently the perturbed guiding-center dynamical system acquires new gyrophase dependence. The perturbation analysis that follows makes extensive use of Lie-transform perturbation techniques. Because the electromagnetic perturbations affect both the Hamiltonian and the Poisson-bracket structure, the Phase-space Lagrangian Lie perturbation method is used.

The description of the reduced test-particle dynamics is given in terms of a noncanonical Hamiltonian gyrocenter theory. The description of the reduced kinetic dynamics is concerned with the self-consistent response of the guiding-center plasma and is described in terms of the nonlinear gyrokinetic Maxwell-Vlasov equations. It is also shown that the gyrokinetic Maxwell-Vlasov system possesses a gyrokinetic energy adiabatic invariant and that, in both the linear and nonlinear (quadratic) approximations, the corresponding energy invariants are exact.

The description of the reduced fluid dynamics is concerned with the derivation of a closed set of reduced fluid equations. Three generations of reduced fluid models are presented: the reduced MHD equations; the reduced FLR-MHD equations; and the gyrofluid equations. After a brief review of the derivation of the first two generations of reduced fluid models, we give a thorough presentation of the derivation of the
gyrofluid equations, viewed as moments of the gyrokinetic Maxwell–Vlasov system. The moment hierarchy is closed by assuming a small \((k_\perp \rho_i)\) ordering, which allows a direct comparison with the other reduced fluid models to be made.
To my parents:

Three passions, simple but overwhelmingly strong, have governed my life: the longing for love, the search for knowledge, and unbearable pity for the suffering of mankind. These passions, like great winds, have blown me hither and thither, in a wayward course, over a deep ocean of anguish, reaching to the very verge of despair.

What I have lived for, by Bertrand Russell
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To Dinah:

I have sought love, first, because it brings ecstasy — ecstasy so great that I would often have sacrificed all the rest of life for a few hours of this joy. I have sought, next, because it relieves loneliness — that terrible loneliness in which one shivering consciousness looks over the rim of the world into the cold unfathomable lifeless abyss. I have sought it, finally, because in the union of love I have seen, in a mystic miniature, the prefiguring vision of the heaven that saints and poets have imagined. This is what I sought, and though it might seem to good for human life, this is what — at last — I have found.

What I have lived for, by Bertrand Russell
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With equal passion I have sought knowledge. I have wished to understand the hearts of men. I have wished to know why the stars shine. And I have tried to apprehend the Pythagorean power by which number holds sway above the flux. A little of this, but not much, I have achieved.

What I have lived for, by Bertrand Russell
PART I: DYNAMICAL REDUCTION IN PLASMA PHYSICS

We present the basic principles involved in the asymptotic dynamical reduction of Hamiltonian systems. Three methods used to derive reduced Hamiltonian equations will be presented: (1) the classical method of phase-averaging in multiple-frequency systems; (2) the Hamiltonian Lie perturbation method; and (3) the Phase-space Lagrangian Lie perturbation method.

The existence of a fast-varying angle (or phase) in any dynamical system leads to a clear separation of time scales. For Hamiltonian systems, there exists a hierarchy of time scales associated with the phase-space coordinates \((Z_I, \theta)\), in which a fast-varying phase angle \(\theta\) is associated with a slow-varying action (or momentum) \(I\), and the other degrees of freedom \(Z\) have intermediate time scales. The purpose of the asymptotic dynamical reduction is to express Hamilton’s equations of motion in terms of new phase-space coordinates \((Y_J, \zeta)\), where the angle \(\zeta\) is an ignorable coordinate and \(J\) is its associated conserved action (momentum). The remaining reduced Hamiltonian dynamics involve the reduced phase-space coordinates \(Y\), a reduced Hamiltonian function \(H(Y; J)\), and a Poisson-bracket structure whose components are independent of the fast-angle coordinate \(\zeta\). (Further reduction can be accomplished if a fast-varying phase angle exits in the reduced dynamical system.)

The classical method known as the phase-averaging method provides us with a simple procedure to describe the slow (reduced) dynamics, in which the fast degree of freedom has been averaged away. The phase-averaging method, however, works best only at the lowest order in the perturbation analysis (see Kruskal [1962]).

The Hamiltonian Lie perturbation method (Cary [1981]) and the Phase-space Lagrangian Lie perturbation method (Littlejohn [1981, 1982a] and Cary and Littlejohn [1983]) are methods that naturally preserve the Hamiltonian properties of the original dynamical system. The new phase-space coordinates \((Y_J, \zeta)\) are obtained from the old phase-space coordinates \((Z_I, \theta)\) by a near-identity (Lie) transformation \(T_\zeta \doteq \exp \epsilon G\), where \(G\) is the generating vector field for the transformation, and \(\epsilon\) is a small parameter. Each coordinate is then represented as an asymptotic (Lie) series of the form \(Y = Z + \epsilon GZ + \mathcal{O}(\epsilon^2)\), etc., where the components of the vector field \(G\) are chosen so that the adiabatic-invariant action \(J\) satisfies the condition \(\dot{J} = \mathcal{O}(\epsilon^2)\).
It is important to remember that the reduced Hamiltonian dynamics correspond to the exact Hamiltonian dynamics expressed with phase-space coordinates for which the fast-varying phase angle is ignorable.

The Hamiltonian Lie perturbation method describes the construction of a momentum coordinate $J$, canonically conjugate to the fast-varying phase $\zeta$, based on the Darboux algorithm, used in the proof of the Darboux theorem (Arnold [1978] and Littlejohn [1979]). The canonical pair $(J, \zeta)$, thus constructed, is also decoupled from the other phase-space coordinates $Y$. Unfortunately, although the dependence on the fast-phase angle has been transformed away, the equations are, nonetheless, not locally symmetric with respect to the fast-phase angle. By this we mean that, given the gauge transformation $\zeta' = \zeta + \psi(r, t)$, where $\psi$ corresponds to a local definition of the fast-phase angle, the reduced equations of motion are not phase-gauge invariant. The requirement that the reduced dynamical equations be independent of the phase and its phase gauge is, therefore, equivalent to requiring that the reduced dynamical equations be locally symmetric with respect to the fast phase. A modified version of the Hamiltonian Lie perturbation method was, therefore, proposed by Hagan and Frieman [1985] to eliminate this unpleasant phase-gauge dependence, which did not appear in the original or phase-averaged dynamical equations.

The phase-space Lagrangian Lie perturbation method recognizes the fact that, in the most general cases, both the Hamiltonian function and the Poisson-bracket (symplectic) structure can be perturbed, whereas the use of the Hamiltonian method assumes that only the Hamiltonian function is perturbed. This remark is especially important in view of the fact that the introduction of the electromagnetic field in the phase-space Lagrangian description of charged particle dynamics affects both the Poisson-bracket structure (through the magnetic field) and the Hamiltonian function (through the electrostatic potential). The application of the Lagrangian method ensures the simultaneous derivation of fast-phase-independent expressions for the Hamiltonian and the Poisson brackets that are phase-gauge invariant.

The phase-space Lagrangian Lie perturbation method will be used extensively in this dissertation, both in regular phase space $(P, Q)$ for time-independent Hamiltonian systems, and in extended phase space $(P, Q, w, t)$ (where $w$ is the numerical value of the Hamiltonian function, and $t$ is time) for time-dependent Hamiltonian systems.
Chapter 1

Introduction

After briefly reviewing the present understanding of nonlinear tokamak physics, the nonlinear gyrokinetic ordering, first introduced by Frieman and Chen [1982], is presented.

The mathematical tools used in Hamiltonian mechanics are introduced, and the phase-space Lagrangian approach to Hamiltonian mechanics is presented. The presentation emphasizes a noncanonical-coordinate formulation of reduced Hamiltonian dynamics.

Finally, a brief description of Lie-transform based Hamiltonian perturbation theory is presented. Two methods are discussed: (1) the Hamiltonian Lie perturbation method; and (2) the phase-space Lagrangian Lie perturbation method. Both of these methods are described in terms of the so-called extended phase space (Lanczos [1970]), which includes time and energy as phase-space coordinates, and the corresponding extended Hamilton’s equations. This is to be contrasted with the state-space formulation which includes only the time variable (Cary and Littlejohn [1983]).

1.1 The Physics of Magnetically Confined Plasmas

Magnetically confined plasmas, found in either fusion-research devices or astrophysical media, exhibit a wide range of spatial and temporal scales. For example, consider the behavior of individual charged particles in a plasma confined by a strong magnetic field. The particles of species $s$ execute three types of (nearly) periodic
motions: (1) a gyromotion about a single magnetic field line, with a time scale associated with the gyrofrequency \( \omega_{ce}^{-1} \); (2) a parallel motion along a magnetic field line, with a time scale associated with the bounce frequency \( \omega_{bs}^{-1} \) for trapped particles or the transit frequency \( \omega_{ts}^{-1} \) for circulating particles (in toroidal geometry); and (3) the drift motion across magnetic field lines, with a time scale associated with the magnetic drift frequency \( \omega_{ds} \). For magnetically confined nonuniform plasmas, these time scales are normally well separated, \( \omega_{ce}^{-1} \ll \omega_{bs}^{-1} \ll \omega_{ts}^{-1} \ll \omega_{ds}^{-1} \).

It has become customary to refer to high-frequency behavior or low-frequency behavior by simply comparing the time scale of interest, \( \omega^{-1} \), with the electron (ion) gyrofrequency, \( \omega_{ce} (\omega_{ci}) \). Hence, high-frequency behavior (resp. low-frequency behavior) refers to processes which satisfy the condition \( \omega \gg |\omega_{ce}| \) (resp. \( \omega \ll \omega_{ci} \)).

1.1.1 Some Aspects of Nonlinear Tokamak Physics

Experimental observations of magnetically confined plasmas indicate that such plasmas represent strongly turbulent systems. (For a review of experimental evidence of plasma turbulence, see the review papers of Liewer [1985], and Wootton et al. [1989].) The observed turbulence is characterized by fluctuation spectra which exhibit the following features: (1) a broad frequency spectrum \( S(\omega, k) \), at fixed wavevector \( (\Delta \omega \sim \omega) \); (2) the characteristic (mean) frequency \( (\omega \simeq \omega_{pe}) \) and wavelength \( (k_{\perp} \simeq \rho_{s}^{-1}) \) of the fluctuation spectrum \( S(\omega, k) \) are typical of drift-wave turbulence theories; (3) high-\( \beta \) plasmas exhibit fluctuations in density \( \bar{n} \), temperature \( \bar{T} \), electrostatic potential \( \bar{\phi} \), and magnetic field \( \bar{B} \), each fluctuating quantity having its own spatial profile across the plasma discharge; and (4) the static spectrum \( S(k) \), or dynamic spectrum \( S(\bar{\omega}, k) \) at fixed frequency \( \bar{\omega} \), is highly anisotropic, i.e., measurements of the components of wavevector reveal that \( k_{\parallel} \ll k_{\perp} \), and indicate that the fluctuations form a 2-D turbulent system in the plane perpendicular to the local magnetic field.

In trying to understand the nonlinear dynamics of magnetically confined plasmas, it is important to address the following crucial issues: (1) the nonlinear evolution of linearly unstable coupled modes towards a saturated strongly turbulent state (including spectrum cascading rules and the nonlinear stability of linearly stable modes); (2) the marginal and nonlinear stability analysis of nonuniform magnetized plasmas with improved confinement properties (e.g., supershot discharges in TFTR, H modes
1.1. *The Physics of Magnetically Confined Plasmas*

in diverted plasma discharges such as DIII-D and ASDEX, and energetic-particle stabilization of MHD modes in several auxiliary-heated tokamak experiments; (3) the confinement degradation studies in auxiliary-heated tokamak plasmas; and (4) the nonlinear dynamics of self-organized convective (vortex) motion in tokamak plasmas.

The observed anomalous transport processes associated with present magnetic confinement devices are thought to be intimately related to the fluctuation-induced transport processes due to the saturated spectra of finite-amplitude, mode-coupled unstable linear modes. Unfortunately, many aspects of the nonlinear dynamics involved in the evolution toward such a saturated state, which often also displays self-organized large-scale motion, are poorly understood.

Studies of anomalous transport processes based on the well-known mixing-length arguments, which assess the impact of turbulent electromagnetic fluctuations, are usually made with the assumption that the tokamak plasma does not display coherent convective plasma motion, or spatial and/or temporal intermittent behavior. Experimental evidence, however, does not exclude these two types of behavior, and a thorough understanding of nonlinear coherent and turbulent dynamics is, therefore, needed. The ordering, presented below, will allow us to focus our attention on low-frequency nonlinear tokamak dynamics.

1.1.2 Gyrokinetic Description of Tokamak Physics

The gyrokinetic formalism is used to provide a reduced description of low-frequency, small-perpendicular-wavelength nonlinear tokamak dynamics. The linear gyrokinetic ordering was, originally, proposed by Hastie et al. [1967] and Rutherford and Frieman [1968] for electrostatic perturbations, and by Catto, Tang, and Baldwin [1981] and Antonsen and Lane [1980] for electromagnetic perturbations of tokamak plasmas. (In addition, the latter two works also used the so-called ballooning-mode representation in giving their gyrokinetic equations.) The nonlinear gyrokinetic ordering was later proposed by Frieman and Chen [1982] in order to investigate electromagnetic perturbations of a magnetized nonuniform plasma.

The gyrokinetic ordering, in both its linear and nonlinear versions, separates the spatial and temporal scaling properties of the unperturbed and perturbed dynamics.

1.1.2.1 Gyrokinetic Ordering of Unperturbed Quantities

The unperturbed tokamak dynamics are characterized by two small parameters,
defined as follows. The first small parameter \( \epsilon_B \equiv \rho_i / \ell_B \) (where \( \rho_i \) is the ion gyroradius and \( \ell_B \) is the equilibrium magnetic nonuniformity length scale) characterizes the (single-particle) guiding-center dynamics. It is well-known that guiding-center dynamics have a critical impact on the description of neoclassical transport processes (c.f., Hinton and Hazeltine [1976]), and on the microscopic stability properties of tokamak plasmas (c.f., Tang [1978]) through trapped-particle effects and effects due to magnetic field curvature and shear. (A rigorous description of guiding-center dynamics is given in chapter 2.)

The second small parameter \( \epsilon \equiv \rho_i / \ell_n \) (where \( \ell_n \) is the density-gradient length scale) characterizes the equilibrium plasma confinement properties, i.e.,

\[
| \rho_i \nabla \ln F_0 | = \mathcal{O}(\epsilon),
\]  

(1.1)

where \( F_0 \) is the equilibrium distribution function.

In toroidal geometry, we have \( \ell_n = a \) and \( \ell_B = qR \simeq R \), where \( a \) and \( R \) are the minor and major radii of the tokamak plasma discharge, and the safety factor \( q \) is of order unity. For most tokamak experiments, we also have \( \epsilon_B = \epsilon \epsilon_0 \ll \epsilon \), where \( \epsilon_0 = a/R \) is the inverse aspect ratio.

Finally, the unperturbed temporal-scale ordering is taken to be the so-called transport ordering (Frieman and Chen [1982] and Hagan [1986]), for which equilibrium quantities evolve on a time scale of order \( \mathcal{O}(\epsilon^{-3}) \) compared to the gyroperiod, i.e.,

\[
\left| \frac{1}{\omega_{ci}} \frac{\partial \ln F_0}{\partial t} \right| = \mathcal{O}(\epsilon^3),
\]  

(1.2)

with a similar ordering applied to the equilibrium magnetic field.

In our presentation, the small parameter \( \epsilon_B \) is used to characterize the guiding-center Hamiltonian dynamics of particles moving in nonuniform magnetic fields. In guiding-center phase space, the coordinates \( (X, U, \mu, \zeta) \) are such that Hamilton's equations are independent of the gyrophase \( \zeta \), and the magnetic moment \( \mu \) is conserved to an arbitrary order in \( \epsilon_B \). Hence, \( \mu \) is given as an asymptotic series

\[
\mu = \mu_0 + \epsilon_B \mu_1 + \mathcal{O}(\epsilon_B^2),
\]

where \( \mu_0 = v_\perp^2 / 2B \) is the well-known lowest-order term, and

\[
\dot{\mu}_0 = -\epsilon_B \omega_{ci} \frac{\partial \mu_1}{\partial \zeta},
\]
so that \( \dot{\mu} = \mathcal{O}(\epsilon_3^2) \). (The true magnetic moment thus acquires gyrophase dependence, but in such a way that it is conserved asymptotically.)

1.1.2.2. Gyrokinetic Ordering for Perturbed Quantities

The perturbed tokamak dynamics are characterized by a third small parameter \( \epsilon_6 \), which gives a relative measure of the perturbation fields, with respect to their equilibrium values

\[
\left| \frac{\delta F}{F_0} \right| = \mathcal{O}(\epsilon_6), \quad \text{etc.} \quad (1.3)
\]

In addition, the electrostatic potential, whose equilibrium value is assumed to be zero, is ordered

\[
\left| \frac{e\delta \phi}{T_e} \right| = \mathcal{O}(\epsilon_6),
\]

where \( T_e \) is an averaged value for the electron temperature.

The gyrokinetic ordering provides us with the following time-scale ordering for the perturbed fields,

\[
\left| \frac{1}{\omega_{ci} G_0} \frac{\partial \delta G}{\partial t} \right| = \mathcal{O}(\epsilon_6 \epsilon), \quad (1.5)
\]
or \( \omega/\omega_{ci} = \mathcal{O}(\epsilon) \), in the eikonal approximation. In addition, the gyrokinetic ordering provides the following spatial-scale ordering for the perturbed fields,

\[
\left| \frac{\rho_i \nabla \| \delta G}{G_0} \right| = \mathcal{O}(\epsilon_6 \epsilon), \quad \text{and} \quad \left| \frac{\rho_i \nabla \perp \delta G}{G_0} \right| = \mathcal{O}(\epsilon_6), \quad (1.6)
\]

where \( \nabla \| \) (resp. \( \nabla \perp \)) refers to a parallel gradient (resp. perpendicular gradient). In the eikonal approximation, we have \( k\| \rho_i = \mathcal{O}(\epsilon) \) and \( k\perp \rho_i = \mathcal{O}(1) \), which corresponds to the experimental situation in which the fluctuation spectrum is highly anisotropic: \( k\| R \approx 1 \) and \( k\perp a \gg 1 \).

Finally, in the nonlinear gyrokinetic ordering of Frieman and Chen [1982], the equivalence \( \epsilon_6 \equiv \epsilon \) was used, which meant that the linear and nonlinear time scales were comparable (i.e., \( k\perp \delta v_B \approx \omega_B \)). This subsidiary ordering allows full nonlinearity to be considered and shall be adopted in this work. In chapters 3 and 4, we shall derive nonlinear gyrokinetic equations, valid up to order \( \mathcal{O}(\epsilon_6 \epsilon_B) \) and \( \mathcal{O}(\epsilon_6^2) \). These low-frequency equations will be expressed in terms of gyrocenter coordinates \((X, U, \bar{\mu}, \bar{\zeta})\), obtained from the guiding-center coordinates \((X, U, \mu, \zeta)\) by a near-identity (Lie) transformation (dependent on the small parameter \( \epsilon_6 \)), where \( \bar{\mu} = \mu_0 + \epsilon_6 \bar{\mu}_1 + \mathcal{O}(\epsilon_6^2) \) is the modified magnetic moment (with \( \mu_0 = \mu \), the guiding-center true magnetic
moment). In the resulting Hamiltonian theory, the new gyrophase angle $\zeta$ is an ignorable coordinate, and its associated action, the magnetic moment $\vec{\mu}$, is a constant of the reduced motion. Because terms of order $\mathcal{O}(e^2)$ are not included in our analysis, the equilibrium tokamak configuration is assumed stationary in time. Furthermore, the $\beta$ value does not enter our ordering scheme and is left arbitrary, for maximal generality.

1.1.3 Overview of the Dissertation

We present here an overview of the dissertation with an emphasis on the original contributions made by the author. Our description of low-frequency nonlinear tokamak dynamics, based on the gyrokinetic ordering, is presented in the form of three reduced descriptions

Test-particle Description : Gyrocenter Dynamics,

Self-consistent Kinetic Description : Gyrokinetic Maxwell-Vlasov System,

Fluid Description : Gyrofluid Equations.

The three reduced descriptions are in fact members of a unified (gyrokinetic) hierarchy in which the gyrokinetic (self-consistent) Maxwell-Vlasov system makes use of the gyrocenter dynamics, and the gyrofluid equations are derived as moments of the gyrokinetic Maxwell-Vlasov system. The derivation of the gyrocenter dynamics is therefore the critical element in the gyrokinetic hierarchy and involves the perturbation of an equilibrium tokamak (guiding-center) plasma by electromagnetic field perturbations that satisfy the gyrokinetic ordering.

The perturbation analysis performed throughout this dissertation is based on the Phase-space Lagrangian Lie perturbation method of Littlejohn [1981, 1982a, 1983] and Cary and Littlejohn [1983]. Our original contribution comes from the application of this method to the extended phase-space Lagrangian dynamics (see sections 1.2. and 1.3.)

The remainder of the dissertation is organized as follows. The pioneering work of Littlejohn [1979, 1981, 1983] on the development of the noncanonical Hamiltonian theory of guiding-center motion is presented in chapter 2. The presentation of this material is made for the purpose of introducing in a familiar setting (i.e., the guiding-center problem) the Phase-space Lagrangian Lie perturbation method and results
from this chapter will be used extensively throughout the dissertation.

The noncanonical Hamiltonian theory of gyrocenter dynamics is given in chapter 3. A perturbative approach based on the Phase-space Lagrangian Lie perturbation method is used in which an (equilibrium tokamak) extended guiding-center phase-space Lagrangian is perturbed by gyrokinetic electromagnetic fields. As a result of the perturbation analysis a transformation to extended gyrocenter phase space is constructed and one obtains a noncanonical Hamiltonian dynamical system which is independent of the gyrocenter gyrophase angle. Our original contributions to the theory of Hamiltonian gyrocenter dynamics are concerned with: (1) the use of the extended Phase-space Lagrangian Lie perturbation method and the inclusion of the full electromagnetic perturbation field \( \delta \phi, \delta A_{\parallel}, \delta B_{\parallel} \); (2) the identification of two different representations for the gyrocenter dynamics where in the symplectic (resp. Hamiltonian) representation the perturbed components of the Poisson-bracket structure have been partially (resp. completely) transferred onto the gyrocenter Hamiltonian; and (3) the consideration of the full toroidal geometry and its effects on the gyrocenter dynamics.

The gyrokinetic Maxwell–Vlasov system, which describes the self-consistent interaction of the gyrocenter particles and the gyrokinetic electromagnetic field perturbations, is presented in chapter 4. In addition, we present the derivation of a gyrokinetic energy invariant which is shown to be adiabatic (i.e., conserved to all orders in our perturbation analysis) and give exact gyrokinetic energy conservation laws for the linear and nonlinear (quadratic) gyrokinetic Maxwell–Vlasov systems. Our original contributions to the derivation of the gyrokinetic Maxwell–Vlasov equations are concerned with: (1) the consideration of the full set of the gyrokinetic Maxwell’s equations; (2) the derivation of an adiabatic gyrokinetic energy invariant and the construction of exact gyrokinetic energy conservation laws which include the full electromagnetic perturbation field. Finally, we present two original contributions in the development of the gyrokinetic formalism: (1) the formalism used in the construction of a gyrokinetic collision operator; and (2) the application of the extended Phase-space Lagrangian Lie perturbation method to the derivation of known equations in the gyrokinetic formalism for arbitrary frequencies.

Three generations of reduced fluid models used in the nonlinear analysis of tokamak dynamics are described in the chapters 5 and 6. The first two generations (the RMHD generation and the FLR-RMHD generation, respectively) are briefly derived
in chapter 5. The third generation, the so-called gyrofluid equations, is obtained from the moments of the gyrokinetic Maxwell–Vlasov equations in the limit of small \( k_{\perp}\rho_i \) and is presented in chapter 6. This chapter essentially represents an original contribution to the formalism of reduced fluid modeling of nonlinear tokamak dynamics. A comparison of the gyrofluid model with two second-generation reduced fluid models is also made in chapter 6.

Chapter 7 contains a summary of the original contributions of this dissertation and a discussion on the possible uses of our reduced equations. Finally, Appendix A describes the use of magnetic coordinates in the Hamiltonian theories of guiding-center and gyrocenter dynamics. Our original contribution is concerned with the generalization of the equations of White and Chance [1984] by considering the full electromagnetic field and by adopting the gyrokinetic ordering. Appendix B introduces some techniques used in the derivation of FLR corrections to the set of MHD equations.
1.2 Geometric Foundations of Hamiltonian Mechanics

We begin this section with two quotes taken from the book of V. I. Arnold [1978]: “A Lagrangian mechanical system is given by a manifold, the configuration space, and a function on its tangent bundle, the Lagrangian function.” (p. 53); “A Hamiltonian mechanical system is given by an even-dimensional manifold, the phase space, a symplectic structure on it, the Poincaré integral invariant, and a function on it, the Hamiltonian function.” (p. 161).

These two quotes emphasize the fact that the mathematical theory of Hamiltonian mechanics has greatly evolved compared to the theory presented in Lanczos [1970] or Goldstein [1980], for example. In order to fully make use of the recent developments in the theory of Hamiltonian dynamics, especially the developments in Hamiltonian perturbation theory, it will therefore be useful to introduce the proper mathematical terminology (Littlejohn [1982a]).

1.2.1 Vector Fields and Differential Forms

This subsection introduces the mathematical concepts used in other parts of this work. Consequently, the presentation is strongly focused and only results are given. For more information the reader is referred to the excellent books by Arnold [1978] and Abraham and Marsden [1978].

1.2.1.1 Vector Fields on a Manifold

The geometrical interpretation of a vector $V_x$ is that it is tangent to a curve $C(t)$ at point $x = C(0)$ in space $M$. The usual physical interpretation of such a vector is that it represents the velocity vector of the curve $C$ at point $x$,

$$\left.\frac{d}{dt}\right|_{t=0} C(t) = V_x.$$  (1.7)

(One can show that $V_x$ is independent of the curve $C$ since an equivalence class of tangent curves can be used.) This definition implies that the tangent vector $V_x$ acts as a derivation. Indeed, for an arbitrary function $f$, we find

$$\left.\frac{d}{dt}\right|_{t=0} f(C(t)) = V^i(x) \frac{\partial f}{\partial x^i}(x) = V_x(f).$$  (1.8)
Hence, $V_x$ is a differential operator (with components $V^i$) on functions, and has the following properties

$$V_x(a f + b g) = a V_x(f) + b V_x(g),$$
$$V_x(f g) = V_x(f) g + f V_x(g),$$

for arbitrary real numbers $(a, b)$ and functions $(f, g)$.

The space of vectors tangent to $M$ at $x$ is a vector space known as the tangent vector space, denoted $T_x M$. If the manifold $M$ has dimension $m$, then so does $T_x M$, and a suitable basis set for this vector space, as can be seen in Eq. (1.8), is given as $\{\partial/\partial x^1, \ldots, \partial/\partial x^m\}$. Thus, for every pair of tangent vectors $V_x$ and $U_x$, we have

$$(a V_x + b U_x)(f) = a V_x(f) + b U_x(f),$$

for arbitrary real numbers $(a, b)$, and function $f$. A vector field $V$ on the space $M$ assigns a tangent vector $V_x$ in $T_x M$ for each point $x$ of $M$, and $V_x$ varies smoothly with $x$. Finally, the collection of all tangent vector spaces on $M$, denoted $TM$, is called the tangent bundle. An element of $TM$ is represented as $(x, V)$, and consequently $TM$ has dimension $2m$.

The dynamical properties of the vector field $V$ are emphasized through the introduction of the concept of flows. Consider the following definition of a one-parameter group of transformations, which has the following properties

$$\Phi(s, \Phi(t, x)) = \Phi(s + t, x),$$
$$\Phi(0, x) = x,$$
$$\left.\frac{d}{dt}\right|_{t=0} \Phi(t, x) = V_x.$$

The mapping $\Phi$ is said to be the flow generated by the vector field $V$. In most cases, it is also useful to define a mapping $\phi^t : M \to M$, by the relation $\phi^t(x) = \Phi(t, x)$. Under normal conditions, the flow $\Phi$ can be solved in terms of $V$ as

$$\Phi(t, x) = \phi^t(x) = \exp(tV)x. \quad (1.9)$$

For an arbitrary function $f$, we have

$$\frac{d}{dt} f \left(\phi^t(x)\right) = V_y(f), \quad (1.10)$$
where \( y = \phi^i(x) \). In particular, we find \( f(\exp(\epsilon V)x) = f(x) + \epsilon V_x(f) + \mathcal{O}(\epsilon^2) \), so that \( V_x(f) \) represents the infinitesimal change in \( f \) under the flow generated by \( V \) at \( x \). The finite change in \( f \) can be represented by the Lie series

\[
f(\exp(tV)x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} V_x^k(f),
\]

where \( V_x^0(f) = f(x) \). For example, if we choose the function \( f = \pi^i \) (the \( i \)-coordinate projection), we obtain from \( y = \exp(tV)x \),

\[
y^i = \pi^i(\exp(tV)x) = x^i + t V_i(x) + \frac{t^2}{2} V_i(V_x) + \mathcal{O}(t^3).
\]

1.2.1.2. Transformation Properties of Dynamical Systems

A dynamical system is defined by giving a flow \( \Phi \) and the manifold \( M \) on which it operates. The dynamical equations are, then, given as

\[
\frac{dx}{dt} = V_x, \quad \text{or} \quad \frac{dx^i}{dt} = V^i(x), \quad \text{for} \quad i = 1, \ldots, m,
\]

where \( x = \Phi(t, x_0) \), and \( x_0 \) is the initial condition \( \Phi(0, x_0) \). It is sometimes very useful to transform this dynamical system \( (\Phi, M) \) into another dynamical system \( (\Psi, N) \), where the mapping \( F : M \to N \) is introduced. We shall see that the mapping \( F \) induces a mapping on tangent vectors such that the tangent vector \( V \), which generates the flow \( \Phi \), is transformed into the tangent vector \( W \), which generates the flow \( \Psi \). Such a transformation may, for instance, be used to simplify the dynamical system given by Eq. (1.13), in which some components of the vector field \( W \) vanish. In this dissertation, we are concerned with a Hamiltonian dynamical system, where \( x = (q, p) \) is a (canonical) point in the \( 2n \)-dimensional phase space and the dynamical equations are given as

\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \text{for} \quad i = 1, \ldots, n.
\]

A mapping \( F : M \to N \), from the \( m \)-dimensional manifold \( M \) to the \( n \)-dimensional manifold \( N \), induces a mapping on tangent vectors as follows. We define the tangent map \( TF : TM \to TN \) by the relation

\[
TF(x, V) = (F(x), DF(x) \cdot V_x),
\]
where $DF(x)$ is the Jacobian matrix evaluated at $x$, and the tangent vector to $N$ at $y = F(x)$ is given as

$$DF(x) \cdot V_x = \left( V^i(x) \frac{\partial F^j}{\partial x^i}(x) \right) \frac{\partial}{\partial y^j}.$$  

(1.14)

A useful notation is given by the formula

$$TF(x, V) = F_* V_x = \left. \frac{d}{dt} \right|_{t=0} \phi^t F(x),$$  

(1.15)

where $F_*$ is known as a push-forward operator on vector fields, and $\phi^t$ is a pull-back operator on functions, defined as

$$\phi^t F(x) = F \left( \phi^t(x) \right).$$  

(1.16)

The push-forward operator $F_*$, induced by the mapping $F$, takes a tangent vector $V_x$ tangent to $M$ at $x$, and gives a tangent vector $F_* V_x$ tangent to $N$ at $F(x)$. Given a function $f$ on $N$, we therefore find

$$[F_* V]_{F(x)}(f) = V_x \left( F^*(f) \right).$$  

(1.17)

1.2.1.3. Differential Forms on a Manifold

The treatment of differential forms on a manifold begins with the concept of the dual of a vector space. Given an $n$-dimensional vector space $V$, with basis $\{e_1, ..., e_n\}$, we define the $n$-dimensional dual vector space $V^*$, with basis $\{\omega^1, ..., \omega^n\}$, as follows.

An element $\alpha$ of $V^*$ is defined as a linear form on $V$, and has the following properties

$$\alpha(ax + by) = a\alpha(x) + b\alpha(y),$$

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x),$$

for arbitrary real numbers $(a, b)$ and elements $(x, y)$ of $V$. (The image of $x$ under the action of $\alpha$ is defined as a real number.) Using the bases for $V$ and $V^*$, we find $x = x^i e_i$, and $\alpha = \alpha^i \omega_i$, so that

$$\alpha(x) = \alpha^i x^i,$$

where we have used the relation $\omega^i(e_j) = \delta^i_j$, which defines the dual basis set.
1.2. Geometric Foundations of Hamiltonian Mechanics

Similarly, the vector space $T_x^*M$ is defined as the dual space of the tangent vector space $T_xM$, and is known as the cotangent space. The elements of $T_x^*M$ are known as differential one-forms $\gamma_x$, and are represented as

$$\gamma_x = \gamma_i(x) \, dx^i,$$  \hspace{1cm} (1.18)

where the local basis set $\{dx^1, ..., dx^m\}$ is dual to the local basis set $\{\partial/\partial x^1, ..., \partial/\partial x^m\}$, i.e., $dx^i(\partial/\partial x^j) = \delta^i_j$. From these definitions, we find

$$\gamma_x(V) = \gamma_i(x)V^i(x).$$  \hspace{1cm} (1.19)

A covector field $\gamma$ on the space $M$ assigns a differential one-form $\gamma_x$ in $T_x^*M$ for each point $x$ of $M$, and $\gamma_x$ varies smoothly with $x$. The collection of all cotangent spaces on $M$, denoted $T^*M$, is called the cotangent bundle. An element of $T^*M$ is represented as $(x, \gamma)$, and consequently $T^*M$ has dimension $2m$.

The geometrical interpretation of a one-form involves the concept of contour lines (see, for example, Misner, Thorne, and Wheeler [1973]). Hence, the expression $\gamma_x(V)$ represents the number of contour lines pierced by the vector $V_x$. This concept is made especially clear when we consider the most common differential one-form given by the differential of a function $f$,

$$df_x = \frac{\partial f}{\partial x^i}(x)dx^i,$$

the components of which are normally associated with the gradient of $f$. The application of $df_x$ on the vector $V_x$ gives

$$df_x(V) = V^i(x)\frac{\partial f}{\partial x^i}(x),$$

which represents the directional derivative of $f$ in the direction of $V_x$ at $x$. Another geometrical interpretation, which is generalized easily, comes from the realization that every (oriented) line integral, e.g.,

$$S_C = \int_C P(x,y)dx + Q(x,y)dy$$

(where $C$ is a path on the plane), is an integral over a differential one-form along a directed path. Likewise, every oriented surface integral involves the integral over a differential two-form on an oriented area. The generalization of differential one-forms to $k$-forms, therefore, involves the consideration of antisymmetric multilinear forms.
Chapter 1. Introduction

To consider the generalization to differential two-forms, we return to our \( n \)-dimensional vector space \( V \), introduced earlier, and consider antisymmetric bilinear forms on \( V \). We define the space of bilinear forms \( L_2(V, R) \) as the collection of bilinear forms which have the properties

\[
\alpha_2(ax + by, z) = a\alpha_2(x, z) + b\alpha_2(y, z) \quad \text{and} \quad \alpha_2(x, ay + bz) = a\alpha_2(x, y) + b\alpha_2(x, z),
\]

for arbitrary real numbers \((a, b)\), and elements \((x, y, z)\) of \( V \). In tensor analysis, these bilinear forms are known as two-covariant tensors, and the basis for the space of two-covariant tensors is composed of the elements \((\omega^i \otimes \omega^j)\), where \(\omega^i (i = 1, \ldots, n)\) is an element of the dual basis for \( V^* \) and \( \otimes \) represents the tensor product defined as

\[
(\omega^i \otimes \omega^j)(e_k, e_l) = \delta_k^i \delta_l^j.
\]

The antisymmetric bilinear forms are defined in terms of the antisymmetric version of the tensor product, known as the wedge product, defined as

\[
\omega^i \wedge \omega^j = (\omega^i \otimes \omega^j - \omega^j \otimes \omega^i).
\]  
(1.20)

This definition implies the following expression

\[
\omega^i \wedge \omega^j(x, y) = x^i y^j - x^j y^i,
\]

which displays the desired antisymmetry with respect to the permutation of \( x \) and \( y \). Hence, antisymmetric bilinear forms \( \alpha^2 \) are represented as

\[
\alpha^2 = \frac{1}{2!} A_{ij} \omega^i \wedge \omega^j,
\]

where \( A_{ij} = \alpha^2(e_i, e_j) = -A_{ji} \).

Similarly, the differential two-forms on a manifold form a vector space \( \Lambda^2_x M \), and its elements are represented as

\[
\omega_x = \frac{1}{2!} \omega_{ij}(x) dx^i \wedge dx^j.
\]  
(1.21)

By definition, we have \( \Lambda^1_x M = T^*_x M \), and the space of zero-forms is the space of functions on \( M \). We also note that the maximum degree a differential form on a \( m \)-dimensional manifold \( M \) is \( m \), i.e., \( d\omega_x^m = 0 \). Hence, every differential \( m \)-form is closed, and for any two differential \( m \)-forms \( \omega_x^m \) and \( \Omega_x^m \), there exists a unique function \( f(x) \) such that \( \omega_x^m = f(x)\Omega_x^m \).
In the present work, we shall only be concerned with differential two-forms which are obtained from one-forms through the application of the so-called exterior derivative. The exterior derivative
\[ d : \bigwedge^k M \rightarrow \bigwedge^{k+1} M, \]
is a fundamental operation in the theory of exterior calculus of E. Cartan (Abraham and Marsden [1978]). Consider a differential one-form \( \gamma_x \) given as
\[ \gamma_x = \gamma_i(x)dx^i. \]
The exterior derivative of \( \gamma_x \), denoted \( d\gamma_x \), is defined as
\[ d\gamma_x = \frac{\partial \gamma_1}{\partial x^2}dx^2 \wedge dx^1 + \left( \frac{\partial \gamma_1}{\partial x^1} - \frac{\partial \gamma_2}{\partial x^1} \right) dx^1 \wedge dx^2 + \cdots. \]
Note that if \( \gamma_x \equiv df_x \), we immediately get \( d^2 f_x \equiv 0 \). (This is a generalization of the three-dimensional expression \( \nabla \times \nabla f = 0 \.) This property is, in fact, a general result. If a differential \( k \)-form \( \omega^k \) is given as the exterior derivative of a \((k-1)\)-form \( \alpha^{k-1} \), then we have
\[ \omega^k_x = d\alpha^{k-1}_x \rightarrow d\omega^k_x = d^2 \alpha^{k-1}_x = 0. \]
The \( k \)-form \( \omega^k_x = d\alpha^{k-1}_x \) is called an exact differential form, and if a \( k \)-form \( \omega^k \) satisfies the condition \( d\omega^k = 0 \), it is then called a closed differential form. Obviously, every exact differential form is closed (Poincaré lemma), but the converse is not necessarily true (Arnold [1978]).

The interior product, also known as the antiderivation operator,
\[ i : T^*_x M \times \bigwedge^k M \rightarrow \bigwedge^{k-1} M \]
is another operation in exterior calculus, and is defined as
\[ i_W^k \omega^k_x(V_1, ..., V_{k-1}) = \omega_x^k(W, V_1, ..., V_{k-1}), \]
where, by definition, we have \( i_W^0 f = 0 \), for any zero-form \( f \). Additional properties are given as: (1) \( i_h \omega^k \alpha = h \omega^k \alpha \), for any arbitrary function \( h \); (2) \( i^2 \omega^k = 0 \), \( k > 0 \); and (3) \( i_V(\alpha^k \wedge \beta^l) = i_V \alpha^k \wedge \beta^l + (-1)^k \alpha^k \wedge i_V \beta^l \). For a differential one-form \( \gamma_x = \gamma_i(x)dx^i \), the expression \( i_V \gamma_x \) is the function
\[ i_V \gamma_x = \gamma_x(V) = \gamma_i(x)V_i(x), \] (1.22)
and for a differential two-form \( \omega^2_x = (1/2)\omega_{ij}(x)dx^i \wedge dx^j \), we have
\[
i^*\omega^2_x = \omega_{ij}(x)W^i(x)dx^j,
\]
and
\[
i^*\omega^2_x(V) = \omega_{ij}(x)W^i(x)V^j(x). \tag{1.23}
\]

Finally, the mapping \( F : M \to N \) induces the pull-back \( F^* : \Lambda^k_F N \to \Lambda^k_M \) on differential \( k \)-forms as follows. Given any set of tangent vectors \( (V_1, ..., V_k) \) in \( T_x M \), and a differential \( k \)-form \( \omega^k \) in \( T^*_F N \), we obtain the \( k \)-form \( (F^* \omega^k) \) in \( T^*_x M \) as
\[
[F^*\omega^k]_x(V_1, ..., V_k) = \omega^k_{F(x)}(F_*V_1, ..., F_*V_k), \tag{1.24}
\]
where \( F_*V_l \) is the push-forward of the tangent vector \( V_l \in T^*_x M \), defined in Eq. (1.17).

For example, the expression for \( [F^*\gamma]_x(V) \) is given as
\[
[F^*\gamma]_x(V) = \gamma_i(F(x)) \frac{\partial F^i}{\partial x^j}(x) V^j(x),
\]
and
\[
(F^*\gamma)_x = \gamma_i(F(x)) \frac{\partial F^i}{\partial x^j}(x) dx^j \in T^*_x M. \tag{1.25}
\]
The application of the pull-back \( F^* \) to a zero-form \( f \), is shown in Eq. (1.10) as
\[
F^*f(x) = f(F(x)).
\]

### 1.2.1.4. Lie Derivatives

The concept of the Lie derivative is closely related to the issue of determining the infinitesimal change in geometric objects such as functions, vector fields, and differential forms brought about by the flow \( \phi^t = \exp(tV) \), generated by \( V \). The concept of Lie derivative acquires a special importance if the flow \( \phi^t \) represents a transformation on the space \( M \), as will be the case in this dissertation.

For instance, the infinitesimal change in the function \( f \) under the flow generated by \( V \) was given in Eq. (1.16) as
\[
\left. \frac{d}{dt} \right|_{t=0} \phi^t f(x) = V_x(f) \doteq LV f(x),
\]
which defines the Lie derivative of \( f \) with respect to \( V \). This definition can be generalized to arbitrary differential \( k \)-forms with
\[
L_V \omega^k_x = \left. \frac{d}{dt} \right|_{t=0} \phi^t \omega^k_x, \tag{1.26}
\]
for \( k = 0, 1, \ldots, m \). As in Eq. (1.9), the pull-back operator \( \phi^* \) induced by the flow \( \phi^t \) is solved in terms of \( V \) as

\[
\phi^* = \exp(tL_V),
\]

(1.27)

where \( L_V \) is the Lie derivative with respect to \( V \).

One can easily verify that the Lie derivative preserves the degree of the differential form it operates on. Furthermore, it is straightforward to verify that the Lie derivative \( L_V \) is given in terms of the homotopy formula

\[
L_V \alpha = d(\mathbf{i}_V \alpha) + \mathbf{i}_V (d \alpha),
\]

(1.28)

which holds for arbitrary \( k \)-forms \( \alpha \). For example, consider the differential one-form \( \gamma_x = \gamma_i(x)dx^i \), then one finds

\[
\mathbf{i}_V d \gamma_x = \mathbf{i}_V \left( \frac{\partial \gamma_j}{\partial x^i} dx^i \wedge dx^j \right) = \frac{\partial \gamma_j}{\partial x^i} (x) \left[ V^i(x)dx^j - V^j(x)dx^i \right],
\]

\[
di_V \gamma_x = d \left( \gamma_i(x)V^i(x) \right) = \left[ V^i(x) \frac{\partial \gamma_i}{\partial x^j}(x) + \gamma_i(x) \frac{\partial V^i}{\partial x^j} \right] dx^j,
\]

so that

\[
L_V \gamma_x = \gamma_i(x) \frac{\partial V^i}{\partial x^j}(x) dx^j + V^j(x) \frac{\partial \gamma_i}{\partial x^j}(x) dx^i.
\]

(1.29)

Finally, the infinitesimal change induced in a vector field \( W \) by the vector field \( V \) is again expressed in terms of the Lie derivative \( L_V \), and we have

\[
L_V W_x = - \frac{d}{dt} \bigg|_{t=0} \phi^*_t W_x = [V, W]_x,
\]

(1.30)

where \( [\cdot, \cdot] \) represents a bilinear, antisymmetric product on \( T_xM \) known as the Lie bracket or commutator, and \( \phi^*_t = (\phi^t)^{-1} \). Expressed in local coordinates, it is given as

\[
[V, W]_x = \left( V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right)(x) \frac{\partial}{\partial x^j},
\]

(1.31)

which means that \( [V, W]_x \) is again a tangent vector in \( T_xM \). An important property of the Lie bracket is that it satisfies the so-called Jacobi identity,

\[
[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0,
\]

which can be easily verified by using the local expression given in Eq. (1.31).
Finally, the push-forward operator $F_*$, induced by the mapping $F$, takes the Lie bracket of two tangent vectors in $T_x M$ and gives a Lie bracket of two tangent vectors in $T_{F(x)} N$ according to the rule

$$(F_* [V, W])_{F(x)} = [F_* V, F_* W]_{F(x)}.$$

### 1.2.2 Hamiltonian Mechanics in Extended Phase Space

#### 1.2.2.1 Symplectic Manifold

A symplectic structure on an even-dimensional manifold $M$ is given by a closed, nondegenerate differential two-form $\omega^2$ which satisfies (Arnold [1978]): $d\omega^2 = 0$ (definition of a closed form); and for nondegeneracy, we require that for all $V \neq 0$, there exists at least one vector $W$ such that $\omega^2(V, W) \neq 0$. A symplectic manifold is defined by giving $(M, \omega^2)$, where $M$ is referred to as the phase space of dimension $2m$ and $m$ is the number of degrees of freedom.

A famous theorem of Darboux (Arnold [1978], Littlejohn [1979]) states that, for every point $z \in M$, one can always find local coordinates for which the symplectic structure is constant. The more familiar statement of this theorem is that one can always find canonical variables $(q^i, p_i) \in M$ such that

$$\omega^2 = dp_i \wedge dq^i.$$  \hspace{1cm} (1.32)

In these canonical coordinates, Hamilton's equations are given as

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i},$$

where $H$ is known as the Hamiltonian function.

If we write $(p_i, q^i) = z \in M$, then Hamilton's equations are given in terms of a Hamiltonian vector field $X_H$,

$$\frac{dz}{dt} = X_H,$$

and we find a correspondence between the one-form $dH$ and the symplectic structure $\omega^2$ as

$$i_{X_H} \omega^2 = dH.$$

Finally, the Poisson bracket of two phase-space functions $f$ and $g$ is given as

$$\{f, g\}(z) = -i_{X_f} i_{X_g} \omega^2 = \omega^2(X_f, X_g),$$

$$= -L_{X_f} g = L_{X_g} f,$$
where the vector fields $X_f$ and $X_g$ are Hamiltonian vector fields, with $f$ and $g$ as Hamiltonian functions, respectively. Hamilton's equations are, therefore, given as
\[
\frac{dz}{dt} = \{z, H\}(z) = L_{X_H}z = dH_z(X_z).
\]

1.2.2. Phase-space Lagrangian Dynamics

From the canonical expression for the symplectic structure given in Eq. (1.32), now denoted $\tilde{\omega}$ in view of the generalization introduced in Eq. (1.34), we find that there is a one-form $\tilde{\gamma}$ which locally satisfies the condition $\tilde{\omega} = d\tilde{\gamma}$, namely,
\[
\tilde{\gamma}_z = p_i(z) dq^i.
\]

We note that this one-form is not unique, since any exact one-form $dS$ can be added to $\tilde{\gamma}$, without changing $\tilde{\omega}$ since $d^2 S = 0$.

Consider the fundamental one-form of Poincaré-Cartan (Arnold [1978])
\[
\gamma = \tilde{\gamma} - H dt,
\]
where $\tilde{\gamma} = p_i dq^i$, and $H(p_i, q^i)$ is the time-independent Hamiltonian function. (There should not be any confusion arising from the fact that we refer to $\tilde{\gamma}$ and $\tilde{\omega} = d\tilde{\gamma}$ as being the symplectic structure.) The two-form obtained from $\gamma$ is given as
\[
\omega = \tilde{\omega} - dH \wedge dt = dp_i \wedge dq^i - dH \wedge dt.
\]

If we use the notation $i_{p_k}$ to indicate $i_V$ where $V = \partial / \partial p_k$ (the local basis component vector), then we have
\[
i_{p_k} \omega = \left( dq^k - \frac{\partial H}{\partial p_k} dt \right),
\]
\[
i_{q^i} \omega = \left( -dp_k - \frac{\partial H}{\partial q^k} dt \right).
\]

The use of the name integral invariant of Poincaré-Cartan, Eq. (1.33), refers to the fact that Hamilton's equations are obtained from the requirement that Eqs. (1.35) and (1.36) vanish. The integral invariant of Poincaré-Cartan, Eq. (1.33), is referred to as the phase-space Lagrangian, in analogy with the Lagrangian function $L = p_i \dot{q}^i - H$ in configuration space.
The important physical case where the Hamiltonian is time-dependent is dealt with as follows. We extend the regular phase space to include the time coordinate $t$ and its (anti)conjugate energy coordinate $w$ by giving the extended phase-space Lagrangian as

$$\gamma_E = \gamma_E - H dt = p_i dq^i - w dt - (\tilde{H} - w) dt,$$

where $\gamma_E$ is the extended symplectic structure, $\tilde{H}(p, q, t)$ is the regular Hamiltonian, and $\tau$ is the extended-time parameter. Defining again a two-form $\omega_E = d\gamma_E$, we obtain the regular Hamilton’s equations by requiring that $i_{p_\tau} \omega_E$ and $i_{q_\tau} \omega_E$ vanish, so we have

$$\frac{dq^k}{d\tau} = \frac{\partial H}{\partial p_k} = \frac{\partial \tilde{H}}{\partial p_k}, \quad \text{and} \quad \frac{dp_k}{d\tau} = -\frac{\partial H}{\partial q^k} = -\frac{\partial \tilde{H}}{\partial q^k}. \quad (1.38)$$

The two remaining Hamilton’s equations are, likewise, obtained by requiring that $i_{\omega_E}$ and $i_{\omega_E}$ vanish. It is simple to show that these two equations are given as

$$\frac{dt}{d\tau} = -\frac{\partial H}{\partial w} = 1, \quad \text{and} \quad \frac{dw}{d\tau} = \frac{\partial H}{\partial t} = \frac{\partial \tilde{H}}{\partial t}. \quad (1.39)$$

The physical interpretation of the extended Hamiltonian $H = \tilde{H} - w$ is that the physical (Hamiltonian) trajectory in extended phase-space occurs on the submanifold $H^{-1}(0) = \{(q^i, p_i, w, t) : w = \tilde{H}(p_i, q^i, t)\}$.

### 1.2.3 Lagrange and Poisson Brackets

The recent developments in Hamiltonian mechanics have shown that the use of canonical coordinates is not necessary, and can even be counterproductive. (See, for example, Littlejohn [1979, 1982a] and Greene [1982].) The expression of Hamiltonian dynamics in phase space require a Hamiltonian function, and a Poisson-bracket (symplectic) structure $\{\ , \}$, which is bilinear and antisymmetric in its entries, and satisfies the Jacobi identity.

In this subsection, we shall derive the Poisson-bracket structure from the symplectic structure of the phase-space Lagrangian, and show that this construction automatically satisfies the Jacobi identity. In addition, in preparation for future considerations, we shall derive the necessary expression that a phase-space Lagrangian must have in order for one of the dynamical coordinates to be ignorable and its associated momentum to be conserved.
1.2. Geometric Foundations of Hamiltonian Mechanics

1.2.3.1. Lagrange Brackets

Consider the extended symplectic structure given by the fundamental Poincaré-Cartan one-form (Arnold [1978])

$$\gamma = \gamma_\alpha dx^\alpha = \tilde{\gamma}_i(x^i, t) dx^i - w dt,$$

(1.40)

where $z^\alpha = (x^1, ..., x^6, w, t)$ are extended phase-space coordinates (not necessarily canonical). The corresponding two-form, $\omega = d\gamma$, has the following components

$$\omega_{\alpha\beta} \doteq [z^\alpha, z^\beta] = \frac{\partial \gamma_\beta}{\partial z^\alpha} - \frac{\partial \gamma_\alpha}{\partial z^\beta},$$

(1.41)

where the non-vanishing ones are given as

$$\omega_{ij} = \tilde{\omega}_{ij} = \frac{\partial \tilde{\gamma}_j}{\partial x^i} - \frac{\partial \tilde{\gamma}_i}{\partial x^j}, \quad \omega_{it} = \frac{\partial \tilde{\gamma}_t}{\partial t}, \quad \text{and} \quad \omega_{tw} = 1.$$  

(1.42)

The two-form $\omega$ is known as the extended Lagrange two-covariant tensor, and its components are known as the extended Lagrange brackets. If we use the canonical representation of extended phase space, with coordinates $(p^i, q^i, w, t)$, we find $\tilde{\gamma}_q^i = p^i$, $\tilde{\gamma}_p^i = 0$, and $\partial \tilde{\gamma}_t/\partial t = 0$.

1.2.3.2. Poisson Brackets

The relationship between the extended Lagrange brackets $[z^\alpha, z^\beta]$, Eq. (1.42), and the extended Poisson brackets $\{z^\alpha, z^\beta\}$ is established as follows. First, we form a matrix $(\omega)$ whose components $(\omega)_{\alpha\beta}$ are the extended Lagrange brackets $[z^\alpha, z^\beta]$

$$(\omega) = \begin{pmatrix} (\tilde{\omega}) & -\omega^T_i & 0 \\ \omega_i & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

(1.43)

where $\omega_i = \omega_{ti} \tilde{e}^i$ is a vector in regular phase space, and $\tilde{e}^i$ is a unit vector. Next, using the matrix inversion formula (given by Littlejohn [1981] for antisymmetric matrices), the inverse matrix $(\omega)^{-1} \doteq (J)$ is given as

$$(J) = \begin{pmatrix} (\tilde{\mathcal{J}}) & 0 & -(\tilde{\mathcal{J}})^T \omega_i^T \\ 0 & 0 & -1 \\ -\omega_i (\tilde{\mathcal{J}}) & 1 & 0 \end{pmatrix},$$

(1.44)

where $(\omega)$ was assumed to be non-singular, and $(\tilde{\mathcal{J}})$ is the inverse matrix of $(\tilde{\omega})$. Finally, the extended Poisson brackets are defined as $\{z^\alpha, z^\beta\} = (J)^{\alpha\beta}$. The two-contravariant tensor $\mathcal{J}$, associated with $(J)$, is known as the extended Poisson tensor.
The non-vanishing basic Poisson brackets are
\[
\{ w, x^i \} = \{ x^i, x^i \} \cdot \frac{\partial \gamma^i_j}{\partial t}, \quad \{ w, t \} = 1,
\]
and the brackets \( \{ x^i, x^j \} \) are the (noncanonical) Poisson brackets in (regular) phase space.

The extended Poisson bracket of two arbitrary functions \( F \) and \( G \) in extended phase space is, therefore, given by the expression
\[
\{ F, G \} = \left( \frac{\partial F}{\partial x^i} - \frac{\partial F}{\partial w} \frac{\partial \gamma^i_j}{\partial t} \right) \cdot \{ x^i, x^j \} \cdot \left( \frac{\partial G}{\partial x^j} - \frac{\partial G}{\partial w} \frac{\partial \gamma^j_i}{\partial t} \right) - \left( \frac{\partial F}{\partial t} \frac{\partial G}{\partial w} - \frac{\partial F}{\partial w} \frac{\partial G}{\partial t} \right). \tag{1.45}
\]
An important application of this formula is obtained when \( F = g(x^i, t) \) and \( G = H \) is the extended Hamiltonian. The Poisson bracket, in this case, has the form
\[
\{ g, H \} = \dot{z}^a \frac{\partial g}{\partial \dot{z}^a} = \frac{\partial g}{\partial t} + \dot{z}^i \frac{\partial g}{\partial x^i}, \tag{1.46}
\]
where we have defined \( \dot{z}^i \) by the expression
\[
\dot{z}^i \equiv \frac{dx^i}{d\tau} = \{ x^i, H \} = \{ x^i, x^j \} \cdot \left( \frac{\partial \bar{H}}{\partial \dot{x}^j} + \frac{\partial \gamma^i_j}{\partial t} \right). \tag{1.47}
\]
The remaining Hamilton's equations are \( dt/d\tau = \{ t, H \} = \{ w, t \} = 1 \), and
\[
\frac{dw}{d\tau} = \{ w, H \} = \frac{\partial \bar{H}}{\partial t} - \frac{\partial \gamma^i_j}{\partial t} \cdot \{ x^i, x^j \} \cdot \frac{\partial \bar{H}}{\partial \dot{x}^j}. \tag{1.48}
\]
The canonical Hamilton's equations are obtained when the extended canonical coordinates are used, for which case \( \partial \gamma^i_j/\partial t = 0 \).

1.2.3.3. Phase Space Variational Principle

As was previously discussed, the equations of motion for \( x^i \), \( w \), and \( t \) can be obtained from the interior product of the extended phase-space Lagrangian
\[
\gamma_E = \tilde{\gamma}_i \cdot dx^i - w \cdot dt - \bar{H} \cdot d\tau, \tag{1.49}
\]
with the basis vectors \( \partial/\partial x^i \), \( \partial/\partial w \), and \( \partial/\partial t \). One then obtains, respectively,
\[
i_x \omega_E = \left( \tilde{\omega}_{ij} \cdot dx^j - \frac{\partial \gamma^i_j}{\partial t} \cdot dt - \frac{\partial \bar{H}}{\partial x^i} \cdot d\tau \right) = 0,
\]
\[
i_w \omega_E = (-dt + d\tau) = 0, \tag{1.50}
\]
\[
i_t \omega_E = \left( \frac{\partial \gamma^i_j}{\partial t} \cdot dx^i + dw - \frac{\partial \bar{H}}{\partial t} \cdot d\tau \right) = 0.
\]
Making use of the fact that the matrix of the $\tilde{\omega}_{ij}$ is the inverse of the matrix of Poisson brackets $\{x^i, x^j\}$, and making use of the Euler-Lagrange equation $dt/d\tau = 1$, the equations of motion for $x^i$ and $w$, given by Eqs. (1.47) and (1.48), follow immediately.

As an example, we consider the noncanonical Hamiltonian formulation of charged-particle motion in nonuniform electromagnetic fields (c.f., Littlejohn [1979, 1981]). Using (noncanonical) extended phase-space coordinates $(r, v, w, t)$, the extended phase-space Lagrangian is given by the expression

$$\gamma_E = \left[ \frac{e}{c} A(r,t) + mw \right] \cdot dr - w dt - \left[ \frac{1}{2} mv^2 + e\phi(r,t) - w \right] d\tau, \quad (1.51)$$

where $(\phi, A)$ are the scalar and vector potentials defining the electromagnetic fields,

$$E = -\nabla\phi - \frac{1}{c} \frac{\partial A}{\partial t}, \quad \text{and} \quad B = \nabla \times A.$$

The components of the extended Lagrange two-form are obtained from Eq. (1.51) as

$$\begin{align*}
[r^i, r^j] &= \frac{e}{c} \epsilon_{ijk} B^k(r,t), \\
[r^i, v^j] &= -m \delta_{ij}, \\
[t, r] &= \frac{e}{c} \frac{\partial A}{\partial t}(r,t), \\
[t, w] &= 1,
\end{align*} \quad (1.52)$$

where $\epsilon_{ijk}$ is the antisymmetric Levi-Civita tensor, and $\delta_{ij}$ is the Kronecker delta. By comparing with Eq. (1.43), we see that the components of $\tilde{\omega}$ are given by the first two Lagrange brackets of Eq. (1.52), while the vector $\omega_t$ has only a nonvanishing spatial component, given by the third Lagrange bracket of Eq. (1.52). The inversion of $\tilde{\omega}$ is simple, and we obtain

$$\begin{align*}
\{r^i, v^j\} &= m^{-1} \delta^{ij}, \quad (1.53) \\
\{v^i, v^j\} &= \frac{e}{mc} \epsilon^{ijk} B_k(r,t), \quad (1.54)
\end{align*}$$

while the other components of the extended Poisson tensor, from Eq. (1.44), are given as

$$\{w, v\} = -\frac{e}{mc} \frac{\partial A}{\partial t}(r,t), \quad \text{and} \quad \{w, t\} = 1. \quad (1.55)$$
The equations of motion are easily obtained, from Eq. (1.50), as follows: the interior product of $\gamma_E$ with $\partial/\partial v$ gives

$$
\frac{dr}{d\tau} = \frac{\partial H}{\partial v} = mv;
$$

(1.56)

the interior product of $\gamma_E$ with $\partial/\partial r$ gives

$$
\frac{dv}{d\tau} = eE(r,t) + \frac{e}{c} \frac{dr}{d\tau} \times B(r,t);
$$

(1.57)

the interior product of $\gamma_E$ with $\partial/\partial t$ gives

$$
\frac{dw}{d\tau} = e \frac{\partial}{\partial t} \left[ \phi(r,t) - \frac{v}{c} \cdot A(r,t) \right];
$$

(1.58)

and the interior product of $\gamma_E$ with $\partial/\partial w$ gives $dt/d\tau = 1$.

Using the expressions for the extended Poisson brackets, given by Eqs. (1.53)–(1.55), the equations of motion, Eqs. (1.56)–(1.57) and (1.58), can be represented by the expressions given in Eqs. (1.47) and (1.48), respectively.

1.2.3.4. Jacobi Identity

The use of noncanonical variables in Hamiltonian theory has a fundamental impact on the appearance of the Poisson-bracket structure. Indeed, the canonical structure given by $\{q^i, p_j\} = \delta^i_j$ and $\{w, t\} = 1$ is invariably transformed into a more complicated structure [such as the structure given by Eqs. (1.53)–(1.55)]. Even if the transformed Poisson-bracket structure still displays the required bilinearity and antisymmetry, it is important to verify that the new structure satisfies the Jacobi identity. We would now like to establish the condition which a Poisson-bracket structure must satisfy in order for the Jacobi identity to hold (Greene [1982]). We remind the reader that the Jacobi identity for Poisson brackets is given as

$$
\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,
$$

(1.59)

where $f$, $g$, and $h$ are arbitrary phase-space functions. (We use the term phase space to mean either regular phase space or extended phase space.) Also, the Poisson-bracket structure is completely described by giving the components $J^{ij}$ and in terms of these components, the Poisson bracket of two arbitrary phase-space functions $f$ and $g$ is given as

$$
\{f, g\} = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} = \partial_i f \partial_j g J^{ij}.
$$
Using this expression, one easily obtains

\[ \{f, \{g, h\}\} = \partial_k f J^{kl} \left[ \partial_i J^{ij} \partial_l g \partial_j h + J^{ij} \left( \partial_k g \partial_j h + \partial_k h \partial_j g \right) \right], \]

and a cyclic permutation of \(f, g, h\) finally gives

\[ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \partial_k f \partial_j g \partial_h h \ S^{ijk} + \left( \partial_k f \partial_j g \partial_h h \ Q^{ilk} + \partial_k f \partial_j g \partial_h h \ Q^{ljk} + \partial_k f \partial_j g \partial_h h \ Q^{kij} \right), \]

where we have defined

\[ S^{ijk} = J^{il} \partial_j J^{jk} + J^{jl} \partial_i J^{ki} + J^{kl} \partial_i J^{ij}, \]
\[ Q^{ilk} = J^{jl} J^{ki} + J^{kl} J^{ij}, \text{etc.} \]

It is simple to show that \(Q^{ilk}\) is antisymmetric with respect to the permutation \((i \leftrightarrow l)\) or \((j \leftrightarrow k)\). In addition, since the terms in Eq. (1.60) associated with \(Q^{ilk}\), in particular \(\partial_k f\), are symmetric with respect to the permutation \((i \leftrightarrow l)\), the last three terms, on the right-hand side of Eq. (1.60), vanish. We are, therefore, left with

\[ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \partial_k f \partial_j g \partial_h h \ S^{ijk}. \] (1.61)

Furthermore, since the three functions \((f, g, h)\) are arbitrary, the right-hand side of Eq. (1.61) vanishes provided the Poisson-bracket structure satisfies the condition

\[ S^{ijk} = J^{il} \partial_j J^{jk} + J^{jl} \partial_i J^{ki} + J^{kl} \partial_i J^{ij} = 0. \] (1.62)

We obtain an equivalent condition for the Lagrange-bracket structure (the components of the Lagrange two-form), by using the identity \(\omega_{ai} J^{ij} = \delta_a^j\). Hence, we find

\[ S_{abc} = \omega_{ai} \omega_{bj} \omega_{ck} S^{ijk} = \partial_a \omega_{bc} + \partial_b \omega_{ca} + \partial_c \omega_{ab}. \] (1.63)

Finally, since the Lagrange-bracket (symplectic) structure is defined in Eq. (1.41) as \(\omega = d\gamma\), or

\[ \omega_{ab} = \partial_a \gamma_b - \partial_b \gamma_a, \]

the Jacobi-identity condition, \(S_{abc} = 0\), follows immediately. Hence, the Poisson-bracket structure, obtained from a phase-space Lagrangian, automatically satisfies the Jacobi identity.
1.2.3.5. Dynamical Reduction of a Hamiltonian System with Symmetry

In classical mechanics, a theorem known as Noether's theorem (Goldstein [1980]) associates to every dynamical symmetry a conserved dynamical quantity. For example, in the Kepler two-body problem, the azimuthal generalized momentum is a conserved quantity due to the azimuthally symmetric nature of the problem. (Such a symmetry allows the reduction of the 2-D Kepler problem into the reduced 1-D Kepler problem). The reader is referred to Cary [1981] and Littlejohn [1984c] for a discussion on Noether's theorem in the context of Hamiltonian Lie perturbation theory and phase-space Lagrangian dynamics.

We are interested in determining the general expression that a phase-space Lagrangian must have in order to represent a dynamical system which is symmetric with respect to a coordinate \( u \). According to Noether's theorem, there exists an associated conserved momentum \( v \), and we denote the remaining phase-space coordinates as \( y^i, i = 1, \ldots, (2n - 2) \). In terms of these phase-space coordinates, let the phase-space Lagrangian be given by the expression

\[
\gamma = \tilde{\gamma}_i dy^i + \gamma_u du + \gamma_v dv - H dt.
\]  

(1.64)

First, we would like to establish a relationship between the \( u \)-symmetry and the \( v \)-conservation law. The equation for \( dv/dt \) is given as the Euler-Lagrange equation, \( i_u \omega \),

\[
\left( \frac{\partial \tilde{\gamma}_i}{\partial u} - \frac{\partial \gamma_u}{\partial y^i} \right) \frac{dy^i}{dt} + \left( \frac{\partial \gamma_v}{\partial u} - \frac{\partial \gamma_u}{\partial v} \right) \frac{dv}{dt} - \frac{\partial H}{\partial u} = 0.
\]  

(1.65)

Next, if we assume that the coordinate \( u \) is ignorable, then Eq. (1.65) becomes

\[
\frac{\partial \gamma_u}{\partial y^i} \frac{dy^i}{dt} + \frac{\partial \gamma_u}{\partial v} \frac{dv}{dt} = 0.
\]  

(1.66)

Finally, a necessary and sufficient condition for \( dv/dt = 0 \) is given as \( \partial \gamma_u/\partial v \neq 0 \), and \( \partial \gamma_u/\partial y^i = 0 \) (for arbitrary \( dy^i/dt \)), i.e., \( \gamma_u = \gamma_u(v) \) is only a function of the conserved momentum \( v \).

We note that the component \( \gamma_v \), in Eq. (1.64), appears only in the Euler-Lagrange equation obtained from the interior product \( i_\omega \), i.e., the equation for \( du/dt \)

\[
\frac{\partial \gamma_u}{\partial v} \frac{du}{dt} = \frac{\partial H}{\partial v} - \frac{dy^i}{dt} \left( \frac{\partial \tilde{\gamma}_i}{\partial u} - \frac{\partial \gamma_v}{\partial y^i} \right).
\]
Because we are only interested in the reduced dynamics involving the coordinates $y^i$, we are free to choose $\gamma_0$. It is, therefore, customary to assume that this component is zero, and the final form for a phase-space Lagrangian, with $u$ as an ignorable coordinate, is given as

$$\gamma = \tilde{\gamma}_i(y^i, v)dy^i + \gamma_u(v)du - H(y^i, v)dt.$$ 

Finally, the reduced dynamics are expressed in terms of the following reduced phase-space Lagrangian

$$\gamma_R = \gamma(y^i, v_0) - \gamma_u(v_0)du = \tilde{\gamma}_i(y; v_0)dy^i - H(y; v_0)dr,$$

(1.67)

where $y \in M_R$ are points in the reduced $2(m-1)$-dimensional phase space, and $v = v_0$ appears as a constant parameter which is determined, in the original phase space, by the initial conditions.
1.3 Lie-transform Hamiltonian Perturbation Theory

The mathematical terminology introduced in section 1.2., is now applied to the formulation of Lie-transform Hamiltonian perturbation theory. Two commonly used perturbation methods, based on the use of Lie transforms (Littlejohn [1982a]), are the Hamiltonian Lie perturbation method and the phase-space Lagrangian Lie perturbation method.

The Hamiltonian method assumes that the phase-space Lagrangian, Eq. (1.49), is perturbed only in its Hamiltonian component \( \tilde{H} \), whereas the Lagrangian method assumes that the phase-space Lagrangian, Eq. (1.49), is perturbed in its symplectic structure \( \gamma_i \) and \( w \) as well as its Hamiltonian component \( \tilde{H} \). In both methods, a transformation on phase space is sought, which allows a simplification of the equations of motion (e.g., the dynamical reduction introduced by the requirement that one of the new phase-space coordinates is an ignorable coordinate for the dynamical equations).

The transformation is expressed in terms of a Lie-transform operator, Eq. (1.9), \( T^\varepsilon_G = \exp(\varepsilon G) \), where \( \varepsilon \) is a parameter and \( G \) is a generating vector field for the transformation. In the case of a near-identity transformation, where the parameter \( \varepsilon \) is vanishingly small, the new phase-space coordinates \( Z \) are expressed in terms of the old phase-space coordinates \( z \) as Lie (asymptotic) series, defined in Eq. (1.12),

\[
Z^i = z^i + \varepsilon G^i(z) + O(\varepsilon^2),
\]

for the \( i \)th coordinate component. The induced pull-back operator \( T^\varepsilon_G^* \), defined in Eq. (1.26), and the push-forward operator \( T^\varepsilon_G_* \), defined in Eq. (1.30), are given in terms of the Lie derivative \( L_G \) along the generating vector field \( G \) by the following expressions

\[
T^\varepsilon_G^* = \exp(\varepsilon L_G), \quad \text{and} \quad T^\varepsilon_G_* = (T^\varepsilon_G^*)^{-1} = \exp(-\varepsilon L_G).
\]

1.3.1 Phase-space Lagrangian Lie Perturbation Method

Let us consider the extended phase-space Lagrangian

\[
\gamma = \tilde{\gamma} - \tilde{H} d\tau = \tilde{\gamma}_i dx^i - w \, dt - (\tilde{H} - w) d\tau,
\]

(1.68)
where \( \tilde{\gamma} = \tilde{\gamma}_0 + \epsilon \tilde{\gamma}_1 + \mathcal{O}(\epsilon^2) \), and \( \tilde{H} = \tilde{H}_0 + \epsilon \tilde{H}_1 + \mathcal{O}(\epsilon^2) \) are given as (infinite) power series in \( \epsilon \). By assumption, the zeroth-order system, represented by \( \tilde{\gamma}_0 \) and \( \tilde{H}_0 \), describes simple (perhaps reduced) dynamics.

Two different transformation scenarios are considered in this work: (1) in the theory of guiding-center motion (in chapter 2), a transformation sequence

\[
(r, v) \rightarrow (x, v_{\parallel}, \mu_0, \theta) \rightarrow (X, U, \mu, \zeta)
\]

is sought in which the coordinate \( \zeta \) is an ignorable coordinate, and its associated action (momentum) \( \mu \) is an adiabatic invariant; and (2) in the theory of gyrocenter motion (in chapter 3), a near-identity transformation

\[
(X, U, \mu, \zeta, w, t) \rightarrow (X, U, \mu, \bar{\zeta}, \bar{w}, t)
\]

is sought in which the new coordinate \( \bar{\zeta} \) is now the ignorable coordinate, and its associated action (momentum) \( \bar{\mu} \) is the new adiabatic invariant.

These transformations will be performed through the use of Lie transforms. The expression for the pull-back of the phase-space Lagrangian, induced by the extended phase-space transformation \( T_G^* \), is given as

\[
\Gamma = (T_G^*)^{-1} \gamma + dS,
\]

(1.69)

where \( S(x^i, t) \) is known as a phase-space gauge function. (In the theory of canonical transformations, the function \( -S \) plays the role of the generating function, Arnold [1978].) Because Hamiltonian dynamics is expressed in terms of the two-form \( d\gamma \), the gauge transformation \( dS \), shown in Eq. (1.69), does not change the equations of motion since \( d^2 S \equiv 0 \).

The expression for the pull-back operator \( (T_G^*)^{-1} \) was given by Eq. (1.27) as

\[
(T_G^*)^{-1} = \exp(-\epsilon L_G),
\]

(1.70)

where \( L_G \), defined in 1.2.1.4., is the Lie derivative along the generating vector field \( G \). In extended phase space, the vector field has components \( G^i, G^t \), and \( G^w \). Using the homotopy formula, Eq. (1.28), the Lie derivative is given as \( L_G = i_G d + d i_G \), where \( i_G \) is the interior product of forms and \( d \) is the exterior derivative, introduced in 1.2.1.3. As was discussed previously, the operator \( d i_G \) has no consequence in Hamiltonian
dynamics and, therefore, we concentrate on the operator \( i_G \). When applied to the exact phase-space Lagrangian \( \gamma \), Eq. (1.68), the expression for \( i_G \) is given as

\[
i_G \gamma = G^i \left[ \frac{\partial \tilde{\gamma}_1}{\partial x^i} - \frac{\partial \tilde{\gamma}_1}{\partial t} \right] dx^i + \frac{\partial \tilde{H}}{\partial t} dt - \frac{\partial \tilde{H}}{\partial x^i} d\tau + G^w (-dt + d\tau) \tag{1.71}
\]

\[
+ G^x \left( \frac{\partial \tilde{\gamma}_1}{\partial t} dx^i + dw - \frac{\partial \tilde{H}}{\partial t} d\tau \right).
\]

The zeroth-order components of the transformed phase-space Lagrangian are easily solved as \( \tilde{\Gamma}_0 = \tilde{\gamma}_0 \) and \( \tilde{\mathcal{H}}_0 = \tilde{\mathcal{H}}_0 \). The expression for the old first-order phase-space Lagrangian is

\[
\gamma_1 = \tilde{\gamma}_1 dx^i + \tilde{\gamma}_1 dt - H_1 d\tau,
\]

where a preliminary phase-space gauge transformation was possibly performed (which would be responsible for the component \( \tilde{\gamma}_1 \)), and the expression for the transformed phase-space Lagrangian is

\[
\Gamma_1 = \left[ \tilde{\gamma}_1 - G^i \left( \frac{\partial \tilde{\gamma}_0}{\partial x^i} - \frac{\partial \tilde{\gamma}_0}{\partial t} \right)_0 + \frac{\partial S_1}{\partial x^i} \right] dx^i + \left( G^w + \tilde{\gamma}_1 + \frac{\partial S_1}{\partial t} \right) dt \tag{1.73}
\]

\[
- \left( H_1 - G^x \frac{\partial \tilde{H}_0}{\partial x^i} + G^w \right) d\tau,
\]

where we have chosen \( G^x \equiv 0 \), i.e., time is unchanged by the transformation. (The application of the phase-space Lagrangian Lie perturbation to relativistic Hamiltonian systems should include a nonvanishing \( G^x \) component.) The purpose of the Lie perturbation analysis that follows is to find, at each order \( n \geq 1 \), the components of the \( n \)th-order Lie generating vector field \( G_n \) which will simplify the expression for \( \Gamma_n \).

At this point, two different representations of phase-space Lagrangian Lie perturbation analysis are possible. The first representation is referred to as the Hamiltonian representation. It uses the condition \( \tilde{\Gamma}_n = 0 \) (for \( n = 1, 2, \ldots \)) in Eq. (1.69), so that at first order we have

\[
G^i = -\{x^i, x^j\}_0 \tilde{\gamma}_1 - \{x^i, S_1(x, t)\}_0 \quad \text{and} \quad G^w = -\{w, t\}_0 \tilde{\gamma}_1 - \{w, S_1(x, t)\}_0. \tag{1.74}
\]

The perturbation of the symplectic structure is, therefore, transferred onto the Hamiltonian function (\( \tau \) component),

\[
\mathcal{H}_1 = H_1 - \tilde{\gamma}_1 \{x^i, H_0\}_0 - \tilde{\gamma}_1 \{t, H_0\}_0 - \{S_1, H_0\}_0, \tag{1.75}
\]
and the new symplectic structure is simply the original structure expressed in terms of the new phase-space coordinates.

The second representation is referred to as the symplectic representation. It uses the condition $\hat{\Gamma}_n \neq 0$ (for $n = 1, 2, \ldots$), so that at first order we have

$$G^i = -\{x^i, x^j\}_0 (\hat{\gamma}_{ii} - \hat{\Gamma}_{ii}) - \{x^i, S_1\}_0, \quad \text{and} \quad G^w = -\{w, t\}_0 (\hat{\gamma}_{tt} - \hat{\Gamma}_{tt}) - \{w, S_1\}_0,$$

(1.76)

and the symplectic structure has retained some of the perturbation through $\hat{\Gamma}_n \neq 0$. Consequently, the first-order Hamiltonian is given by the expression

$$\mathcal{H}_1 = H_1 - (\hat{\gamma}_{ii} - \hat{\Gamma}_{ii}) \{x^i, H_0\}_0 - (\hat{\gamma}_{tt} - \hat{\Gamma}_{tt}) \{t, H_0\}_0 - \{S_1, H_0\}_0.$$  

(1.77)

The choice of which representation should be used is normally based on convenience. More will be said about this issue in section 3.2.

1.3.2 Hamiltonian Lie Perturbation Method

If the symplectic structure is not perturbed (i.e., $\hat{\gamma}_n \equiv 0$ for $n \geq 1$), we can then restrict the perturbation analysis to the Hamiltonian. Because the transformation must be canonical, in order to preserve the unperturbed Poisson-bracket structure, we can always choose the Lie generating vector field $G$ to be a Hamiltonian vector field, $G = \{ , g \}$ where $g(x^i, t)$ is the Hamiltonian function. Through the use of the Jacobi identity, Eq. (1.59), it is easy to verify that the (extended) Poisson bracket satisfies (Dragt and Finn [1976])

$$\exp \epsilon G (\{f, h\}) = \{\exp \epsilon G(f), \exp \epsilon G(h)\}.$$

The Hamiltonian Lie perturbation analysis is, therefore, concerned with the formal expression

$$\mathcal{H} = (T_G^e)^{-1} H,$$

(1.78)

where $H = \tilde{H} - w$, and $\tilde{H}$ possesses a known power series expansion in $\epsilon$. For instance, the first-order expression is

$$\tilde{\mathcal{H}}_1 = \tilde{H}_1 + \frac{\partial g}{\partial t} + \{g, \tilde{H}_0\}.$$  

(1.79)

We should point out that the application of the Hamiltonian representation of Phase-space Lagrangian Lie perturbation method, Eq. (1.75), and the Hamiltonian Lie perturbation method, Eq. (1.79), give the same results.
1.3.3 Iterative Sequence of Lie Transforms

Our use of Lie-transform techniques (Dragt and Finn [1976]) involves a succession of Lie-transforms

$$T^n_G = \cdots T^n_2 T^n_1,$$  \hspace{1cm} (1.80)

where $T^n_n = \exp \varepsilon^n G_n$ effectively enters into our perturbation analysis at the $n$th order. For our purposes, it is also convenient to consider the generating vector field $G_n$ to be given as a power series in $\varepsilon$,

$$\varepsilon^n G_n = \sum_{m=n}^{\infty} \varepsilon^m G_{n,m-n},$$

with $G_{n0}$ non-vanishing. The specific expansion in powers of $\varepsilon$ for each $G_n$ depends on the ordering used in the analysis (e.g., the gyrokinetic ordering given in 1.1.2.).

The pull-back operator $(T^n_G)^{-1}$, given by Eq. (1.70), now has the following form

$$(T^n_G)^{-1} = \cdots (T^n_2)^{-1} (T^n_1)^{-1},$$

where $(T^n_1)^{-1} = \exp -\varepsilon^1 L_n$ and $L_n \equiv L_{G_n}$ is the Lie derivative with respect to the $n$th-order generating vector field $G_n$.

We consider, first, the case of the Hamiltonian Lie perturbation analysis of the equation

$$\mathcal{H} = (T^n_G)^{-1} H,$$

where the ordering of the Poisson bracket gives the expansion

$$G_n = \{ , g_n \} + \varepsilon \left( \{ , g_n \}_{1} + \frac{\partial g_n}{\partial \xi} \frac{\partial}{\partial \psi} \right) + \cdots = G_{n0} + \varepsilon G_{n1} + \cdots,$$

so that

$$(T^n_G)^{-1} = 1 - \varepsilon G_{10} - \varepsilon^2 \left( G_{20} + G_{11} - \frac{1}{2} G_{10}^2 \right) + \mathcal{O}(\varepsilon^3).$$

The three lowest terms in powers of $\varepsilon$ are

$$\tilde{\mathcal{H}}_0 = H_0,$$

$$\tilde{\mathcal{H}}_1 = \tilde{H}_1 + \{ g_1, \tilde{H}_0 \}_0,$$

$$\tilde{\mathcal{H}}_2 = \tilde{H}_2 + \{ g_2, \tilde{H}_0 \}_0 + \{ g_1, \tilde{H}_1 \}_0 + \{ g_1, \tilde{H}_0 \}_1 + \frac{\partial g_1}{\partial t} \tilde{H}_0 + \frac{1}{2} \{ g_1, \{ g_1, \tilde{H}_0 \}_0 \}_0.$$  \hspace{1cm} (1.81)
Finally, we consider the case of the phase-space Lagrangian Lie perturbation analysis of the equation

\[ \Gamma = (T_0^\omega)^{-1} \gamma + dS, \]

where \( S = \epsilon S_1 + \epsilon^2 S_2 + \cdots \), we find

\[
\begin{align*}
\Gamma_0 &= \gamma_0, \\
\Gamma_1 &= \gamma_1 - i_1 \omega_0 + dS_1, \\
\Gamma_2 &= \gamma_2 - i_2 \omega_0 - i_1 \omega_1 + \frac{1}{2} i_1 d(i_1 \omega_0) + dS_2,
\end{align*}
\]

(1.82)

where \( \omega_n = d\gamma_n \). If we define \( \Omega_n = d\Gamma_n \), the expression for \( \Gamma_2 \) can, also, be given as

\[
\Gamma_2 = \gamma_2 - i_2 \omega_0 - \frac{1}{2} i_1 (\omega_1 + \Omega_1) + dS_2',
\]

(1.83)

where the expression for \( \Gamma_1 \) was used.

The expressions given by Eqs. (1.82) and (1.83) form the basis upon which perturbation theory will be carried out in this work.
Chapter 2

Single-particle Guiding-center Theory

The effects of magnetic field nonuniformity (e.g., magnetic drift, and magnetic trapping) are known to have quite an impact on the various issues related to the stability properties of magnetically confined plasmas (c.f., Tang [1978]). The study of the complicated dynamics of charged-particle motion in nonuniform magnetic geometries is greatly facilitated by the reduced description known as the guiding-center description (Northrop [1963]). The general theory of the guiding-center motion of charged particles in electromagnetic fields is presented in 2.1. The special case of confining electromagnetic fields is presented in 2.2.

2.1 Adiabatic Motion of Charged Particles in Electromagnetic Fields

We present a review of the general theory of guiding-center motion as derived through the use of three different methods: (1) the gyrophase averaging principle, (2) the Hamiltonian Lie perturbation method, and (3) the Phase-space Lagrangian Lie perturbation method.

2.1.1 Classical Theory of Guiding-center Drift Motion

We begin with a simple presentation on the theory of guiding-center drift motion, taken almost entirely from the paper by A. Baños [1967]. We then follow with a
derivation of a simple expression for the guiding-center drift velocity, due to Morozov and Solov'ev [1960, 1966]. (This expression will prove to be useful, when combined with the use of magnetic coordinates, in Appendix A.)

2.1.1.1. Asymptotic Analysis of Charged Particle Motion

Consider the Lorentz force equation, describing the motion of charged particles in prescribed (external) electromagnetic fields \((E, B)\),

\[
\ddot{r} = \frac{e}{m} \left( E + \frac{\dot{r} \times B}{c} \right)
\]

If we use the following dimensionless variables

\[
b = \frac{B}{B_0}, \quad e = \frac{cE}{v_0 B_0}, \quad \tau = \frac{v_0 t}{L}, \quad \text{and} \quad x = \frac{r}{L},
\]

where \(B_0, v_0,\) and \(L\) are characteristic quantities in our problem, Eq. (2.1) takes on the dimensionless form (Northrop [1963])

\[
\epsilon \frac{d^2 x}{d\tau^2} = e + \frac{dx}{d\tau} \times b,
\]

where the dimensionless parameter \(\epsilon = v_0/\Omega_0 L\) (where \(\Omega_0 = eB_0/mc\)) represents the ratio of the radius of gyration of the particle to the characteristic length. We remark that if we make the following substitution

\[
\epsilon \rightarrow \frac{B}{\Omega}, \quad \text{and} \quad (x, \tau, e, b) \rightarrow (r, t, cE, B),
\]

in Eq. (2.2), we regain the original equation of motion, Eq. (2.1). Consequently, the solutions of the dimensionless equation can be easily mapped onto the solutions of the original equation.

The significance of this remark centers on the dimensionless parameter \(\epsilon\), which is normally a small quantity. Ignoring for the moment the electric field \((e = 0\) in Eq. (2.2)), the smallness of this parameter implies, from the point of view of the right-hand side of Eq. (2.2), that the particle strictly moves along the magnetic field line, \(dx/d\tau \times \dot{b} = O(\epsilon)\). Any motion away from a field line must, therefore, be viewed as a higher-order motion. However, as is well known, the leading-order perpendicular motion is represented by the fast gyration of the charged particle about a magnetic field line. The elimination of this fast gyromotion, thereby bringing forth the next-order drift motion, is the goal of guiding-center theory. In addition, in guiding-center
Chapter 2. Single-particle Guiding-center Theory

theory, we have \( v_0 = V_{\text{thermal}} \) and \( L = \ell_B \) (the magnetic nonuniformity length scale), and thus \( \epsilon \equiv \epsilon_B \). (We note that the usual expansion in powers of \( m/e \) is now understood as an expansion in powers of \( \epsilon \). Northrop [1963])

2.1.1.2. Instantaneous Particle Velocity

When considering the inclusion of an electric field, it is customary to assume that the parallel component of the electric field, \( E_\parallel \), is of order \( \mathcal{O}(\epsilon) \). This constraint prevents rapid parallel acceleration over short periods of time, which would violate the guiding-center approximation of slow time developments compared to the gyroperiod. The issue of how the perpendicular component of the electric field scales with \( \epsilon \) is normally avoided by simply going into a frame that drifts with the velocity \( u_E \), defined as

\[
u_E = \frac{E \times \tilde{b}}{B},
\]

where \( \tilde{b} = B/B \).

If we take \( \dot{r} = v + u_E \), then the right-hand side of Eq. (2.1) becomes

\[
\left( e/mc \right) (\tilde{b} c E_\parallel + v \times B).
\]

To continue the analysis, it is desirable to define two orthonormal basis sets: a fixed basis set \((\hat{1}, \hat{2}, \hat{b})\), and a gyrating basis set \((\tilde{\theta}, \tilde{b}, \tilde{1})\), each set given in right-hand-rule order. Furthermore, we define local-magnetic coordinates \((x, v_\parallel, v_\perp, \theta)\) as follows: \( r = x \), and \( v = v_\parallel \tilde{b} + v_\perp \tilde{1} \), where \( \tilde{1} = -(\hat{1} \sin \theta + \hat{2} \cos \theta) \). The velocity component \( v_\parallel \) is known as the parallel component, \( v_\perp \) as the perpendicular component, and \( \theta \) as the gyrophase. (We note that all these field quantities are functions of \( x(t) \) and \( t \).)

Finally, because the last term in Eq. (2.3) gives the \( \mathcal{O}(\epsilon^0) \) gyrophase-dependent term \((-\Omega v_\perp \tilde{\theta})\), we shall consider a procedure, in 1.1.1.3., used to eliminate the gyrophase dependence from our equations of motion. The resulting \( \mathcal{O}(\epsilon) \) perpendicular motion is defined as the (leading-order) guiding-center drift motion.

We are now ready to consider the left-hand side of Eq. (2.1) and, thereby, obtain the equations of motion for \( v_\parallel, v_\perp, \) and \( \theta \). First, we write \( \ddot{x} \) as

\[
\ddot{x} = v_\parallel \ddot{b} + v_\parallel \dot{\tilde{b}} + u_E + v_\perp \tilde{1} + v_\perp \tilde{1},
\]

where the notation \( \dot{u} \) is used to denote

\[
\dot{u} = \frac{\partial u}{\partial t} + \dot{x} \cdot \nabla u = \frac{du}{dt} + v_\perp \tilde{1} \cdot \nabla u.
\]
2.1. Adiabatic Motion of Charged Particles in Electromagnetic Fields

The scalar product of Eq. (2.1) with \( \hat{b} \), using Eqs. (2.3) and (2.4), gives the equation for \( \hat{v}_\parallel \)

\[
\hat{v}_\parallel = \frac{e}{m} E_\parallel + u_E \cdot \hat{b} + v_\perp \cdot \hat{b},
\]

(2.5)

where the identity \( \hat{a} \cdot \hat{c} = -\hat{c} \cdot \hat{a} \) was used. Similarly, the equation for \( \hat{v}_\perp \) is obtained from the scalar product with \( \hat{I} \), and we obtain

\[
\hat{v}_\perp = -\hat{I} \cdot \hat{u}_E - v_\parallel \hat{I} \cdot \hat{b}.
\]

(2.6)

Finally, we obtain the equation for \( \hat{\theta} \) by first, writing the expression for \( \hat{\perp} \) as

\[
\hat{\perp} = -\hat{\theta} \hat{\theta} - \hat{\hat{I}} \sin \theta - \hat{\hat{b}} \cos \theta.
\]

By taking the scalar product with \( \hat{\theta} \), we obtain

\[
\hat{\theta} = \Omega + \hat{\hat{I}} \cdot \hat{\hat{b}} + \frac{\hat{\theta}}{v_\perp} \cdot (v_\parallel \hat{b} + \hat{u}_E),
\]

(2.7)

where \( \Omega \) denotes the gyrofrequency.

2.1.1.3. Averaged Equations of Motion

We note that the equations of motion, Eqs. (2.5)–(2.7), have gyrophase-independent and gyrophase-dependent parts. It is our purpose to obtain the gyrophase-independent parts by defining the following averaging operator

\[
\langle F \rangle(x) = \int \frac{d\theta}{2\pi} F(x, \theta),
\]

where \( x \) is a slowly-varying variable. The assumption used here is that, to lowest order, \( d\theta = \Omega \, dt \), i.e., during one gyration period, we regard as rigorously constant all slowly varying functions of time which do not explicitly contain \( \cos \theta \) or \( \sin \theta \). An important application of this averaging operator is that, for example,

\[
\langle \hat{u} \rangle = \frac{du}{dt} = \frac{\partial u}{\partial t} + (u \hat{b} + u_E) \cdot \nabla u,
\]

where all field variables, after such an operation, are to be evaluated at the instantaneous guiding-center position \( X(t) = \langle x(t) \rangle \).

Making use of the following basic gyrophase-averaged expressions

\[
\langle \hat{I} \rangle = 0 = \langle \hat{\theta} \rangle,
\]

\[
\langle \hat{\perp} \hat{I} \rangle = \frac{1}{2} \hat{I} \hat{\perp} = \langle \hat{\theta} \hat{\theta} \rangle,
\]

\[
\langle \hat{\theta} \hat{\perp} \rangle = \frac{1}{2} \hat{I} \hat{\perp} \hat{\theta} = -\langle \hat{I} \hat{\theta} \rangle,
\]

(2.8)
where $I_\perp = \hat{1} \hat{1} + \hat{2} \hat{2}$ is the perpendicular unit dyadic, we obtain the averaged equations of motion

$$\frac{dv_\parallel}{dt} = \frac{e}{m} E_\parallel + u_E \left( \frac{\partial}{\partial t} + v_\parallel \hat{b} \cdot \nabla \hat{b} \right) - \frac{v_\perp^2}{2B} \frac{\partial B}{\partial s},$$

$$\frac{dv_\perp}{dt} = \frac{v_\perp dB}{2B \frac{dt}{dt}},$$

$$\frac{d\theta}{dt} = \Omega + \left( \frac{\partial \hat{1}}{\partial t} + v_\parallel \hat{b} \cdot \nabla \hat{1} \right) \cdot \hat{2} + \frac{1}{2} \hat{b} \cdot \nabla \times (v_\parallel \hat{b} + u_E).$$

(2.9)

We see that the expressions for $dv_\parallel / dt$, $dv_\perp / dt$, and $(d\theta / dt - \Omega)$ are of order $O(\epsilon)$. One also observes that, to lowest order, the quantity $v_\perp^2 / 2B$ is conserved. Such a quantity is well-known as the magnetic moment $\mu_0$. The subscript 0 is used to indicate that $\mu_0$ is really the lowest-order term of an asymptotic series representing the magnetic moment.

2.1.1.4. Guiding-center Drift Motion

We now wish to obtain the equation for $dX/dt$, where $X$ is the guiding-center position at time $t$, defined as $\langle x(t) \rangle = X(t)$. The lowest-order expression is simply

$$\frac{dX}{dt} = \langle \dot{x} \rangle = v_\parallel \hat{b} + u_E.$$

In order to derive the next-order corrections to this expression, we let

$$X = x - \hat{\theta} \frac{v_\perp}{\Omega},$$

(2.10)

where terms of order $O(\epsilon)$ have been neglected. The expression for $\dot{X}$, using Eqs. (2.5)–(2.7), is given as

$$\dot{X} = v_\parallel \hat{b} + u_E + v_\perp \hat{1} + \frac{v_\perp}{\Omega B} \dot{\theta} \hat{b} - \frac{1}{\Omega} (v_\perp \hat{\vartheta} + v_\perp \hat{\vartheta}).$$

A more suitable expression, before we perform the averaging process, is

$$\dot{X} = \left( v_\parallel + \frac{v_\perp \hat{\vartheta} \hat{b}}{\Omega} \right) \hat{b} + u_E + \frac{1}{\Omega} \left[ \frac{v_\perp}{B} \dot{\theta} \hat{b} + v_\parallel (\hat{\theta} \hat{\vartheta} - \hat{\vartheta} \hat{\vartheta}) \cdot \hat{b} + (\hat{\theta} \hat{\vartheta} - \hat{\vartheta} \hat{\vartheta}) \cdot u_E \right].$$

(2.11)

The averaging procedure, using Eq. (2.8), is simple, and we finally obtain

$$\frac{dX}{dt} = \left( v_\parallel + \frac{\mu_0 B}{\Omega} \hat{b} \cdot \nabla \times \hat{b} \right) \hat{b} + u_E + \frac{\hat{b}}{\Omega} \times \left[ \mu_0 \nabla B + \frac{d}{dt} (v_\parallel \hat{b} + u_E) \right].$$

(2.12)
2.1. Adiabatic Motion of Charged Particles in Electromagnetic Fields

The second term appearing in the parallel component of Eq. (2.12) is known as the Baños parallel drift term. It indicates that the instantaneous particle parallel velocity and the instantaneous guiding-center parallel velocity are not equal in a nonuniform magnetic field. Northrop and Rome [1978] have given an example illustrating this behavior. The last three terms, on the right-hand side of Eq. (2.12), have their usual meanings. The $\nabla B$-term represents the well-known $\text{grad } B$ drift, while, for static fields, the $\partial b/\partial t$-term gives the curvature drift. The last term represents the so-called polarization drift.

Finally, the energy conservation property can also be expressed in terms of guiding-center quantities. Let $(E_k) = \left< m\dot{x}^2/2 \right> \equiv \mathcal{E}_{g_k}$ be the gyrophase-averaged kinetic energy. It is a simple task to show that

$$\frac{d\mathcal{E}_{g_k}}{dt} = \mu \frac{\partial B}{\partial t} + eE \cdot \frac{dX}{dt},$$

where the expression for $dX/dt$ is given in Eq. (2.12). For static fields (i.e., $\partial/\partial t = 0$ and $E = -\nabla \Phi$), we obtain the guiding-center energy conservation law

$$\mathcal{E}_{g_k} + e\Phi = \text{constant.}$$

2.1.1.5. Equation of Morozov and Solov'ev

Morozov and Solov'ev [1960, 1966] have considered the relativistic form of the expression for the guiding-center drift velocity (e.g., Eq. (2.12)), for a charged particle moving in static fields $(E, B)$, where the magnetic field is assumed to satisfy the condition $B \cdot \nabla \times B = 0$. This expression is given as

$$\frac{dX}{dt} = v_{||} \tilde{b} + \frac{c}{B} E \times \tilde{b} + \frac{mc}{eB} \tilde{b} \times \left( \frac{v_{||}^2}{2B} \nabla B + v_{||}^2 \tilde{b} \cdot \nabla \tilde{b} \right),$$

where $m = m_0(1 - v^2/c^2)^{-1/2}$, and $v^2 = v_{||}^2 + v_{\perp}^2$.

Two conserved quantities associated with this motion are identified as the relativistic guiding-center energy, $(mc^2 + e\Phi)$, and the relativistic form for the magnetic moment, $(m^2 v_{\perp}^2/m_B B)$. These quantities are used to derive the following expression

$$\frac{v_{||}}{B} \nabla \times \left( \frac{v_{||}}{\Omega} B \right) = \frac{mc}{eB} \tilde{b} \times \left( \frac{-e}{m} E + \frac{v_{||}^2}{2B} \nabla B + v_{||}^2 \tilde{b} \cdot \nabla \tilde{b} \right),$$

which is then used to obtain the following expression for the drift velocity

$$\frac{dX}{dt} = \frac{v_{||}}{B} \left[ B + \nabla \times \left( \frac{v_{||}}{\Omega} B \right) \right].$$
Because the magnetic field can be expressed as $B = \nabla \times \mathbf{A}$, the expression contained in the square bracket of Eq. (2.16) can also be given as $\nabla \times \mathbf{A}^\ast$, where $\mathbf{A}^\ast = \mathbf{A} + (v_{||}/\Omega)B$ represents the gyrophase-averaged generalized canonical momentum, as expressed in Eq. (2.57). Morozov and Solov'ev [1960, 1966] have used the modified field $\mathbf{A}^\ast$ in their discussion of drift surfaces and trapped-particle orbits in various magnetic confinement geometries. Finally, as was shown by White, Boozer, and Hay [1982], Boozer [1980], and White and Chance [1984], the magnetic-coordinate representation of $\mathbf{A}^\ast$ allows for a canonical Hamiltonian formulation of the guiding-center equations of motion. (See Appendix A for more details.)
2.1.2 Hamiltonian Theory of Guiding-center Drift Motion

The work presented in this subsection essentially reproduces the work of Littlejohn [1979, 1981] on the guiding-center Hamiltonian theory. Even though our work concentrates on the use of the phase-space Lagrangian Lie perturbation method, the application of the Hamiltonian Lie perturbation method to the guiding-center problem will help us compare the two methods.

2.1.2.1. Noncanonical Hamiltonian Theory

We begin by writing down the well-known forms for the canonical Poisson bracket, and the Hamiltonian describing the motion of a charged particle in a nonuniform magnetic field. For simplicity, we shall also assume that there is no electric field present, and that the magnetic field is static but spatially nonuniform. The canonical Poisson bracket of two phase-space functions $F(p, q), G(p, q)$ is given as

$$\{F, G\} = \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial q}, \quad (2.17)$$

and the Hamiltonian is given as

$$H(p, q) = \frac{1}{2} |p - e^{-1} A(q)|^2, \quad (2.18)$$

where the units used here are such that $\epsilon = m = c = 1$. The appearance of the small parameter $\epsilon$ ensures that the (Hamilton's) equations of motion will take on the dimensionless form given by Eq. (2.2).

The derivation of the guiding-center Hamiltonian and Poisson bracket will proceed in four steps, each step corresponding to a transformation on phase space. Our goal is to derive phase-space coordinates such that the Hamiltonian and Poisson bracket, when expressed in terms of these final coordinates, will be gyrophase independent, in the sense of asymptotics. These four transformations are identified as follows:

$$(p, q) \rightarrow (r, v) \rightarrow (x, u_\parallel, u_\perp, \theta) \rightarrow (X, U, \mu, \zeta) \rightarrow (\bar{X}, \bar{U}, \bar{\mu}, \bar{\zeta}). \quad (2.19)$$

The first two transformations will be referred to as preliminary transformations, while the last two transformations will be referred to as the Darboux–Lie transformations.

The purpose of the Darboux transformation is to construct a coordinate $\mu$, canonically conjugate to the gyrophase $\zeta$, with the use of an algorithm based on the proof of the Darboux theorem, see Littlejohn [1979]. The use of this algorithm also ensures
that in the expression for the new Poisson bracket, the canonical pair \((\mu, \zeta)\) will not be coupled with the remaining phase-space coordinates \((X, U)\). Furthermore, another consequence of the use of this algorithm is that the coordinates \((X, U)\) are not, in general, canonical, and that the Poisson bracket is itself gyrophase-independent. Unfortunately, as a result of the Darboux transformation, the Hamiltonian acquires gyrophase dependence when expressed in terms of these Darboux coordinates.

The purpose of the Lie transformation is, therefore, to place the explicit gyrophase dependence of the Hamiltonian beyond any desired order, in the sense of asymptotics. Because the Lie transformation is canonical, the form of the (Darboux) Poisson bracket is unchanged. We, therefore, obtain a guiding-center Hamiltonian and a guiding-center Poisson bracket in terms of which the guiding-center dynamics are expressed.

2.1.2.2. Preliminary Transformations

The first transformation \((q, p) \rightarrow (r, v)\) is defined as

\[
q = r, \quad \text{and} \quad p = v + \frac{1}{\epsilon} A(r),
\]

and gives the following expression for the Hamiltonian

\[
H(r, v) = \frac{1}{2} |v|^2,
\]

and the noncanonical Poisson bracket, given by Eqs. (1.53) and (1.54),

\[
\{F, G\} = \frac{\partial F}{\partial r} \frac{\partial G}{\partial v} - \frac{\partial F}{\partial v} \frac{\partial G}{\partial r} + \frac{B}{\epsilon} \left( \frac{\partial F}{\partial v} \times \frac{\partial G}{\partial v} \right).
\]

We note that the Poisson bracket is now the sum of the usual canonical bracket, involving the coordinates \((x, v)\), and a new bracket referred to as the gyroscopic-bracket. One easily verifies that the Lorentz equations of motion, Eq. (2.2), can be expressed as

\[
\dot{r} = \{r, H\} = v, \quad \text{and} \quad \dot{v} = \{v, H\} = \frac{1}{\epsilon}(v \times B).
\]

The second transformation \((r, v) \rightarrow (x, v_\parallel, v_\perp, \theta)\) is defined as

\[
r = x, \quad \text{and} \quad v = v_\parallel \hat{b} + v_\perp \hat{1},
\]

where \(v_\parallel = v \cdot \hat{b}, \ v_\perp = |v \times \hat{b}|\), and

\[
\hat{1} = - (\sin \theta \hat{1} + \cos \theta \hat{2}) = \frac{\partial \hat{b}}{\partial \theta} = \hat{b} \times \hat{b}.
\]
Finally, we use the same orthonormal basis sets \((\hat{1}, \hat{2}, \hat{b})\) and \((\hat{\theta}, \hat{b}, \hat{\perp})\) defined previously. Using these coordinates, the Hamiltonian, Eq. (2.21), becomes

\[
H(x, v_\|, v_\perp) = \frac{1}{2} (v_\|^2 + v_\perp^2),
\]

while the gyroscopic-bracket in Eq. (2.22) becomes

\[
\frac{B}{ev_\perp} \left( \frac{\partial F}{\partial \theta} \frac{\partial G}{\partial v_\perp} - \frac{\partial F}{\partial v_\perp} \frac{\partial G}{\partial \theta} \right).
\]

The canonical bracket in Eq. (2.22) acquires significant complexity in a nonuniform magnetic field. Indeed, the differential operator \(\partial/\partial v\) becomes

\[
\frac{\hat{1}}{v_\perp} \frac{\partial}{\partial v_\perp} + \hat{b} \frac{\partial}{\partial v_\|} - \frac{\hat{\theta}}{v_\perp} \frac{\partial}{\partial \theta},
\]

and the differential operator \(\partial/\partial r\) becomes

\[
\frac{\partial}{\partial x} + \frac{\partial v_\|}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial v_\perp}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta},
\]

where

\[
\frac{\partial v_\|}{\partial x} = \frac{v_\perp}{v_\perp} \hat{1} \cdot \nabla \hat{b},
\]

\[
\frac{\partial v_\perp}{\partial x} = -v_\| \nabla \hat{\theta} \cdot \hat{1}, \text{ and}
\]

\[
\frac{\partial \theta}{\partial x} = \frac{v_\|}{v_\perp} \nabla \hat{b} \cdot \hat{\theta} + R.
\]

The vector \(R \equiv (\nabla \hat{1}) \cdot \hat{2}\) in the last expression of Eq. (2.29) is a vector whose geometrical significance will be discussed later (2.1.3.2.) in the context of gyrogaugel invariance.

2.1.2.3. Darboux Transformation

While the expression for the Hamiltonian, Eq. (2.25), has no gyrophase dependence, the expression for the Poisson bracket has acquired explicit gyrophase dependence. The purpose of the third (Darboux) transformation in Eq. (2.19) will be to eliminate this gyrophase dependence through the use of the Darboux algorithm (Littlejohn [1979]).
We define the Darboux coordinates $Z = (X, U, \mu, \zeta)$ through the use of the following Poisson-bracket relations:

$$\{\zeta, \mu\} = \epsilon^{-1},$$

$$\{\zeta, X\} = 0 = \{\zeta, U\}, \quad \text{and} \quad \{\mu, X\} = 0 = \{\mu, U\},$$

where the $i$th-component of the Darboux coordinate $Z$ is expressed as $Z^i(X, v_{\|}, v_{\perp}, \theta)$, and $\zeta \equiv \theta$. The first Poisson-bracket relation serves as a definition of the (canonically conjugate) coordinate $\mu$. The remaining relations are solved by obtaining the $\zeta$- and $\mu$-characteristics, i.e., the trajectories which result from using $\zeta$ and $\mu$ as a Hamiltonian, respectively. Further details are contained in the work of Littlejohn [1979, 1981].

Using the operator $d/d\lambda \equiv \{ \cdot, \theta \}$, the differential equations for the $\zeta$ characteristics are given as

$$\frac{d\mu}{d\lambda} = -\frac{1}{\epsilon},$$

$$\frac{dx}{d\lambda} = -\frac{\hat{\theta}}{v_{\perp}},$$

$$\frac{dv_{\|}}{d\lambda} = -(\theta b_{\perp}) - \hat{b} \cdot R + \frac{v_{\|}}{v_{\perp}} \hat{a} \cdot \nabla \times \hat{b},$$

$$\frac{dv_{\perp}}{d\lambda} = -\frac{B}{ev_{\perp}} \left(1 + \epsilon \frac{v_{\|}}{B} \hat{b} \cdot \nabla \times \hat{b} + \epsilon \frac{v_{\perp}}{B} \hat{a} \cdot R\right), \quad \text{and}$$

$$\frac{dx}{d\lambda} = 0 = \frac{dU}{d\lambda},$$

where the notation $(abc) = \hat{a} \cdot \nabla \hat{b} \cdot \hat{c}$ was used. Following Littlejohn [1979], provided $v_{\perp}(\lambda)$ is monotonic in $\lambda$ we replace $\lambda$ by $v_{\perp}$ as the characteristic parameter and obtain

$$\frac{d\mu}{dv_{\perp}} = \frac{v_{\perp}}{B} \left(1 - \epsilon \frac{v_{\|}}{B} \hat{b} \cdot \nabla \times \hat{b} - \frac{v_{\perp}}{B} \hat{a} \cdot R + \mathcal{O}(\epsilon^2)\right), \quad \mu = 0 \quad \text{at} \quad v_{\perp} = 0,$$

$$\frac{dv_{\|}}{dv_{\perp}} = \epsilon \frac{v_{\perp}}{B} \left((\theta b_{\perp}) + \hat{b} \cdot R - \frac{v_{\|}}{v_{\perp}} \hat{a} \cdot \nabla \times \hat{b}\right) + \mathcal{O}(\epsilon^2), \quad v_{\|} = U \quad \text{at} \quad v_{\perp} = 0,$$

$$\frac{dx}{dv_{\perp}} = \frac{\hat{\theta}}{B} + \mathcal{O}(\epsilon^2), \quad x = X \quad \text{at} \quad v_{\perp} = 0.$$

Integrating these expressions, subject to the corresponding initial conditions, it is easy to obtain the following expressions (correct to order $\epsilon$)

$$X(x, v_{\|}, v_{\perp}, \theta) = x - \frac{v_{\perp}}{B} \hat{\theta},$$
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\[ U(x, v_\parallel, v_\perp, \theta) = v_\parallel \left( 1 + \frac{v_\parallel}{B} \vec{\nabla} \times \vec{b} \right) - \frac{v_\parallel^2}{2B} ((\theta b_\perp) + \vec{b} \cdot \vec{R}), \]  
(2.32)

\[ \mu(x, v_\parallel, v_\perp, \theta) = \frac{v_\parallel^2}{2B} \left( 1 - \frac{v_\parallel}{B} \vec{b} \cdot \vec{\nabla} \times \vec{b} \right) + \frac{v_\parallel^2}{3B^2} \left( \frac{1}{2} \vec{b} \cdot \vec{\nabla} \ln B - \vec{b} \cdot \vec{R} \right), \]

where we made use of the fact that for an arbitrary field quantity \( \psi \) we have \( \psi(r) = \psi(X) + \epsilon v_\perp/B \vec{b} \cdot \vec{\nabla} \psi + \mathcal{O}(\epsilon^2) \). Wimmel [1983] has considered the possibility that \( dv_\perp/d\lambda \) may vanish and, consequently, developed a regularized version of Littlejohn's work. We will not present this version but, instead, we will refer the interested reader to Wimmel [1983].

As was pointed by Littlejohn [1979, 1981], since the last Poisson-bracket relations in Eq. (2.30), defining the \( \mu \) characteristics, are trivially satisfied, we can now use the expressions for \( X \) and \( U \), and the Poisson bracket, expressed in local-magnetic coordinates, to obtain the Poisson-bracket relations involving the coordinates (\( X, U \)). Because \( X \) and \( U \) are constant along the \( \zeta \) characteristics, so will be the Poisson brackets \( \{X, U\} \) and \( \{X_i, X_j\} \). It is, therefore, straightforward to find the Poisson bracket of two arbitrary functions \( F, G \) in Darboux phase space

\[ \{F, G\}_D = \frac{1}{\epsilon} \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \zeta} \right) - \frac{\vec{b}}{B} \left( \frac{\partial F}{\partial X} \times \frac{\partial G}{\partial X} \right) \]
(2.33)

\[ + \left( \vec{b} + \epsilon \frac{U}{B} \vec{b} \times \vec{b} \cdot \vec{\nabla} \vec{b} \right) \cdot \left( \frac{\partial F}{\partial X} \frac{\partial G}{\partial U} - \frac{\partial G}{\partial X} \frac{\partial F}{\partial U} \right). \]

As was previously indicated, we note that the canonical pair \((\mu, \zeta)\), and the non-canonical coordinates \((X, U)\) are decoupled.

Finally, the inverse transformation \((X, U, \mu, \zeta) \to (x, v_\parallel, v_\perp, \theta)\), is given by the expressions

\[ x(X, U, \mu, \zeta) = X + \epsilon \left( \frac{2\mu}{B} \right)^{1/2} \vec{\zeta}, \]

\[ v_\parallel(X, U, \mu, \zeta) = U + \epsilon \left[ \mu \left( (\zeta b_\perp) + \vec{b} \cdot \vec{R} \right) - U \left( \frac{2\mu}{B} \right)^{1/2} \vec{b} \cdot \vec{\nabla} \times \vec{b} \right], \]

\[ v_\perp(X, U, \mu, \zeta) = (2\mu B)^{1/2} \left( 1 + \epsilon \frac{U}{2B} \vec{b} \cdot \vec{\nabla} \times \vec{b} \right) + \epsilon \frac{2\mu}{3} (\vec{\zeta} \cdot \vec{\nabla} \ln B + \vec{b} \cdot \vec{R}). \]

With these (inverted) expressions, the Darboux Hamiltonian becomes

\[ H_D(X, U, \mu, \zeta) = \frac{1}{2} U^2 + \mu B + \epsilon U \mu \left( \vec{b} \cdot \vec{\nabla} \times \vec{b} + \vec{\zeta} \cdot \vec{b} \vec{b} \cdot \vec{\nabla} \times \vec{b} \right), \]

\[ + \epsilon \frac{(2\mu B)^{1/2}}{B} \left[ \frac{2\mu}{3} (\vec{\zeta} \cdot \nabla B + B \vec{\nabla} \times \vec{b}) - U^2 \vec{b} \cdot \vec{\nabla} \times \vec{b} \right]. \]
The Darboux transformation has, therefore, transferred the gyrophase dependence from the Poisson-bracket structure onto the Hamiltonian.

2.1.2.4. Lie Transformation

Because we assume that the Lie transformation is a canonical transformation, the Darboux Poisson bracket \( \{ \cdot, \cdot \}_D(Z) \), given by Eq. (2.33), simply becomes the guiding-center Poisson bracket \( \{ \cdot, \cdot \}_G(\overline{Z}) = \{ \cdot, \cdot \}_D(\overline{Z}) \), where \( \overline{Z} = (\overline{X}, \overline{U}, \overline{\mu}, \overline{\zeta}) \) are the guiding-center coordinates.

The purpose of the Lie transformation is, therefore, to derive an expression for the guiding-center Hamiltonian such that the gyrophase-dependent terms are present only beyond a specified order. As explained in 1.3.3., we consider a sequence of Lie transformations \( T = T_n \ldots T_2 T_1 \), so that \( \overline{Z} = T \overline{Z} \) induces \( \overline{H} = (T^*)^{-1} H \), where \( H = H_D \) and \( \overline{H} = H_G \) is the desired Hamiltonian, and \( (T^*)^{-1} = (T_1)^{-1} (T_2)^{-1} \ldots \), with \( (T_n^*)^{-1} = \exp \left( -\epsilon^n L_n \right) \).

In Hamiltonian Lie perturbation theory, we assume the original Hamiltonian to be of the form

\[
H(X, U, \mu, \zeta) = H_0(X, U, \mu) + \sum_{n=1}^\infty \epsilon^n H_n(X, U, \mu, \zeta),
\]

and assume that the Lie transformation generating vector field is given in terms of a generating function \( g \) and the original Poisson bracket

\[
L_n f \equiv \epsilon \{ g_n, f \} = L_{n0} f + \epsilon L_{n1} f + \mathcal{O}(\epsilon^2).
\]

In the present case, the Darboux Poisson bracket, Eq. (2.33), gives

\[
L_{n0} f = \frac{\partial g_n}{\partial \zeta} \frac{\partial f}{\partial \mu} - \frac{\partial g_n}{\partial \mu} \frac{\partial f}{\partial \zeta},
\]

\[
L_{n1} f = \overline{b} \cdot \left( \frac{\partial g_n}{\partial X} \frac{\partial f}{\partial U} - \frac{\partial g_n}{\partial U} \frac{\partial f}{\partial X} \right),
\]

\[
L_{n2} f = -\frac{\overline{b}}{B} \cdot \left( \frac{\partial g_n}{\partial X} \times \frac{\partial f}{\partial X} \right) + \frac{U}{B} \overline{b} \times \overline{b} \cdot \nabla \overline{b} \left( \frac{\partial g_n}{\partial X} \frac{\partial f}{\partial U} - \frac{\partial g_n}{\partial U} \frac{\partial f}{\partial X} \right), \text{ etc.}
\]

The expression \( \overline{H} = (T^*)^{-1} H \), when expanded as a power series in \( \epsilon \), gives an infinite set of inhomogeneous linear equations for the generating functions. At order \( n \), for example, the equation involves \( g_n, H_D^n, \) all the previous \( g_k \)'s, and \( H_G^n \). The requirement that \( H_G^n \) be gyrophase independent gives a generic equation of the form

\[
L_{n0} H_D^n = B \frac{\partial g_n}{\partial \zeta} = H_D^n - H_G^n + \mathcal{R}_n(H; g),
\]
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where \( \mathcal{R}_1 = 0 \), and \( \mathcal{R}_2 = -L_{10} H_{11} + (1/2)L_{10}^2 H_{00} - L_{11} H_{00} \), etc. We remark that, at order \( \mathcal{O}(\epsilon^0) \), we have \( H_{G0} = H_{D0} \). Separating gyrophase-independent terms from the gyrophase-dependent terms, we obtain

\[
H_{Gn} = \langle H_{Dn} \rangle + \langle \mathcal{R}_n \rangle, \tag{2.36}
\]

and

\[
B \frac{\partial g_n}{\partial \zeta} = \frac{\mathcal{H}_{Dn}}{\mathcal{H}_{Dn} + \mathcal{R}_n}, \tag{2.37}
\]

where the gyrophase-independent (resp. gyrophase-dependent) terms are identified with the notation \( \langle \ldots \rangle \) (resp. \( \mathcal{R} \)).

Going back to the Darboux Hamiltonian, Eq. (2.35), we obtain at zeroth order \( H_{G0} = (1/2) \mathcal{U}^2 + \mathcal{B}B \). At first order, we have

\[
\langle H_{D1} \rangle = \mathcal{U} \mathcal{B} \left( \frac{1}{2} \mathcal{B} \cdot \nabla \times \mathcal{B} + \mathcal{B} \cdot \mathcal{R} \right), \tag{2.38}
\]

and the equation for \( g_1 \), obtained from Eq. (2.35) and Eq. (2.37) with \( n = 1 \), becomes

\[
B \frac{\partial g_1}{\partial \zeta} = \frac{1}{2} \mu U (\mathcal{C} + \mathcal{B} \cdot \mathcal{R}) + \frac{(2 \mu B)^{3/2}}{3 \mathcal{B}} (\mathcal{C} \cdot \nabla \ln \mathcal{B} + \mathcal{B} \cdot \mathcal{R}) \tag{2.39}
- (2 \mu B)^{1/2} U \mathcal{B} \cdot \nabla \times \mathcal{B}.
\]

Assuming that \( g_1 \) is purely oscillatory in gyrophase, we give the solution of Eq. (2.39) as

\[
g_1 = \frac{\mu U}{4B} (\mathcal{C} - \mathcal{B} \cdot \mathcal{R}) : \nabla \mathcal{B} - \frac{(2 \mu B)^{3/2}}{3 \mathcal{B}^2} (\mathcal{B} \cdot \nabla \ln \mathcal{B} - \mathcal{C} \cdot \mathcal{R}) - (2 \mu B)^{1/2} \frac{U^2 \mathcal{B}}{3 \mathcal{B}^2} \mathcal{C} \cdot \nabla \times \mathcal{B},
\]

so that the guiding-center coordinates are given as \( \mathcal{Z} = Z + \epsilon \{ g_1, Z \} + \mathcal{O}(\epsilon^2) \), or

\[
\mathcal{X} = X + \mathcal{O}(\epsilon^2), \quad \mathcal{U} = U + \mathcal{O}(\epsilon^2), \quad \mathcal{B} = \mu + \epsilon \frac{\partial g_1}{\partial \mu} + \mathcal{O}(\epsilon^2), \tag{2.40}
\]

\[
\mathcal{C} = \zeta - \epsilon \frac{\partial g_1}{\partial \mu} + \mathcal{O}(\epsilon^2).
\]

We note that a gyrophase-independent term \( \langle g_1 \rangle \) would only appear in the definition of the gyrophase \( \mathcal{C} \), whose presence was eliminated from the guiding-center theory.
Finally, up to order $\mathcal{O}(\epsilon)$, the guiding-center Hamiltonian is given by the expression

$$H_G = \frac{1}{2}U^2 + \mu B + \epsilon \mu U \left( \frac{1}{2} \hat{b} \cdot \nabla \times \hat{b} + \hat{b} \cdot \mathbf{R} \right),$$

(2.41)

and the guiding-center equations of motion are

$$\dot{\mathbf{X}} = \hat{b} \left[ U + \epsilon \mu \left( \frac{1}{2} \hat{b} \cdot \nabla \times \hat{b} + \hat{b} \cdot \mathbf{R} \right) \right] + \epsilon \frac{\hat{b}}{B} \times (\mu \nabla B + U^2 \hat{b} \cdot \nabla \hat{b}) + \mathcal{O}(\epsilon^2),$$

$$\dot{U} = -\mu \hat{b} \cdot \nabla B + \mathcal{O}(\epsilon),$$

$$\dot{\zeta} = \Omega + \epsilon \frac{U \Omega}{B} \left( \frac{1}{2} \hat{b} \cdot \nabla \times \hat{b} + \hat{b} \cdot \mathbf{R} \right).$$

(2.42)

The equations for $\dot{\mathbf{X}}_\perp$, $\dot{U}$, and $\dot{\zeta}$ are similar to the ones derived classically, as given in Eqs. (2.9) and (2.12). The equation for $\hat{b} \cdot \dot{\mathbf{X}}$ shows, however, a possible discrepancy. This discrepancy disappears when one considers the gyrophase-averaged expression $\langle \nu_\parallel(X, U, \mu, \zeta) \rangle$ in Eq. (2.34), given as

$$\langle \nu_\parallel \rangle = U + \epsilon \mu \left( -\frac{1}{2} \hat{b} \cdot \nabla \times \hat{b} + \hat{b} \cdot \mathbf{R} \right).$$

The expression for $\hat{b} \cdot \dot{\mathbf{X}}$, combined with $\langle \nu_\parallel \rangle$, becomes

$$\hat{b} \cdot \dot{\mathbf{X}} = \langle \nu_\parallel \rangle + \epsilon \mu \hat{b} \cdot \nabla \times \hat{b},$$

which is precisely the expression we derived classically, in Eq. (2.12), with $\epsilon = B/\Omega$. 
2.1.3 Phase-space Lagrangian Theory of Guiding-center Drift Motion

The work presented in this subsection is a slightly different version of the work of Littlejohn [1981, 1983] on the guiding-center phase-space Lagrangian theory. (Our version avoids a preliminary transformation considered by Littlejohn [1981, 1983].) The method of phase-space Lagrangian Lie perturbation will prove to be quite useful in that it will provide us with guiding-center equations of motion that are simpler than those derived through the application of the Hamiltonian Lie perturbation method [compare Eq. (2.42) with Eq. (2.60)]. In addition, the Phase-space Lagrangian Lie perturbation method allows us to perform, simultaneously, operations on the Poisson-bracket structure and the Hamiltonian.

2.1.3.1 Phase-space Lagrangian for Charged Particle Motion

Our goal remains the derivation of a guiding-center Hamiltonian and Poisson bracket. The same preliminary transformations, defined in Eq. (2.19), will be used

\[(p, q) \rightarrow (r, v) \rightarrow (x, v_\parallel, \mu_0, \theta),\]

where \(\mu_0 = \frac{v_\parallel^2}{2B}\). As a consequence of these two transformations, the phase-space Lagrangian, describing the motion of a charged particle in a nonuniform magnetic field,

\[\gamma = \frac{1}{\epsilon}(A + \epsilon v) \cdot dr - \frac{1}{2} |v|^2 dt, \quad (2.43)\]

is transformed into

\[\gamma = \frac{1}{\epsilon} \left[ A(x) + \epsilon \left( v_\parallel \vec{b} + B \frac{\partial \rho_0}{\partial \theta} \right) \right] \cdot dx - \left( \frac{1}{2} v_\parallel^2 + \mu_0 B \right) dt \equiv \sum_{n=0}^{\infty} \epsilon^{n-1} \gamma_n, \quad (2.44)\]

where \(\gamma_n = \tilde{\gamma}_n dz^i - H_n dt\), and the gyroradius vector is denoted \(\rho_0 = (2\mu_0 B)^{1/2} \tilde{\theta} = \rho_{\parallel 0} \tilde{\theta}\).

The only nonvanishing Lagrange bracket \([z^i, z^j]_0\) for \(\gamma_0 = A \cdot dx\) is

\[[x, x]_0 = -B \times I, \quad (2.45)\]

while the Lagrange brackets \([z^i, z^j]_1\) for

\[\gamma_1 = \left( v_\parallel \vec{b} + B \frac{\partial \rho_0}{\partial \theta} \right) \cdot dx - \left( \frac{1}{2} v_\parallel^2 + \mu_0 B \right) dt, \]
are given as \( \hat{1} = \hat{\theta} \times \hat{b} \)

\[
\begin{align*}
[x, x]_1 &= -\left\{ v_\parallel \nabla \times \hat{b} + \nabla \times (B \rho_{\perp 0} \hat{1}) \right\} \times I, \\
[v_\parallel, x]_1 &= \hat{b}, \\
[\mu_0, x]_1 &= \hat{1} / \rho_{\perp 0}, \\
[\theta, x]_1 &= -B \rho_0, \\
[v_\parallel, t]_1 &= -v_\parallel, \\
[\mu_0, t]_1 &= -B, \\
[x, t]_1 &= -\mu_0 \nabla B,
\end{align*}
\] (2.46)

2.1.3.2. Gyrogauche Invariance

The goal of the Hamiltonian guiding-center theory is to derive equations that are locally symmetric with respect to the gyrophase angle. In a nonuniform magnetic field, this means that the equations of motion are invariant with respect to a local redefinition of how the gyrophase angle is measured. We briefly present the theory of gyrogauche invariance, due to Littlejohn [1981, 1984b].

The gyrogauche transformation involves a redefinition of the orthonormal basis set \( (\hat{1}, \hat{2}, \hat{b}) \), whereby the unit vectors \( \hat{1} \) and \( \hat{2} \) are rotated by an angle \( \psi(x) \), in the plane perpendicular to the local unit vector \( \hat{b} \). Hence, the gyrogauche transformation is defined as

\[
\begin{align*}
\hat{b}' &= \hat{b}, \\
\hat{1}' &= \hat{1} \cos \psi + \hat{2} \sin \psi, \\
\hat{2}' &= -\hat{1} \sin \psi + \hat{2} \cos \psi.
\end{align*}
\]

In order to ensure that the (original) Lorentz force equation of motion, Eq. (2.1), be invariant with respect to this gyrogauche transformation, i.e., \( \hat{\theta}' = \hat{\theta} \) and \( \hat{1}' = \hat{1} \), we also transform the gyrophase \( \theta \) according to the rule: \( \theta' = \theta + \psi(x) \). The gyrogauche transformation, therefore, amounts to a local redefinition of the perpendicular axis from which the gyrophase angle is measured.

In order to ensure local gyrophase symmetry, it is important that dynamical quantities which are independent of the gyrophase should, also, be gyrogauche invariant. It is, however, interesting to note that the former property (gyrophase independence) does not imply the latter property (gyrogauche independence). Consider, for example,
the vector $R = (\nabla \hat{1}) \cdot \hat{2}$ appearing in various expressions in 2.1.1. and 2.1.2. It is simple to show that, under gyrogauged transformation, this vector becomes $R' = R + \nabla \psi (x)$. Thus, any expression containing $R$, which may be independent of the gyrophase, will nonetheless display gyrogauged dependence. This is a rather undesirable feature since the original dynamical quantities, and their classical guiding-center versions, do not show this dependence. Examples of this feature appear in the expressions for $H_G$ and $\dot{X}$, given by Eqs. (2.41) and (2.42), respectively.

It is, therefore, our purpose to develop an asymptotic theory of guiding-center drift motion which is explicitly gyrogauged invariant at every order. We note that the derivation of a guiding-center theory which is independent of the gyrophase and its gyrogauged corresponds to the constraint of local gyrophase symmetry.

The constraint of local gyrophase symmetry also imposes certain conditions on the form taken by gyrogauged-invariant vector fields and one-forms. Consider, first, a vector field $G$ acting on a gyrogauged-invariant function $F$

$$G(F) = G^r \frac{\partial F}{\partial r} + G^q \frac{\partial F}{\partial q} + G^\mu \frac{\partial F}{\partial \mu} + G^\theta \frac{\partial F}{\partial \theta}.$$ 

When the gyrogauged transformation: $r' = r$ and $\theta' = \theta + \psi (r)$, is applied to $G(F)$, the expression for $G(F)$ becomes

$$G'(F) = G^r \frac{\partial F}{\partial r'} + \cdots + G^\theta \frac{\partial F}{\partial \theta'},$$

where the components not shown are not affected by the gyrogauged transformation. Using the following chain-rule identities

$$\frac{\partial}{\partial r'} = \frac{\partial}{\partial r} - \nabla \psi \frac{\partial}{\partial \theta'} \quad \text{and} \quad \frac{\partial}{\partial \theta'} = \frac{\partial}{\partial \theta},$$

the vector field $G'$ becomes

$$G'(F) = G^r \frac{\partial F}{\partial r} + \cdots + (G^\theta - G^r \nabla \psi) \frac{\partial F}{\partial \theta}.$$ 

The requirement that the vector field be gyrogauged invariant, $G'(F) = G(F)$, implies that

$$G' = G, \quad \text{and} \quad G^\theta = G^\theta + G \cdot \nabla \psi.$$ 

The latter expression, in turn, implies that

$$G^\theta = g^\theta + G \cdot R,$$
where $g^\theta$ is the gyrogaug-e-invariant part of $G^\theta$. Similar considerations applied to the one-form
\[ \Gamma = \Gamma_r \cdot dr + \cdots + \Gamma_\theta d\theta, \]
result in the gyrogaug-e-invariant expression
\[ \Gamma = \gamma_r \cdot dr + \cdots + \Gamma_\theta (d\theta - R \cdot dr), \]
where $\gamma_r$ is the gyrogaug-e-invariant part of $\Gamma_r$. We shall use these expressions in what follows as necessary and sufficient conditions that vector fields and one-forms must satisfy in order to maintain gyrogaug e invariance.

We refer the reader to the work of Littlejohn [1981, 1984, 1988] on the time-dependent gyrogaug e transformation and its relationship to the so-called Berry's phase, discovered in the context of the study of adiabatic classical and quantum systems — see Lochak and Meunier [1988].

2.1.3.3. Guiding-center Transformation

Instead of using a two-step (Darboux–Lie) approach — as in the last two transformations of Eq. (2.19) — in order to derive the guiding-center Hamiltonian and Poisson bracket we consider the transformation
\[ (x, v_\parallel, \mu_0, \theta) \rightarrow (X, U, \mu, \zeta), \]
defined by the Lagrange-bracket relations
\[ [\mu, \zeta] = \epsilon, \tag{2.47} \]
\[ [\zeta, Z] = 0, \quad \text{where } Z = X, \ U, \text{ or } t. \]
The last Lagrange-bracket relation indicates that the new phase-space Lagrangian is gyrophase independent, so that the guiding-center Hamiltonian and Poisson bracket are readily available. The first Lagrange bracket is similar to the Poisson bracket involving $\mu$ and $\zeta$, defined in Eq. (2.30). However, because the Lagrange brackets involving $\mu$ and the other coordinates $(X, U, t)$ are unspecified, the coordinate pair $(\mu, \zeta)$ will not, in general, be canonically conjugate. This extra feature allows us additional freedom in the derivation of a guiding-center theory, and, in particular, allows us to successfully ensure gyrogaug e invariance. (We note that Littlejohn [1981, 1983] has introduced a preparatory transformation to the lowest-order guiding-center position
\[ (x, v_\parallel, \mu_0, \theta) \rightarrow (X, v'_\parallel, \mu'_0, \theta') \rightarrow (X, U, \mu, \zeta), \]
which is not necessary.

2.1.3.4. Lie Perturbation Analysis

In the analysis that follows, we will make use of Lie perturbation techniques as applied to one-forms (e.g., the phase-space Lagrangian). This analysis involves the use of the following equations, correct up to order $\varepsilon^3$

\[
\begin{align*}
\Gamma_0 &= \gamma_0, \\
\Gamma_1 &= \gamma_1 - i_1\omega_0 + dS_1, \\
\Gamma_2 &= -i_1\omega_1 - i_2\omega_0 + \frac{1}{2}i_1(dx_1\omega_0) + dS_2, \\
\Gamma_3 &= -i_2\omega_1 + \frac{1}{2}(i_1d)\gamma_1 + i_2d(i_1\omega_0) - \frac{1}{6}(i_1d)^3\gamma_0 + dS_3.
\end{align*}
\]

The zeroth-order term for the guiding-center phase-space Lagrangian is $\Gamma_0 = A(X) \cdot dX$. The first-order term $\Gamma_1$ will be determined from the expression

\[
\Gamma_1 = \left[ v_{\|} \hat{b} + B \frac{\partial \rho_0}{\partial \theta} - (B \times G_1) \right] \cdot dx - \left( \frac{v_{\parallel}^2}{2} + \mu_0 B \right) dt,
\]

where $i_1\omega_0 = (B \times G_1) \cdot dX$, obtained from Eq. (2.45), and $S_1 = 0$ were used. The requirement that $\Gamma_1$ be gyrophase-independent is easily satisfied if $G_1 = -\rho_0$. The other components of the first-order generating vector field are to be defined through higher-order expressions. However, to satisfy gyrogauge invariance, we must also have $G^\theta_1 = -\rho_0 R + g^\theta_1$, where $g^\theta_1$ is yet to be determined. The first-order term for the guiding-center phase-space Lagrangian is, therefore,

\[
\Gamma_1 = U\hat{b} \cdot dX - \left( \frac{1}{2} U^2 + \mu B \right) dt.
\]

We note that since $t$ is not to be transformed, we use $G^\xi_n \equiv 0$, for all $n$.

The expressions for $\Gamma_2$ and $\Gamma_3$, after some straightforward manipulations, are given as

\[
\begin{align*}
\Gamma_2 &= - \left[ \hat{b} \left( G^U_1 + U\hat{b} \cdot \nabla \hat{b} \cdot \rho_0 + \mu (\zeta_b \perp) \right) + \mu R \\
&\quad + B \times G_2 + U \frac{\partial \rho_0}{\partial \zeta} \hat{b} \cdot \nabla \hat{b} + \frac{1}{2} \left( \frac{G^\mu_1}{\rho_1} - \frac{\mu}{B} \hat{\zeta} \cdot \nabla B \right) - \frac{1}{2} B\rho_0 g^\xi_1 \right] \cdot dx \\
&\quad + \mu d\zeta + (UG^U_1 + BG^\mu_1 - \mu \rho_0 \nabla B) dt, \\
\Gamma_3 &= \left( \frac{\partial S_3}{\partial \zeta} - \frac{2}{3} \frac{\mu}{B \rho_0} \partial_\zeta g^\xi_1 - G^\mu_1 - \frac{\mu U}{B} \hat{b} \cdot \nabla \hat{b} + \frac{\mu}{3B} \rho_0 \cdot \nabla B \right) d\zeta
\end{align*}
\]
where \( S_2 = 0 \) was used in Eq. (2.50), and only the relevant terms are shown in Eq. (2.51). We note that the latter expression, with \( S_2 \neq 0 \), is needed in order to self-consistently derive, up to order \( O(\epsilon_B) \), the generating vector field \( G_1 \) and the spatial component of the generating vector field \( G_2 \).

Choosing \( S_3 = (2\mu/3)g_1^\zeta + s_3 \) and \( G_1^\mu = (\mu/B)\rho_0\cdot\nabla B + g_1^\mu \), Eqs. (2.50) and (2.51) take on the simplified form

\[
\Gamma_2 = \mu(d\zeta - R\cdot dX) - \left[ \frac{1}{2}\mu\hat{b}\cdot\nabla b - \frac{1}{2}\mu\hat{\zeta}\hat{\zeta} \frac{\partial S_3}{\partial \mu} \right] \cdot dX + (U G_1^{\mu} + B g_1^{\mu})dt, \tag{2.52}
\]

\[
\Gamma_3 = \left( \frac{\partial S_3}{\partial \zeta} + G_2 \cdot \hat{b} \right) dU + \left( \frac{\partial S_3}{\partial \mu} + g_1^\zeta \right) d\mu
+ \left( \frac{\partial S_3}{\partial \zeta} - g_1^{\mu} - \frac{2\mu}{3B}\rho_0\cdot\nabla B - \frac{\mu U}{B}\hat{b}\cdot\nabla \hat{b} \right) d\zeta. \tag{2.53}
\]

The expression for \( \tilde{G}_1^U \), in Eq. (2.52), is obtained by requiring that the parallel spatial component of \( \Gamma_2 \) be gyrophase-independent, while the expression for \( G_2 \) is obtained by cancelling the perpendicular spatial component of \( \Gamma_2 \), except for the gyrogauge term \( \mu R \) as discussed in 2.1.3.2. The expression for \( \tilde{g}_1^{\mu} \) is obtained by requiring that the second-order Hamiltonian be gyrophase independent. Finally, using \( \tilde{g}_1^{\mu} \) in the \( \zeta \)-component of \( \Gamma_3 \), we can solve for \( s_3 \) (which is assumed to be purely oscillatory in gyrophase), and obtain expressions for \( g_1^\zeta \) and \( G_2 \cdot \hat{b} \), both of which are assumed to be purely gyrophase dependent.

The resulting expressions for \( \Gamma_2 \) and \( \Gamma_3 \) are

\[
\Gamma_2 = \mu(d\zeta - R\cdot dX) - (U G_1^U - \frac{1}{2}\mu \hat{b}\cdot\nabla \hat{b}) \cdot dX \tag{2.54}
+ (U G_1^{\mu} + B g_1^{\mu})dt,
\]

\[
\Gamma_3 = -(g_1^{\mu} + \frac{\mu U}{B} \hat{b}\cdot\nabla \hat{b}) d\zeta. \tag{2.55}
\]

The guiding-center Hamiltonian, given by Eq. (2.41), is obtained by setting

\[
\langle G_1^U \rangle = \frac{1}{2}\mu \hat{b}\cdot\nabla \hat{b}, \quad \text{and} \quad \langle g_1^\mu \rangle = -\frac{\mu U}{B} \hat{b}\cdot\nabla \hat{b}.
\]
2.1. Adiabatic Motion of Charged Particles in Electromagnetic Fields

As can be seen in Eq. (2.55), it cancels the contribution of \( \Gamma_3 \), and and leaves \( \Gamma_2 \) as

\[
\Gamma_2 = \mu \left( d\zeta - R \cdot dX \right) - \left( \frac{\mu U}{B} \hat{b} \cdot \nabla \times \hat{b} \right) dt.
\]

A more physical choice, which simplifies the phase-space Lagrangian and transforms to a frame that drifts with the Baños parallel drift, is given as

\[
U\langle G_1'^\mu \rangle = \mu U \hat{b} \cdot \nabla \times \hat{b} = -B\langle g_1'^\mu \rangle,
\]  

(2.56)

so that the guiding-center phase-space Lagrangian, up to order \( \mathcal{O}(\epsilon^2) \), is given as

\[
\Gamma = \frac{1}{\epsilon} \left( A(X) + \epsilon U \hat{b} - \epsilon^2 \mu W \right) \cdot dX + \epsilon \mu d\zeta - \left( \frac{1}{2} U^2 + \mu B \right) dt,
\]  

(2.57)

where \( W = R + \frac{1}{2} \hat{b} \cdot \nabla \times \hat{b} \). The transformation \((x, v, \mu, \theta) \to (X, U, \mu, \zeta)\) is, therefore, represented by the following expressions

\[
X = x - \rho_0 + \rho_1,
\]

\[
U = v_\parallel + \mu_0 \hat{b} \cdot \nabla \times \hat{b} + \frac{B}{4} \rho_0 \rho_0 : (\hat{b} \times \nabla \hat{b} - \nabla \hat{b} \times \hat{b}) - v_\parallel \hat{b} \cdot \nabla \hat{b} \cdot \rho_0,
\]  

(2.58)

\[
\mu = \mu_0 - \mu_0 \frac{v_\parallel}{B} \hat{b} \cdot \nabla \hat{b} - \frac{v_\parallel}{4} \rho_0 \rho_0 : (\hat{b} \times \nabla \hat{b} - \nabla \hat{b} \times \hat{b}) - \frac{\partial \rho_0}{\partial \theta} \cdot v_d,
\]

\[
\zeta = \theta - \rho_0 \cdot R - \frac{v_\parallel}{4B} (\hat{\theta} - \hat{I}) : \nabla \hat{b} + \frac{1}{B} \frac{\partial \rho_0}{\partial \mu_0} \frac{1}{B} \frac{\partial}{\partial \mu_0} (\mu_0 v_d),
\]

where \( \rho_1 = G_1^2(x)/2 + G_2 \), and

\[
v_d = \frac{\hat{b}}{B} \times (\mu_0 \nabla B + v_\parallel \hat{b} \cdot \nabla \hat{b})
\]

is the unperturbed guiding-center (magnetic) drift velocity. Finally, we note that the gyrogaugedependent term \( R \) is present only in the definition of the gyrophase angle. The removal of the dependence on this angle from the guiding-center equations of motion, therefore, ensures that these equations are also gyrogauged-independent.

2.1.3.5. Guiding-center Equations of Motion

Once the guiding-center phase-space Lagrangian, Eq. (2.57),

\[
\Gamma = \frac{1}{\epsilon} \left( A + \epsilon U \hat{b} - \epsilon^2 \mu W \right) \cdot dX + \epsilon \mu d\zeta - H dt,
\]

with Hamiltonian \( H = \frac{U^2}{2} + \mu B \), has been derived, it is easy to verify that the Lagrange-bracket relations, Eq. (2.47), used to define the transformation are indeed satisfied.
Using the inversion algorithm of Littlejohn [1981] given by Eqs. (1.43) and (1.44),
used to construct the Poisson-bracket relations from the Lagrange-bracket relations, we obtain the following guiding-center Poisson bracket

$$\{F, G\} = \frac{1}{\epsilon} \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \zeta} \right) - \epsilon \frac{\hat{b}}{B_{\parallel}^*} \cdot \left[ \left( \nabla F + \mathbf{w} \frac{\partial F}{\partial \zeta} \right) \times \left( \nabla G + \mathbf{w} \frac{\partial G}{\partial \zeta} \right) \right] + \frac{\mathbf{B}^*}{B_{\parallel}^*} \cdot \left[ \left( \nabla F + \mathbf{w} \frac{\partial F}{\partial \zeta} \right) \frac{\partial G}{\partial U} - \left( \nabla G + \mathbf{w} \frac{\partial G}{\partial \zeta} \right) \frac{\partial F}{\partial U} \right], \tag{2.59}$$

where $\mathbf{B}^* = \mathbf{B} + \epsilon U \nabla \times \hat{b} + \mathcal{O}(\epsilon^2)$, and $B_{\parallel}^* = \hat{b} \cdot \mathbf{B}^*$.

The guiding-center equations of motion are

$$\dot{X} = \{X, H\} = \frac{\mathbf{B}^*}{B_{\parallel}^*} \cdot \frac{\partial H}{\partial U} + \epsilon \frac{\hat{b}}{B_{\parallel}^*} \times \nabla H$$

$$= U \hat{b} + \epsilon v_d + \mathcal{O}(\epsilon^2),$$

$$\dot{U} = \{U, H\} = -\frac{\mathbf{B}^*}{B_{\parallel}^*} \cdot \nabla H \tag{2.60}$$

$$= -\mu \hat{b} \cdot \nabla B + \mathcal{O}(\epsilon),$$

$$\dot{\zeta} = \{\zeta, H\} = \frac{1}{\epsilon} \frac{\partial H}{\partial \mu} + U \hat{b} \cdot \mathbf{w} + \mathcal{O}(\epsilon),$$

$$\dot{\mu} = \{\mu, H\} = 0.$$

Finally, we comment on the $\epsilon$ ordering appearing in the expressions for the Poisson bracket, Eq. (2.59), and the guiding-center equations of motion, Eq. (2.60). First, we note in Eq. (2.59) that the gyromotion is assigned the dominant scaling ($\epsilon^{-1}$), the parallel motion is assigned the intermediate scaling ($\epsilon^0$), and the perpendicular (drift) motion is assigned the subdominant scaling ($\epsilon$). This $\epsilon$ scaling establishes the hierarchy of time scales for the problem of charged-particle motion in electromagnetic fields, which will be covered in more details in the next section.

The theory of guiding-center motion, which is concerned with the reduced dynamics in $(X, U)$ phase space, can still display fast periodic (bounce) motion in the parallel motion along the magnetic field, and a slow drift motion in the direction perpendicular to the magnetic field lines. The Hamiltonian theory of bounce-averaged guiding-center motion will be presented in subsection 2.2.2.
2.2 Adiabatic Motion of Magnetically Confined Charged Particles

The theory of the guiding-center motion of magnetically confined charged particles is reviewed. A brief description of the three well-known adiabatic invariants is given. Particular emphasis is put on the bounce-averaged guiding-center motion, due to the importance of trapped-particle dynamics in tokamak stability.

2.2.1 Adiabatic Invariants and Guiding-center Drift Motion

2.2.1.1. Exact and Adiabatic Invariants

The motion of charged particles, confined through the use of a magnetic field configuration, is inherently a three-dimensional problem. As is well-known, symmetries of the confining field can help reduce the number of degrees of freedom, and one very often talks of 1-D or 2-D magnetic geometries. The dynamical invariants, related to these field symmetries, are exact to the extent of the exactness of the underlying symmetries. For example, the axisymmetric tokamak configuration is a 2-D geometry, with the toroidal generalized canonical momentum acting as the corresponding dynamical invariant. (The finite number of toroidal-field coils leads to the breaking of the axisymmetry, and consequently the destruction of the dynamical invariant produces the so-called ripple transport.)

The confinement of charged particles implies the existence of a compact configuration-space volume which generically allows the existence of three incommensurable periodic motions. In addition, due to the presence of gradients in the confining field geometry — necessary for confinement — these periodic motions have typical time scales which are widely separated.

The first of these periodic motions exists even in uniform (unconfining) geometries, and is well-known as the gyromotion of a charged particle about a magnetic field line. The second periodic motion requires longitudinal confinement, i.e., $v_{||}$ periodically going through zero, and is known as the bounce motion of trapped particles along a magnetic field line. The third periodic motion, known as the precession (drift) motion, comes as a result of the compact nature of the configuration space and, consequently, the recurring nature of the drift motion perpendicular to the magnetic
field lines. Finally, we note that some magnetic geometries (e.g., toroidal geometry with nested magnetic surfaces) allow for the existence of confined, untrapped (or circulating) particles whose parallel motion is referred to as transit motion along a magnetic field line.

2.2.1.2. Hierarchy of Time Scales

We denote the periods for gyromotion, bounce motion, and precession motion, with \( T_c, T_b, \) and \( T_p \), respectively.

Let us consider the problem of studying the time evolution of a quantity \( F \). For this purpose, it is useful to define a time scale of interest, \( \omega^{-1} \), during which \( F \) changes in a specified fashion. Generically, we assume that \( F \) also changes on the time scales \( T_c, T_b, \) and \( T_p \). Because of the large separation of time scales, \( T_c \ll T_b \ll T_p \), it is possible to define an adiabatic invariant for each kind of periodic motion, through the use of an averaging principle.

The first application of the averaging principle to the exact equations of motion involves averaging over the gyration period, provided the condition \( \omega T_c \ll 1 \) is satisfied. The gyrophase-average operation on \( F \) is performed on the \( T_c \) time scale, during which \( F \) is considered frozen on the time scales \( T_b, T_p, \) and \( \omega^{-1} \). We loosely refer to this new quantity as the guiding-center \( F_g \).

Schematically, gyrophase averaging is described as follows

\[
\{\text{Exact Motion}\} \xrightarrow{\text{Gyration Period}} \left\{ \text{Guiding-center Motion} \right\} \xrightarrow{\text{(Magnetic Moment)}} \]

where the first adiabatic invariant is known as the magnetic moment \( \mu \). The guiding-center equations of motion were derived in the previous subsections. They describe the motion of the guiding center in \( (X, U) \) space, where \( X \) is the guiding-center position and \( U \) is the parallel velocity. The gyrophase degree of freedom has been averaged out (transformed away), and the gyromomentum (magnetic moment) is a conserved (adiabatic) quantity. The phase space for guiding-center motion is, therefore, four-dimensional.

The second application of the averaging principle involves averaging over the bounce period, provided the condition \( \omega T_b \ll \omega T_c \ll 1 \) is satisfied. If one is not interested in resonances between the gyration and bounce motions (c.f., Dubin and Krommes [1982]), we apply this averaging operation to the guiding-center equations of motion. The bounce-average operation on \( F_g \) is performed on the \( T_b \) time scale,
during which \( F_b \) is considered frozen on the time scales \( T_p \) and \( \omega^{-1} \). We loosely refer to this new quantity as the bounce-averaged \( F_b \).

Schematically, bounce averaging is described as follows

\[
\begin{align*}
\{ & \text{Guiding-center Motion} \} \quad \text{Bounce Period} \quad \{ & \text{Bounce-averaged,} \\
( & \text{magnetic Moment}) \}& \quad \{ & \text{Guiding-center Motion} \\
& \text{(Magnetic Moment)} \}& \quad \{ & \text{Guiding-center Motion} \\
& (\text{Longitudinal Invariant}) \}& \\
\end{align*}
\]

where a second adiabatic invariant, known as the longitudinal invariant \( J \), appears. The bounce-averaged guiding-center equations of motion will be given below, and will be derived in the next subsection. They describe the motion of the bounce-averaged guiding center in \((\alpha, \beta, w, t)\) space, where \((\alpha, \beta)\) are magnetic coordinates \((B = \nabla \alpha \times \nabla \beta)\). We have also included the possibility of time dependence, with canonical energy \( w \) and time \( t \) as the additional phase-space coordinates. The parallel (longitudinal) degree of freedom (bounce motion) has been averaged out (transformed away), and the longitudinal action integral is a conserved (adiabatic) quantity. The phase space for the time-dependent bounce-averaged guiding-center motion is four-dimensional.

The third application of the averaging principle involves averaging over the precession period, provided the condition \( \omega T_p \ll \omega T_b \ll \omega T_c \ll 1 \) is satisfied. If one is, again, not interested in resonance effects, we apply this averaging operation to the bounce-averaged guiding-center equations of motion. The precession-average operation on \( F_b \) is performed on the \( T_p \) time scale, during which \( F_b \) is considered frozen on the time scale \( \omega^{-1} \). We loosely refer to this new quantity as the precession-averaged \( F_p \).

Schematically, precession averaging is described as follows

\[
\begin{align*}
\{ & \text{Bounce-averaged,} \\
& \text{Guiding-center Motion} \} \quad \text{Precession Period} \quad \{ & \text{Bounce/Precession-averaged,} \\
( & \text{Magnetic Moment) } \}& \quad \{ & \text{Guiding-center Motion} \\
& \text{(Magnetic Moment) } \}& \quad \{ & \text{Guiding-center Motion} \\
& (\text{Longitudinal Invariant) } \}& \quad \{ & \text{Guiding-center Motion} \\
& (\text{Flux Invariant) } \}& \\
\end{align*}
\]

where a third adiabatic invariant, known as the flux invariant \( \Phi \), appears. The precession-averaged guiding-center equations of motion will given below. They describe the motion of the bounce/precession-averaged guiding center in the \((w, t)\) space. The periodic motion in the \( \alpha-\beta \) plane has been averaged out (transformed away), and
the flux through a $\alpha$-$\beta$ (closed) loop is a conserved (adiabatic) quantity. Finally, the phase space is two-dimensional. (Obviously the problem is trivial if there is no time dependence.)

2.2.1.3. Classical Proofs of Adiabatic Invariance

The adiabatic invariants $(\mu, J, \Phi)$ are classically given as asymptotic series

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots,$$

which, in general, do not converge. We shall present here the simplified derivation for $\mu_1$, taken from the direct proof of invariance for $\mu = \mu_0 + \epsilon \mu_1$, and compare this expression to the one derived in 2.1.3. [see Eq. (2.58)]. The classical proofs of invariance for $J_0 = \oint mv_\parallel ds$ and $\Phi_0 = \oint \alpha d\beta$ are well-known and will not be given here. They are to be found, for example, in Northrop [1963]. We will simply give the expressions for the bounce-averaged guiding-center equations of motion and $dJ/dt$, and for the bounce/precession-averaged guiding-center equations of motion and $d\Phi/dt$.

First, we consider a simple derivation of the expression for $\mu_1$, as presented in Catto et al. [1981], with $\mu_0 = mv_\perp^2/2B$. For time-independent field quantities, the operator $d/dt$ is given as

$$\frac{d}{dt} = v \cdot \nabla + \left( \Omega v \times \hat{b} - \frac{e}{m} \nabla \phi \right) \cdot \frac{\partial}{\partial v},$$

where $\phi$ is an equilibrium electrostatic potential. The expression for $\dot{\mu}_0$ is, therefore, given as

$$\dot{\mu}_0 = -\mu_0 v \cdot \nabla \ln B - \frac{mv_\parallel}{B} v \cdot \nabla \hat{b} \cdot v_\perp - \frac{e}{m} \nabla \phi \cdot v_\perp,$$  \hspace{1cm} (2.61)

which does not vanish, even in uniform magnetic fields ($\nabla \phi \neq 0$). We note, however, that the gyrophase-averaged expression $\langle \dot{\mu}_0 \rangle$ does vanish, which is a feature we also find with $J_0$ and $\Phi_0$, when averaged over their respective periodic degree of freedom.

Our purpose is to find an expression for $\mu_1$ so that $\dot{\mu}$ does indeed vanish, to order $O(\epsilon)$. The expression for $\dot{\mu}_1$ is

$$\dot{\mu}_1 = \Omega v \times \hat{b} \cdot \frac{\partial \mu_1}{\partial v} + \text{h.o.t.} = -\Omega \frac{\partial \mu_1}{\partial \zeta},$$  \hspace{1cm} (2.62)

where higher-order terms (h.o.t.) have been neglected. The solution to $\dot{\mu} = 0$, at the desired order, is now trivial, and we obtain $\tilde{\mu}_1$ (the gyrophase-dependent part of $\mu_1$)
as given by the expression

$$\bar{\mu}_1 = -\frac{v_{||}}{4\Omega} \frac{m v_{\perp} v_{\perp}}{B} : (\hat{b} \times \nabla \hat{b} - \nabla \hat{b} \times \hat{b}) - \frac{m}{B} v_{\perp} \cdot v_D,$$

(2.63)

where \(v_D\) is the total guiding-center drift velocity

$$v_D = \frac{\hat{b}}{\Omega} \times \left( \frac{e}{m} \nabla \phi + \frac{\mu_0}{m} \nabla B + v_{||} \hat{b} \cdot \nabla \hat{b} \right).$$

The gyrophase-independent part, \(\langle \mu_1 \rangle\), is obtained from the energy equation

$$\frac{m}{2} v_{||}^2 + e \phi(r) + \mu_0 B(r) = \frac{m}{2} U^2 + e \phi(X) + \mu B(X),$$

(2.64)

where quantities on the left-hand side are expressed in physical-space coordinates, and quantities on the right-hand side are expressed in guiding-center coordinates. The gyrophase average of Eq. (2.64), with the use of \(X = r + v \times \hat{b}/\Omega\) and the Baños relation \(U = v_{||} + (\mu_0 B/m\Omega) \hat{b} \cdot \nabla \times \hat{b}\), gives

$$\langle \mu_1 \rangle = -\frac{\mu_0}{\Omega} \hat{b} \cdot \nabla \times \hat{b}.$$  

(2.65)

The final expression for \(\mu_1\) is, therefore, given as

$$\mu_1 = -\frac{\mu_0 v_{||}}{\Omega} \hat{b} \cdot \nabla \times \hat{b} - \frac{m}{B} v_{\perp} \cdot v_D - \frac{v_{||}}{4\Omega} \frac{m v_{\perp} v_{\perp}}{B} : (\hat{b} \times \nabla \hat{b} - \nabla \hat{b} \times \hat{b}),$$

(2.66)

which is the same equation given in Eq. (2.58).

Next, the lowest-order term for the longitudinal invariant is given as \(J_0 = \oint m v_{||} ds\), where \(ds\) is the length element along the magnetic field and the integration interval is a bounce cycle. Following Northrop [1963], we use magnetic coordinates \((\alpha, \beta, s)\) where

$$B = \nabla \alpha \times \nabla \beta, \quad \text{and} \quad \hat{b} = \frac{B}{B} = \frac{\partial r}{\partial s}.$$  

(2.67)

The expression for the parallel velocity \(v_{||}\) is taken as

$$\frac{m}{2} v_{||}^2 = W - e (\phi + \psi) - \mu B,$$

(2.68)

where \(W\) is the total particle energy and the electric field is given as

$$E = -\nabla (\phi + \psi) + \frac{1}{c} \left( \frac{\partial \beta}{\partial t} \nabla \alpha - \frac{\partial \alpha}{\partial t} \nabla \beta \right),$$

(2.69)

so that \(E_{||} = -\partial (\phi + \psi)/\partial s\) and \(\psi = (\alpha/c) \partial \beta/\partial t\).
We assume that the fields $\phi$, $\psi$, and $B$ are functions of $(\alpha, \beta, s, t)$, and

$$J_0(\alpha, \beta, W, \mu, t) = \oint \{2m[W - e(\phi + \psi) - \mu B]\}^{1/2} ds.$$  \hspace{1cm} (2.70)

The equations describing the bounce-averaged guiding-center motion make use of the implicit relation $W = W(\alpha, \beta, \mu, J_0, t)$, and are given as

$$\langle \dot{\alpha} \rangle_b = -\frac{c}{e} \frac{\partial W}{\partial \beta},$$

$$\langle \dot{\beta} \rangle_b = \frac{c}{e} \frac{\partial W}{\partial \alpha},$$

$$\langle \dot{W} \rangle_b = \frac{\partial W}{\partial t},$$

$$1 = T_b \frac{\partial W}{\partial J_0},$$  \hspace{1cm} (2.71)

where $\langle \cdot \rangle_b$ is the bounce-average operator, defined as follows

$$\langle f \rangle_b = T_b^{-1} \oint \frac{ds}{v_\parallel} f(s).$$  \hspace{1cm} (2.72)

The lowest-order expression for $dJ_0/dt$ is given as

$$\frac{dJ_0}{dt}(s) = \frac{eT_b}{c} \left[ (\dot{\alpha})_b \dot{\alpha}(s) - \langle \dot{\beta} \rangle_b \dot{\alpha}(s) \right] + T_b \left[ \dot{W}(s) - \langle \dot{W} \rangle_b \right].$$  \hspace{1cm} (2.73)

We again note that, even though $\dot{J}_0$ does not vanish, its bounce average

$$\oint \frac{ds}{v_\parallel} d\dot{J}_0(s) = \oint \frac{d\beta}{v_\parallel} \oint \frac{ds}{v_\parallel} \left\{ \frac{e}{c} \left[ (s') \dot{\alpha}(s) - \dot{\beta}(s') \dot{\alpha}(s) \right] + \left[ \dot{W}(s) - \dot{W}(s') \right] \right\},$$

obviously does, due to the antisymmetric nature of the integrand. We also note that the equations of motion, Eq. (2.71), clearly have a Hamiltonian canonical structure, a fact which will be clarified in the next subsection.

Finally, the lowest-order term for the flux invariant is given as $\Phi_0 = \oint \alpha d\beta$, where the integration is performed along a closed loop in the $(\alpha, \beta)$ plane. The equations describing the bounce/precession-averaged guiding-center motion are given as

$$\langle \dot{W} \rangle_{bp} = -\frac{e}{cT_p} \frac{\partial \Phi}{\partial t}(\mu, J, W, t),$$

$$1 = \frac{e}{cT_p} \frac{\partial \Phi}{\partial W}(\mu, J, W, t),$$  \hspace{1cm} (2.74)
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where \( \langle \cdot \rangle_{bp} \) is the bounce/precession-average operator, defined as

\[
\langle f \rangle_{bp} = T_p^{-1} \int \frac{d\beta}{\langle \beta \rangle_b} \langle f \rangle_b.
\]  

(2.75)

The lowest-order expression for \( d\Phi/dt \) is given as

\[
\frac{d\Phi_0}{dt} = \frac{cT_p}{e} \left[ \langle \dot{W} \rangle_b - \langle \dot{W} \rangle_{bp} \right].
\]  

(2.76)

Again, we note that the expression for \( \dot{\Phi}_0 \) does not vanish, but that its precession-average, \( \langle \dot{\Phi}_0 \rangle_{bp} \), does. We also note that for time-independent systems, the conservation of \( \Phi \) is trivial.

Further discussion on these adiabatic invariants is found in Northrop [1963].
2.2.2 Theory of Bounce-averaged Guiding-center Drift Motion

This subsection gives an improved version of the work of Littlejohn [1982] on the Hamiltonian theory of guiding-center bounce motion. Littlejohn uses a state-space formulation, in which time only is added to the regular phase-space coordinates, concludes in his Eq. (2.46a) that the first-order Hamiltonian vanishes, and that the expression corresponding to \( F_r \), in his Eq. (2.62), had not been calculated previously. We will show in fact that the first-order Hamiltonian, Eq. (2.104), does not vanish and that the expression corresponding to \( F_r \), Eq. (2.116), is given in Northrop [1963], for instance.

We use extended phase-space coordinates \((X, U, w, t; \mu)\), appropriate for the study of time-dependent Hamiltonian systems, where \( \mu \) appears only as a parameter. We note that Dubin and Krommes [1982] have considered the possibility of gyro-bounce resonances in their theory of Hamiltonian bounce-averaged motion. In the present work, we consider only the simplest case where the analysis is carried out sufficiently far away from such resonances.

2.2.2.1. Extended Guiding-center Phase-space Lagrangian

The expression for the extended phase-space Lagrangian was given previously as

\[
\gamma = \frac{1}{\epsilon} A^* \cdot dX - w dt - h dT,
\]

(2.77)

where the term \( \epsilon \mu \partial d\zeta \) has been omitted, and \( T \) is the extended-time parameter. The expressions for \( A^* \) and \( h \) were derived in Eq. (2.57), and are given as

\[
A^* = A + \epsilon \hat{U} \hat{b} - \epsilon^2 \mu W, \quad \text{and} \quad h = \frac{1}{2} U^2 + \mu B + \phi - w.
\]

Note that we have considered the addition of an electrostatic potential \( \phi \). Also, as was explained in 1.2.2.2., the physical motion in extended phase space occurs on the submanifold \( h^{-1}(0) \), or \( w = U^2/2 + \mu B + \phi \), a fact which will be frequently used in what follows.

Instead of using the guiding-center position \( X \), it is customary (Northrop [1963]) to use the following magnetic coordinates \((\alpha, \beta, s)\) where [Eq. (2.67)]

\[
B = \nabla \alpha \times \nabla \beta, \quad \text{and} \quad \hat{b} = \frac{\partial X}{\partial s}.
\]
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Each magnetic coordinate is assumed to be a function of $X$ and $t$. [The time dependence is assumed to involve the combination $\tau = \epsilon t$, so that $w \, dt$ becomes $w \, \epsilon^{-1} \, d\tau$ in Eq. (2.77).] It is, also, quite convenient to introduce a 2-D vector field $\chi$, where $\chi^1 = \alpha$ and $\chi^2 = \beta$. A suitable expression for the vector potential $A = \alpha \nabla \beta$, is the symmetric form

$$A = \frac{\eta^{ab}}{2} \chi^a \nabla \chi^b,$$

where $\eta^{12} = 1 = -\eta^{21}$, and $\eta^{11} = 0 = \eta^{22}$.

Before giving the expression for $A \cdot dX$, we must first introduce the orthogonality relations for $X = X(\alpha, \beta, s, \tau)$. (For more details, see Boozer [1986].) The expression for $dX$ is

$$dX = \frac{\partial X}{\partial \chi^a} d\chi^a + \hat{b} \, ds + \frac{\partial X}{\partial \tau} \, d\tau,$$  \hspace{1cm} (2.78)

and the orthogonality relations are

$$\frac{\partial X}{\partial \chi^a} \cdot \nabla \chi^b = \delta^b_a,$$
$$\frac{\partial X}{\partial \chi^a} \cdot \nabla s = 0 = \hat{b} \cdot \nabla \chi^a,$$  \hspace{1cm} (2.79)
$$\frac{\partial X}{\partial \tau} \cdot \nabla \chi^a = 0 = \frac{\partial s}{\partial \tau} + \frac{\partial X}{\partial \tau} \cdot \nabla s.$$

The derivation of these relations is well known and will not be given here. With these relations, the expression for $A \cdot dX$ is, therefore, given as

$$A \cdot dX = \alpha \, d\beta - \alpha \frac{\partial \beta}{\partial \tau} \, d\tau.$$  \hspace{1cm} (2.80)

If we now make the following substitution

$$w = w + \alpha \frac{\partial \beta}{\partial \tau},$$  \hspace{1cm} (2.81)
$$H = \frac{1}{2} U^2 + \mu B + \Phi - w,$$

the phase-space Lagrangian [Eq. (2.77)] is, finally, given as

$$\gamma = \frac{1}{\epsilon} \left( \frac{\eta^{ab}}{2} \chi^a d\chi^b - w \, d\tau \right) + (U \hat{b} - \epsilon \mu W) \cdot dX - \left( \Phi + \frac{1}{2} U^2 + \mu - w \right) \, dT.$$  \hspace{1cm} (2.82)
2.2.2.2. Bounce Action-Angle Variables

The periodic bounce motion in the \( U-s \) plane is naturally described in terms of the following, leading-order, action-angle variables \( I \) and \( \psi \), defined by the expressions

\[
I(\chi, w, \tau) = \frac{1}{\pi} \int_{s_0}^{s_1} [2(w - \Phi - \mu B)]^{1/2} ds',
\]

\[
\psi(s, U; \chi, w, \tau) = \pi + \text{sgn}(U) \omega_0 \int_{s_0}^{s} [2(w - \Phi - \mu B)]^{-1/2} ds',
\]

where the lowest-order expression for the bounce frequency is given as \( \omega_0^{-1} = \partial I / \partial w \).

By definition, the transformation \( (s, U; \chi, w, \tau) \rightarrow (\psi, I; \chi, w, \tau) \) is canonical and therefore satisfies the condition

\[
\frac{\partial s}{\partial \psi} \frac{\partial U}{\partial I} - \frac{\partial s}{\partial I} \frac{\partial U}{\partial \psi} = 1.
\]

The new expression for \( dX \) is now given as

\[
dX = \frac{\partial X}{\partial \chi^a} d\chi^a + b \left( \frac{\partial s}{\partial \psi} d\psi + \frac{\partial s}{\partial I} dI \right) + \frac{\partial X}{\partial \tau} d\tau,
\]

and the expression for \( \tilde{U} \cdot dX \) is given as

\[
\tilde{U} \cdot dX = \tilde{U} \frac{\partial X}{\partial \chi^a} d\chi^a + U \left( \frac{\partial s}{\partial \psi} d\psi + \frac{\partial s}{\partial I} dI \right) + \tilde{U} \frac{\partial X}{\partial \tau} d\tau.
\]

The phase-space Lagrangian, expressed in terms of the coordinates \( (\chi, \psi, I, w, \tau) \), is given as \( \gamma = \tilde{\gamma} - H dT \), where

\[
\tilde{\gamma} = \left( \frac{\eta^{ab}}{2\epsilon} \chi^a + \tilde{U} b \frac{\partial X}{\partial \chi^b} \right) d\chi^b - \left( \frac{w}{\epsilon} - \tilde{U} b \frac{\partial X}{\partial \tau} \right) d\tau + U \left( \frac{\partial s}{\partial \psi} d\psi + \frac{\partial s}{\partial I} dI \right)
\]

is the extended symplectic (Poisson-bracket) structure, showing the leading-order terms \( O(\epsilon^{-1}) \) and the next-order \( O(1) \) terms, and the extended Hamiltonian \( H \) was given in Eq. (2.81), with \( U = U(\psi, I) \).

From the extended symplectic structure, we obtain the extended Lagrange two-form whose components are given as follows

\[
\tilde{\omega}_{ab} = \frac{\eta^{ab}}{\epsilon} + \left[ \frac{\partial}{\partial \chi^a} \left( \tilde{U} b \frac{\partial X}{\partial \chi^b} \right) - \frac{\partial}{\partial \chi^b} \left( \tilde{U} b \frac{\partial X}{\partial \chi^a} \right) \right] = \frac{\eta^{ab}}{\epsilon} + \Lambda_{ab},
\]

\[
\tilde{\omega}_{Ia} = \frac{\partial}{\partial I} \left( \tilde{U} b \frac{\partial X}{\partial \chi^a} \right) - \frac{\partial}{\partial \chi^a} \left( U \frac{\partial s}{\partial I} \right) = \Lambda_{Ia},
\]
\[\begin{align*}
\tilde{\omega}_{\phi a} &= \frac{\partial}{\partial \phi} \left( \hat{U}_b \frac{\partial \chi}{\partial \chi^a} \right) - \frac{\partial}{\partial \chi^a} \left( \hat{U} \frac{\partial \phi}{\partial \phi} \right) = \Lambda_{\phi a}, \\
\tilde{\omega}_{I\phi} &= \frac{\partial}{\partial I} \left( \hat{U} \frac{\partial \phi}{\partial \phi} \right) - \frac{\partial}{\partial \phi} \left( \hat{U} \frac{\partial I}{\partial I} \right) = 1, \\
\tilde{\omega}_{ra} &= \frac{\partial}{\partial \tau} \left( \hat{U}_b \frac{\partial \chi}{\partial \chi^a} \right) - \frac{\partial}{\partial \chi^a} \left( \hat{U}_b \frac{\partial \chi}{\partial \tau} \right) = \Lambda_{ra}, \\
\tilde{\omega}_{rI} &= \frac{\partial}{\partial \tau} \left( \hat{U} \frac{\partial I}{\partial I} \right) - \frac{\partial}{\partial I} \left( \hat{U}_b \frac{\partial \chi}{\partial \tau} \right) = \Lambda_{rI}, \\
\tilde{\omega}_{r\psi} &= \frac{\partial}{\partial \tau} \left( \hat{U} \frac{\partial \psi}{\partial \psi} \right) - \frac{\partial}{\partial \psi} \left( \hat{U}_b \frac{\partial \chi}{\partial \tau} \right) = \Lambda_{r\psi}, \\
\tilde{\omega}_{rw} &= \frac{1}{\epsilon}.
\end{align*}\]

The determinant of the extended Lagrange matrix, whose elements are given in Eq. (2.88), is \(\epsilon^{-4}(1 + \epsilon \Delta)^3\), where

\[\Delta = \eta^{ab} \left( \frac{1}{2} \Lambda_{ab} + \Lambda_{a\psi} \Lambda_{b\psi} \right).\]  

The matrix is, therefore, invertible and its inverse, the extended Poisson matrix, provides us with the expressions for the basic Poisson brackets. These brackets are given as follows

\[\begin{align*}
\{ \chi^a, \chi^b \} &= -\frac{\epsilon}{(1 + \epsilon \Delta)} \eta^{ab}, \\
\{ \chi^a, I \} &= \frac{\epsilon}{(1 + \epsilon \Delta)} \eta^{ab} \Lambda_{b\psi}, \\
\{ \chi^a, \psi \} &= \frac{\epsilon}{(1 + \epsilon \Delta)} \eta^{ab} \Lambda_{bI}, \\
\{ \psi, I \} &= \frac{(1 + \epsilon \Lambda_{12})}{(1 + \epsilon \Delta)}, \\
\{ w, \chi^a \} &= \frac{\epsilon^2}{(1 + \epsilon \Delta)} \eta^{ab} (\Lambda_{r\psi} + \Lambda_{bI} \Lambda_{r\psi} - \Lambda_{b\psi} \Lambda_{rI}), \\
\{ w, I \} &= -\frac{\epsilon}{(1 + \epsilon \Delta)} \left[(1 + \epsilon \Lambda_{12}) \Lambda_{r\psi} - \epsilon \eta^{ab} \Lambda_{r\psi} \Lambda_{b\psi}\right], \\
\{ w, \psi \} &= \frac{\epsilon}{(1 + \epsilon \Delta)} \left[(1 + \epsilon \Lambda_{12}) \Lambda_{rI} - \epsilon \eta^{ab} \Lambda_{r\psi} \Lambda_{bI}\right], \\
\{ w, \tau \} &= \epsilon.
\end{align*}\]

Using these Poisson brackets and the extended Hamiltonian \(H = \tilde{H} - w\), where

\[\frac{\partial \tilde{H}}{\partial I} = \omega_b, \quad \frac{\partial \tilde{H}}{\partial \psi} = 0,\]
(we commented earlier on the equivalence between $\tilde{H}$ and $w$) the equations of motion are

$$\frac{d\chi^a}{dT} = \{\chi^a, H\} = -\frac{\epsilon}{(1 + \epsilon\Delta)} \eta^{ab} \left[ \frac{\partial H}{\partial \chi^b} + \omega_\psi \Lambda_{b\psi} - \epsilon(\Lambda_{\tau b} + \Lambda_{bI} \Lambda_{\tau \psi} - \Lambda_{b\psi} \Lambda_{\tau I}) \right],$$

$$\frac{d\psi}{dT} = \{\psi, H\} = \omega_\psi + \epsilon \left[ \Lambda_{\tau I} - \eta^{ab} \left( \frac{\partial H}{\partial \chi^a} + \Lambda_{bI} \right) \right],$$

$$\frac{dI}{dT} = \{I, H\} = -\frac{\epsilon}{(1 + \epsilon\Delta)} \left( \Lambda_{\psi} + \eta^{ab} \frac{\partial H}{\partial \chi^a} \Lambda_{b\psi} \right),$$

$$\frac{dw}{dT} = \{w, H\} = \epsilon \left( \omega_\psi \Lambda_{\psi \tau} + \frac{\partial H}{\partial \tau} \right).$$

(2.91)

Notice that these equations have bouncephase dependent and independent parts.

2.2.2.3. Phase-space Gauge Transformation

Our goal is to obtain bounce-averaged equations of motion. However, before we proceed with the application of the phase-space Lagrangian Lie perturbation method on $\gamma$, we must perform two phase-space gauge transformations in order to facilitate the use of Lie perturbation techniques. (We remind the reader that phase-space gauge transformations are normally used to simplify the form of the phase-space Lagrangian, and do not affect the Hamiltonian dynamics.)

The first transformation is applied on the first-order term of the phase-space Lagrangian

$$\tilde{\gamma}_i = U \dot{\psi} \frac{\partial \chi^a}{\partial \chi^a} d\chi^a + U \dot{\psi} \frac{\partial \chi^a}{\partial \tau} d\tau + U \left( \frac{\partial s}{\partial \psi} d\psi + \frac{\partial s}{\partial I} dI \right),$$

(2.92)

and is expressed as $\gamma'_i = \tilde{\gamma}_i + dK_1$. We choose $K_1$ so that the component $\tilde{\gamma}_i'$ vanishes, which is easily satisfied with

$$K_1(\chi, \tau, \psi, I) = - \int_0^I dI' \tilde{\gamma}_i(\chi, \tau, \psi, I').$$

(2.93)

This choice is particularly useful since the component $\gamma'_{i\psi}$ takes on the form

$$\gamma'_{i\psi} = U \frac{\partial s}{\partial \psi} - \int_0^I dI' \frac{\partial}{\partial \psi} \left( U \frac{\partial s}{\partial I'} \right) = I,$$

by virtue of Eq. (2.84). This form is precisely the form needed for a bouncephase-independent Hamiltonian theory, as discussed in 1.2.3.5. The application of this transformation, therefore, produces the expression

$$\tilde{\gamma}'_i = F_a d\chi^a + Id\psi + F_\tau d\tau,$$

(2.94)
2.2. Adiabatic Motion of Magnetically Confined Charged Particles

where

\[ F_a = \int_0^1 dI' \Lambda_{Ia} = \int_0^{\psi} d\psi' \Lambda_{\phi a}, \]  \hspace{1cm} (2.95)  

\[ F_\tau = \int_0^1 dI' \Lambda_{I\tau} = \int_0^{\psi} d\psi' \Lambda_{\phi \tau}. \]  

The latter expressions for \( F_a \) and \( F_\tau \) show that they are odd in \( \psi \).

The second phase-space gauge transformation is applied on the second-order term \( \gamma_2 = -\mu W \cdot dX \), and is expressed as \( \gamma_2' = \gamma_2 + dK_2 \). Again, we choose \( K_2 \) so that the component \( \gamma_{2I} \) vanishes, which is easily satisfied with

\[ K_2 = -\int_0^1 dI' \gamma_{2I} = \int_0^1 dI' \mu \tilde{b} \cdot W \frac{\partial s}{\partial I'} = \mu \tilde{b} \cdot W (s - s_0). \]  \hspace{1cm} (2.96)  

The last expression on the right-hand side of Eq. (2.96) easily follows because \( \tilde{b} \cdot W \) is a function of \( \chi \) and \( \tau \) only. We note that the component \( \gamma_{2b} \) also vanishes. We are, therefore, left with

\[ \gamma_2' = \gamma_{2a} d\chi^a + \gamma_{2\tau} d\tau, \]

which we shall neglect in the analysis that follows.

2.2.2.4. Phase-space Lagrangian Lie Perturbation Analysis

The first-order analysis involves the expression

\[ \Gamma_1 = \gamma_1 - i_1 \omega_0, \]

where we have omitted the prime on \( \gamma_1 \), and have chosen \( S_1 = 0 \). The expression for \( i_1 \omega_0 \) is given as

\[ i_1 \omega_0 = \eta^{ab} G_1^b d\chi^a - G_1^W d\tau - \left( G_1^a \frac{\partial \tilde{H}_0}{\partial \chi^a} + G_1^\omega \omega_b - G_1^W \right) dT, \]  \hspace{1cm} (2.97)  

where \( \tilde{H} = \tilde{H}_0 + O(c^2) \) was used.

If we wish to adopt a Hamiltonian representation for our bounce-averaged phase-space Lagrangian, i.e., \( \Gamma_{1a} = 0 = \Gamma_{1\tau} \), the first-order generating vector field is, then, given as

\[ G_1^a = -\eta^{ab} F_b, \text{ and } G_1^W = -F_\tau, \]  \hspace{1cm} (2.98)  

with \( G_1^\tau \) assumed to be zero, and \( G_1^I \) and \( G_1^\psi \) yet to determined.
Finally, the expression for the first-order Hamiltonian is given as

$$\widetilde{H}_1 = -F_r - \omega_b G_1^r + \eta^{ab} \frac{\partial \widetilde{H}_0}{\partial \chi^a} F_b. \quad (2.99)$$

As usual, we require \( \widetilde{H}_1 \) to be equal to the bounce-averaged expression of the right-hand side of Eq. (2.99), i.e., \( \widetilde{H}_1 = -\omega_b (G_1^r) \). We have used the fact that \( F_b \) and \( F_r \) are both odd in \( \psi \), as shown in Eq. (2.95), and consequently, their bouncephase averages vanish. From Eq. (2.99) again, the bouncephase-dependent part of \( G_1^r \) is, therefore, solved as

$$\tilde{G}_1^r = \frac{1}{\omega_b} \left( \eta^{ab} \frac{\partial \widetilde{H}_0}{\partial \chi^a} F_b - F_r \right). \quad (2.100)$$

The second-order analysis involves the expression

$$\Gamma_2 = \gamma_2 - i_2 \omega_0 - \frac{1}{2} i_1 (\omega_1 + \omega_1) + dS_2,$$

where \( \omega_1 = d\gamma_1 \) and \( \omega_1 = d\Gamma_1 \). The expression for the \( \psi \)-component of \( \Gamma_2 \) is

$$\Gamma_{2\psi} = \frac{1}{2} \eta^{ab} \frac{\partial F_a}{\partial \psi} - G_1^r + \frac{\partial S_2}{\partial \psi}. \quad (2.101)$$

If we require this component to vanish, then, \( G_1^r \) is also given as [see Eq. (2.100)]

$$G_1^r = \frac{\partial S_2}{\partial \psi} - \frac{1}{2} \eta^{ab} F_b \frac{\partial F_a}{\partial \psi}. \quad (2.102)$$

The bounce-averaged expression is, therefore,

$$\langle G_1^r \rangle = -\frac{1}{2} \eta^{ab} \left< F_b \frac{\partial F_a}{\partial \psi} \right> = - \left< F_1 \frac{\partial F_2}{\partial \psi} \right>, \quad (2.103)$$

and, hence, the first-order Hamiltonian is given as

$$\widetilde{H}_1 = \frac{1}{2} \omega_b \eta^{ab} \left< F_b \frac{\partial F_a}{\partial \psi} \right>. \quad (2.104)$$

(We note that from Eqs. (2.88) and (2.95), the first-order Hamiltonian does not vanish.) The bouncephase-dependent part of Eq. (2.102) defines \( S_2 \), which is given as

$$S_2 = \frac{1}{\omega_b} \int_0^\psi d\psi' \left( \eta^{ab} \frac{\partial \widetilde{H}_0}{\partial \chi^a} F_b - F_r + \frac{1}{2} \eta^{ab} \omega_b F_b \frac{\partial F_a}{\partial \psi'} \right). \quad (2.105)$$
Finally, the expression for the $I$ component of $\Gamma_2$ is

$$\Gamma_{2I} = \frac{1}{2} G^a_I \frac{\partial F_a}{\partial I} + G^{\psi}_I + \frac{\partial S_2}{\partial I},$$

where $S_2$ is given above. If we require this component to vanish, then $G^{\psi}_I$ is given by the expression

$$G^{\psi}_I = \frac{1}{2} \eta^{ab} F_b \Lambda_{Ia} - \frac{\partial S_2}{\partial I} = \frac{1}{\omega_b} \int_0^\psi d\psi' \left( - \dot{\psi}_1 + G^a_I \frac{\partial \omega_b}{\partial \chi^a} + G^I_I \frac{\partial \omega_b}{\partial I} \right), \quad (2.106)$$

where $\dot{\psi}_1$ is the $\mathcal{O}(\epsilon)$ term in the expression for $d\psi/dT$, given in Eq. (2.91).

2.2.2.5. Bounce-averaged Phase-space Coordinates

Once the first-order generating vector field $G_1$ has been obtained, we can express the transformation $(\chi, I, \psi, w) \rightarrow (\xi, J, \Psi, W)$ as follows

$$\xi^a = \chi^a - \epsilon \eta^{ab} F_b + \mathcal{O}(\epsilon^2),$$

$$J = I + \epsilon \omega_b \left[ \eta^{ab} \left( F_b \frac{\partial F_a}{\partial \chi^b} - \frac{1}{2} \left( F_b \frac{\partial F_a}{\partial \psi} \right) \right) - F_r \right] + \mathcal{O}(\epsilon^2), \quad (2.107)$$

$$\Psi = \psi + \epsilon G^{\psi}_1 + \mathcal{O}(\epsilon^2),$$

$$W = w + \epsilon F_r + \mathcal{O}(\epsilon^2).$$

The bounce-averaged guiding-center phase-space Lagrangian is given by the expression

$$\Gamma = \frac{1}{2\epsilon} \eta^{ab} \xi^a d\xi^b + J d\Psi - W d\tau - (\tilde{\mathcal{H}} - W) dT, \quad (2.108)$$

and the Hamiltonian is given as

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \epsilon \frac{\eta^{ab}}{2} \omega_b \left( F_b \frac{\partial F_a}{\partial \psi} \right) + \mathcal{O}(\epsilon^2). \quad (2.109)$$

The new extended symplectic structure, in Eq. (2.108), gives the Poisson bracket expression

$$\{ F, G \} = \left( \frac{\partial F}{\partial \psi} \frac{\partial G}{\partial J} - \frac{\partial F}{\partial J} \frac{\partial G}{\partial \psi} \right) - \epsilon \left[ \eta^{ab} \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\partial \xi^b} + \left( \frac{\partial F}{\partial \tau} \frac{\partial G}{\partial W} - \frac{\partial F}{\partial W} \frac{\partial G}{\partial \tau} \right) \right], \quad (2.110)$$
and the bounce-averaged guiding-center equations of motion are

\[
\frac{d\xi^a}{dT} = \{\xi^a, \mathcal{H}\} = -\epsilon \eta^{ab} \frac{\partial \mathcal{H}}{\partial \xi^b}, \\
\frac{d\Psi}{dT} = \{\Psi, \mathcal{H}\} = \omega_b + \epsilon \frac{\eta^{ab}}{2} \frac{\partial}{\partial J} \left( F_b \frac{\partial F_a}{\partial \Psi} \right), \\
\frac{dW}{dT} = \{W, \mathcal{H}\} = \epsilon \frac{\partial \mathcal{H}}{\partial \tau}, \\
\frac{d\tau}{dT} = \{\tau, \mathcal{H}\} = 1,
\] (2.111)

with \(dJ/dT = \{J, \mathcal{H}\} = 0\). We note that these equations possess the aforementioned Hamiltonian canonical structure.

Finally, we shall look at the \(\mathcal{O}(\epsilon)\) term for the bounce invariant \(J\). More specifically, let us consider the bounce-phase-dependent part

\[
\tilde{J}_1 = \frac{1}{\omega_b} \left( \eta^{ab} F_b \frac{\partial \mathcal{H}_0}{\partial \chi^a} - F_r \right).
\] (2.112)

The first term on the right-hand side can be given as \(\langle \dot{\chi}^b \rangle F_b / \epsilon \omega_b\), where we have used the bounce-averaged expression for \(d\chi^b / dT\), given in Eq. (2.91). The bounce-phase-dependent part of \(d\chi^b / dT\) is given as

\[
\dot{\chi}^b - \langle \dot{\chi}^b \rangle = -\epsilon \omega_b \eta^{ab} \frac{\partial F_a}{\partial \Psi},
\]

which can be integrated to give the expression

\[
F_b = -\frac{1}{\epsilon \omega_b} \int_0^\phi d\psi' \eta^{bc} \langle \dot{\chi}^c - \langle \dot{\chi}^c \rangle \rangle.
\] (2.113)

The first term on the right-hand side of Eq. (2.112), therefore, is given by the expression

\[
\frac{1}{\epsilon^2 \omega_b^2} \int_0^\phi d\psi' \eta^{bc} \dot{\chi}^b (\dot{\chi}^c - \langle \dot{\chi}^c \rangle) = \frac{1}{\epsilon^2 \omega_b^2} \int_0^\phi d\psi' [\dot{\alpha}(\dot{\beta}) - \dot{\beta}(\dot{\alpha})],
\] (2.114)

where the antisymmetry of \(\eta^{ab}\) was used.

The second term on the right-hand side of Eq. (2.112), \(F_r\), can be obtained from the expression for the energy equation, given in Eq. (2.91) as

\[
\dot{w} = \epsilon \omega_b \frac{\partial F_r}{\partial \Psi} + \epsilon \frac{\partial \mathcal{H}_0}{\partial \tau}.
\] (2.115)
Since the last term on the right-hand side of Eq. (2.115) is simply \( \langle \dot{\mathbf{w}} \rangle \), \( F_\tau \) is solved as

\[
F_\tau = \frac{1}{\epsilon \omega_b} \int_0^\psi d\psi' \left( \dot{\mathbf{w}} - \langle \dot{\mathbf{w}} \rangle \right),
\]

which corresponds to the classical result given in Northrop [1963], for example.

The classical expression for \( \epsilon \mathbf{j}_1 \) is, therefore, regained and is given as

\[
\epsilon \mathbf{j}_1 = \frac{1}{\epsilon \omega_b^2} \int_0^\psi d\psi' \left[ (\dot{\mathbf{\alpha}}(\dot{\mathbf{\beta}}) - \dot{\mathbf{\beta}}(\dot{\mathbf{\alpha}})) + (\langle \dot{\mathbf{w}} \rangle - \dot{\mathbf{w}}) \right].
\]

Using the classical expression for \( dJ_0/dT \), given in Eq. (2.93), and the expression for \( J_1 \), we arrive at the well-known result

\[
\frac{dJ_0}{dT} + \epsilon \omega_b \frac{\partial J_1}{\partial \psi} = \mathcal{O}(\epsilon^2).
\]
PART II : REDUCED GYROKINETIC DESCRIPTION

There are three different methods used to derive nonlinear gyrokinetic equations. The first method, the gyrophase-averaging method, is exemplified by the work of Frieman and Chen [1982]. It consists of explicitly gyrophase-averaging the Vlasov equation, expressed in lowest-order guiding-center coordinates. Separating the equilibrium and perturbed parts of the guiding-center distribution function, one obtains a nonlinear gyrokinetic equation for the nonadiabatic part of the perturbed distribution function. Unfortunately, this method involves rather complex algebraic manipulations, fails to provide a clear algorithm indicating how to proceed to higher order, and provides equations that are not directly suitable for particle simulation studies.

The second (Hamiltonian) and third (Phase-space Lagrangian) methods require the use of Lie perturbation techniques (Littlejohn [1982a]). The unperturbed problem is assumed to represent the guiding-center motion of charged particles in a nonuniform equilibrium magnetic field, and the transformation from physical phase-space coordinates \((r, v)\) to guiding-center phase-space coordinates \((X, \rho, \mu, \zeta)\) is assumed to be given.

The Hamiltonian Lie perturbation method involves the following procedure. Under perturbation, the guiding-center Hamiltonian acquires gyrophase dependence, and consequently, the guiding-center magnetic moment \(\mu\) is no longer invariant. We obtain gyrocenter equations of motion by deriving a gyrocenter Hamiltonian which has no gyrophase dependence, to any desired order. This provides us with a transformation from guiding-center coordinates \((X, \rho, \mu, \zeta, w, t)\) to extended gyrocenter coordinates \((\overline{X}, \overline{\rho}, \overline{\mu}, \overline{\zeta}, \overline{w}, t)\), where now \(\overline{\mu}\) is the adiabatic invariant for the gyrocenter dynamics, to any desired order. This method is exemplified by the works of Dubin et al. [1983], Yang and Choi [1985], Hagan and Frieman [1985], and Hahm, Lee, and Brizard [1988].

The Phase-space Lagrangian Lie perturbation method represents a generalized form of the Hamiltonian method, as it allows for more general transformations from guiding-center phase space to gyrocenter phase space. It recognizes the fact that the Lagrange (and Poisson) brackets and the Hamiltonian can be perturbed simultaneously. We again obtain gyrocenter equations of motion by removing the gyrophase dependence from the Lagrange (and Poisson) brackets and the Hamiltonian. New
gyrocenter coordinates are, also, produced. This method was developed by Littlejohn [1982a] and Cary and Littlejohn [1983] and is exemplified by the works of Hahm [1988] and Brizard [1989a].
Chapter 3

Single-particle Gyrocenter Theory

The theory of gyrocenter motion, which describes the dynamics of guiding-centers under the influence of external (non self-consistent) electromagnetic fields, is presented. The perturbation fields are low-frequency, small-perpendicular-wavelength electromagnetic fields which satisfy the gyrokinetic ordering, introduced in 1.1.2. The unperturbed problem considered in our perturbation analysis represents the dynamics of guiding-center motion in a nonuniform equilibrium magnetic field. The gyrocenter theory presented here generalizes the guiding-center theory of White and Chance [1984].

3.1 Gyrokinetic Perturbation of Guiding-center Dynamics

The guiding-center dynamics of charged particles in nonuniform magnetic fields naturally evolve in (extended) guiding-center phase space. As the perturbation electromagnetic fields are first introduced in the original (physical) phase space, we must express these perturbation quantities in terms of the extended guiding-center coordinates \((X, \rho_\parallel, \mu, \zeta, w, t)\). The application of the Phase-space Lagrangian Lie perturbation method then proceeds with a derivation of the theory of gyrocenter dynamics in terms of the gyrocenter coordinates \((\bar{X}, \bar{\rho}_\parallel, \bar{\mu}, \bar{\zeta}, \bar{w}, \bar{t})\), where \(\bar{\mu}\) is the gyrocenter magnetic moment.
3.1.1 Electromagnetic Perturbation in Guiding-center Phase Space

We start with the extended guiding-center phase-space Lagrangian, given by Eq. (2.57)

$$\gamma_0 = \left[ \frac{\Omega}{B} (A + \rho_B B) - \frac{\mu B}{\Omega} W \right] \cdot dX + \frac{\mu B}{\Omega} d\zeta - w dt - H_0 d\tau,$$

where $H_0 = \frac{1}{2} \rho_\parallel^2 \Omega^2 + \mu B - w$. The extended guiding-center Poisson bracket is also given as [compare with the regular bracket, Eq. (2.59)]

$$\{F, G\} = \frac{\Omega}{B} \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \zeta} \right) + \left( \frac{\partial F}{\partial w} \frac{\partial G}{\partial \zeta} - \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial w} \right)$$

$$- \frac{B}{\Omega B^*_{\parallel}} \left[ \left( \nabla F + W \frac{\partial F}{\partial \zeta} \right) \times \left( \nabla G + W \frac{\partial G}{\partial \zeta} \right) \right]$$

$$+ \frac{B^*}{\Omega B^*_{\parallel}} \left[ \left( \nabla F + W \frac{\partial F}{\partial \zeta} \right) \frac{\partial G}{\partial \rho_\parallel} - \left( \nabla G + W \frac{\partial G}{\partial \zeta} \right) \frac{\partial F}{\partial \rho_\parallel} \right],$$

where $B^* = B + \epsilon_B \rho_\parallel \nabla \times B + O(\epsilon_B^2)$, and $B^*_{\parallel} = \tilde{b} \cdot B^*$.

3.1.1.1 Electromagnetic Perturbation

Consider perturbing the guiding-center phase-space Lagrangian with the one-form

$$\gamma_1 = \frac{\Omega}{B} \delta A(T_{GC}^{-1}X, t) \cdot d(T_{GC}^{-1}X) - \frac{\epsilon}{m} \delta \phi(T_{GC}^{-1}X, t) dt,$$

where $(\delta \phi, \delta A)$ are electromagnetic perturbation potentials, $T_{GC}^{-1}X = X + \rho_0 + \epsilon_B \rho_1 + O(\epsilon_B^2)$, and

$$\rho_1 \doteq -G_{2gc} - \frac{1}{2} G_{1gc}(\rho_0)$$

$$= - \left( \rho_\parallel \tilde{b} \cdot \nabla \tilde{b} + 2 \frac{\mu B}{\Omega^2} \nabla \text{ln} B \right) - \tilde{b} \left[ \frac{\mu B}{4 \Omega^2} \nabla \cdot \tilde{b} + 2 \rho_\parallel \left( \tilde{b} \cdot \nabla \tilde{b} \cdot \frac{\partial \rho_0}{\partial \zeta} \right) \right]$$

$$- \frac{1}{2} \rho_\parallel \rho_0 \cdot \left[ I (\tilde{b} \cdot \nabla \tilde{b}) + \frac{1}{2} (\tilde{b} \times \nabla \tilde{b} - \nabla \tilde{b} \times \tilde{b}) \right],$$

where the Lie generating vector fields $G_{1gc}$ and $G_{2gc}$ were evaluated in 2.1.3.4.

Using the notation $\delta \psi_0 = \delta \psi(X + \rho_0, t)$ for arbitrary perturbation field $\delta \psi$, and expanding Eq. (3.3) in powers of $\epsilon_B$, we obtain, up to order $O(\epsilon_B)$,

$$\gamma_1 = (\delta A_0 + \rho_1 \cdot \nabla \delta A_0 + \nabla \rho_0 \cdot \delta A_0) \cdot \frac{\Omega}{B} dX + \left( \frac{\delta A_{\rho_1}}{\partial \rho_\parallel} \right) \frac{\Omega}{B} d\rho_\parallel$$

(3.5)
+ \left[ (\delta A_0 + \rho_1 \cdot \nabla \delta A_0) \frac{\partial \rho_0}{\partial \mu} + \delta A_0 \frac{\partial \rho_1}{\partial \mu} \right] \frac{\Omega}{B} d\mu + \left[ (\delta A_0 + \rho_1 \cdot \nabla \delta A_0) \frac{\partial \rho_0}{\partial \zeta} \right] \frac{\Omega}{B} d\zeta - \frac{e}{m} (\delta \phi_0 + \rho_1 \cdot \nabla \delta \phi_0) d\tau.

Note that, because of the gyrokinetic ordering in perpendicular wavelength, Eq. (3.1.6), the perturbation potentials should not be expanded in powers of \( \rho_0 \).

### 3.1.1.2 Preliminary Phase-space Gauge Transformation

Considering the invariance of Hamiltonian dynamics under the phase-space gauge transformation \( \gamma \rightarrow \gamma + dS \) (where \( S \) is known as a phase-space gauge function), we redefine the perturbation one-form, Eq. (3.5), to be given by the expression

\[
\gamma' = \gamma - \frac{\Omega}{B} d(\rho_1 \cdot \delta A_0),
\]

which becomes (omitting the prime on \( \gamma_1 \))

\[
\gamma_1 = \frac{\Omega}{B} (\delta A_0 + \nabla \rho_0 \cdot \delta A_0 - \rho_1 \times \delta B_0) \cdot dX + \frac{\Omega}{B} (\delta A_0 - \rho_1 \times \delta B_0) \cdot \left( \frac{\partial \rho_0}{\partial \mu} + \frac{\partial \rho_0}{\partial \zeta} \right) dt + \frac{e}{m} (\delta \phi_0 + \rho_1 \cdot \nabla \delta \phi_0) d\tau,
\]

where \( \delta B_0 = \nabla \times \delta A_0 \) is the perturbed magnetic field. By adding Eq. (3.6) to Eq. (3.1) we obtain a total phase-space Lagrangian which exhibits gyrophase dependence. The purpose of the perturbation that follows is to remove this dependence to all orders in perturbation.

### 3.1.2 Phase-space Lagrangian Lie Perturbation Method

#### 3.1.2.1 First-Order Analysis

The application of the Lie perturbation method to the one-form \( \gamma \) gives the following first-order expression [see 1.3.3., Eq. (1.82)]

\[
\Gamma_1 = \gamma_1 - i_1 \omega_0 + dS_1.
\]

When we substitute the electromagnetic perturbation one-form, Eq. (3.6), with

\[
i_1 \omega_0 = \frac{B}{B} (B^* \times G_1 + C_1^w B - \frac{B^2}{\Omega^2} G_1^u W) \cdot dX - \frac{\Omega}{B} G_1 \cdot B d\rho_{||} - G_1^w dt
\]

\[
+ \frac{\Omega}{B} G_1^w d\zeta - \frac{B}{\Omega} (G_1^\zeta - G_1 \cdot W) d\mu - (G_1 \cdot \nabla \tilde{H}_0 + \rho_{||} \Omega^2 G_1^\rho_{||} + B G_1^u - G_1^w) d\tau,
\]
we obtain the following expressions ($\hat{\Gamma}_1 \omega \equiv 0$)

\[
\begin{align*}
\hat{\Gamma}_{1X} + \frac{\Omega}{B} (B^* \times G_1 + G_1^{\rho\parallel} B - \frac{B^2}{\Omega^2} G_1^w W) &= \frac{\Omega}{B} (\delta A_0 + \nabla \rho_0 \cdot \delta A_0 - \rho_1 \times \delta B_0) + \nabla S_1, \\
\hat{\Gamma}_{1\rho\parallel} - \frac{\Omega}{B} G_1 \cdot B &= \frac{\partial S_1}{\partial \rho_\parallel}, \\
\hat{\Gamma}_{1\mu} - \frac{B}{\Omega} (G_1^\mu - G_1 \cdot W) &= \frac{\Omega}{B} (\delta A_0 - \rho_1 \times \delta B_0) \cdot \frac{\partial \rho_0}{\partial \mu} + \frac{\partial S_1}{\partial \mu}, \\
\hat{\Gamma}_{1\zeta} + \frac{B}{\Omega} G_1^\mu &= \frac{\Omega}{B} (\delta A_0 - \rho_1 \times \delta B_0) \cdot \frac{\partial \rho_0}{\partial \zeta} + \frac{\partial S_1}{\partial \zeta}, \\
\hat{\Gamma}_{1t} - G_1^w &= -\frac{\Omega}{B^2} \rho_1 \cdot \frac{\partial \delta A_0}{\partial t} + \frac{\partial S_1}{\partial t}, \\
H_1 + (G_1 \cdot \nabla \tilde{H}_0 + \rho_\parallel \Omega^2 G_1^{\rho\parallel} + B G_1^\mu - G_1^w) &= \frac{e}{m} (\delta \phi_0 + \rho_1 \cdot \nabla \delta \phi_0).
\end{align*}
\]

(3.7)

Given a desired form for the components of the new first-order phase-space Lagrangian, Eq. (3.7) represents a set of inhomogeneous equations expressing the components of the first-order generating vector field $G_1$ in terms of the magnetic field perturbation $\delta A$, the phase-space gauge function $S_1$, and the components of $\hat{\Gamma}_1$. The equation for $S_1$ will be obtained from the $\tau$ component of the new phase-space Lagrangian, i.e., the Hamiltonian function.

Finally, it will be clear from Eq. (3.7) that in the case of a purely electrostatic perturbation (i.e., $\delta A = 0 = \hat{\Gamma}_1$), the components of the vector field $G_1$ are uniquely determined by the phase-space gauge function, which means that the transformation generated by $G_1$, etc., is a canonical transformation.

3.1.2.2. Second-Order Analysis

The application of the Lie perturbation method to the one-form $\gamma$ gives the following second-order expression (see 1.3.3.)

\[
\Gamma_2 = -i_2 \omega_0 - \frac{1}{2} i_1 (\omega_1 + \tilde{\omega}_1) + dS_2,
\]

(3.8)

where $\tilde{\omega}_1 \equiv d\Gamma_1$. In this work, the second-order analysis is always conducted at order $O(e^2 e_B^0)$. The first term on the right-hand side of Eq. (3.8) is given as

\[
i_2 \omega_0 = \frac{\Omega}{B} (B \times G_2 + G_2^{\rho\parallel} B) \cdot dX - \frac{\Omega}{B} G_2 \cdot B d\rho_\parallel - G_2^w dt \\
+ \frac{B}{\Omega} (G_2^\mu d\zeta - G_2^\zeta d\mu) - (\rho_\parallel \Omega^2 G_2^{\rho\parallel} + B G_2^\mu - G_2^w) \, dr,
\]
while the second term is given as

\[ i_1 \omega_1 = \frac{\Omega}{B} \left[ d_0 \times G_1 + G_1^\xi \left( \frac{\partial \delta A_0}{\partial \lambda^i} - \nabla \delta A_0 \cdot \frac{\partial \rho_0}{\partial \lambda^i} \right) \right] \cdot dX \]

\[ - \frac{\Omega}{B} \left[ G_1 \left( \frac{\partial \delta A_0}{\partial \lambda^i} - \nabla \delta A_0 \cdot \frac{\partial \rho_0}{\partial \lambda^i} \right) + \eta^{1i} G_1 \left( \frac{\partial \delta A_0}{\partial \lambda^i} \cdot \frac{\partial \rho_0}{\partial \lambda^j} - \frac{\delta \delta A_0}{\partial \lambda^j} \cdot \frac{\partial \rho_0}{\partial \lambda^i} \right) \right] d\lambda^i \]

\[ - \frac{\Omega}{B} \frac{\partial \delta A_0}{\partial t} \left( G_1 + G_1^\xi \frac{\partial \rho_0}{\partial \lambda^i} \right) dt - \frac{e}{m} \left( G_1 \cdot \nabla \delta \phi_0 + G_1^\xi \frac{\partial \delta \phi_0}{\partial \lambda^i} \right) \cdot d\tau, \]

where \( \lambda^1 = \mu, \lambda^2 = \zeta, \) and \( \eta^{12} = -\eta^{21} = 1 \) and \( \eta^{11} = \eta^{22} = 0. \) Given the spatial dependence \((X + \rho_0)\) of the perturbation potentials \((\delta \phi, \delta A)\), we easily obtain the following identities

\[ \frac{\partial \delta \phi_0}{\partial \lambda^i} = \frac{\partial \rho_0}{\partial \lambda^i} \cdot \nabla \delta \phi_0, \quad (3.9) \]

\[ \frac{\partial \delta A_0}{\partial \lambda^i} - \nabla \delta A_0 \cdot \frac{\partial \rho_0}{\partial \lambda^i} = -\frac{\partial \rho_0}{\partial \lambda^i} \times \delta B_0, \]

for \( i = 1 \) or \( 2, \)

\[ \frac{\Omega^2}{B} \left( \frac{\partial \delta A_0}{\partial \mu} \cdot \frac{\partial \rho_0}{\partial \mu} - \frac{\partial \delta A_0}{\partial \zeta} \cdot \frac{\partial \rho_0}{\partial \zeta} \right) = \delta B_{||0}. \quad (3.10) \]

Using these identities, the expression for \( i_1 \omega_1 \) therefore becomes

\[ i_1 \omega_1 = -G_1^* \left[ \frac{\Omega}{B} \delta B_0 \times \left( dX + \frac{\partial \rho_0}{\partial \mu} d\mu + \frac{\partial \rho_0}{\partial \zeta} d\zeta \right) + \frac{\Omega}{B} \frac{\partial \delta A_0}{\partial t} dt + \frac{e}{m} \nabla \delta \phi_0 dt \right], \]

where

\[ G_1^* \equiv G_1 + G_1^\mu \frac{\partial \rho_0}{\partial \mu} + G_1^\zeta \frac{\partial \rho_0}{\partial \zeta}. \]

Finally, \( i_1 \bar{\omega}_1 \) depends on the choice for the new one-form \( \Gamma_1. \) In general, for \( \Gamma_1 = \tilde{\Gamma}_{1\alpha} dz^\alpha - H_1 d\tau \) where \( z = (X, \rho_{||}, \mu, \zeta), \) and with \( \tilde{\Gamma}_{1\alpha} \) and \( H_1 \) functions of \((z, t),\) we have

\[ i_1 \bar{\omega}_1 = \tilde{\Gamma}_1^\beta \left( \frac{\partial \tilde{\Gamma}_{1\alpha}}{\partial z^\alpha} - \frac{\partial \tilde{\Gamma}_{1\beta}}{\partial z^\alpha} \right) dz^\alpha - \left( G_1^\alpha \frac{\partial \tilde{\Gamma}_{1\alpha}}{\partial t} \right) dt - \left( G_1^\alpha \frac{\partial H_1}{\partial z^\alpha} \right) d\tau. \]

The components of the new second-order one-form \( \tilde{\Gamma}_2 \) are, therefore, given by the following expressions \((\tilde{\Gamma}_{2\alpha} \equiv 0)\)

\[ \tilde{\Gamma}_{2X} + \frac{\Omega}{B} (B \times G_2 + G_2^\alpha B) = \frac{\Omega}{2B} G_1^* \times \delta B_0 - \frac{1}{2} G_1^\alpha \left( \frac{\partial \tilde{\Gamma}_{1\alpha}}{\partial z^\alpha} - \nabla \tilde{\Gamma}_{1\alpha} \right) + \nabla S_2, \]
\[ \hat{\Gamma}_{2\rho} - \frac{\Omega}{B} G_2 \cdot \mathbf{B} = -\frac{1}{2} G_1^2 \left( \frac{\partial \hat{\Gamma}_{1\rho}}{\partial x^\alpha} - \frac{\partial \hat{\Gamma}_{1\alpha}}{\partial \rho} \right) + \frac{\partial S_2}{\partial \rho}, \]

\[ \hat{\Gamma}_{2\mu} - \frac{B}{\Omega} G_2^\mu = \frac{\Omega}{2B} G_1^\mu \left( \delta B_0 \times \frac{\partial \rho_0}{\partial \mu} \right) - \frac{1}{2} G_1^\mu \left( \frac{\partial \hat{\Gamma}_{1\mu}}{\partial x^\alpha} - \frac{\partial \hat{\Gamma}_{1\alpha}}{\partial \mu} \right) + \frac{\partial S_2}{\partial \mu}, \quad (3.11) \]

\[ \hat{\Gamma}_{2\zeta} + \frac{B}{\Omega} G_2^\zeta = \frac{\Omega}{2B} G_1^\zeta \left( \delta B_0 \times \frac{\partial \rho_0}{\partial \zeta} \right) - \frac{1}{2} G_1^\zeta \left( \frac{\partial \hat{\Gamma}_{1\zeta}}{\partial x^\alpha} - \frac{\partial \hat{\Gamma}_{1\alpha}}{\partial \zeta} \right) + \frac{\partial S_2}{\partial \zeta}, \]

\[ \hat{\Gamma}_{2t} - G_2^w = \frac{\Omega}{2B} G_1^w \frac{\partial A_0}{\partial t} + \frac{1}{2} G_1^w \frac{\partial \hat{\Gamma}_{1\alpha}}{\partial t} + \frac{\partial S_2}{\partial t}, \]

\[ H_2 + (\rho_0 G_2^\rho + BG_2^\mu - G_2^w) = -\frac{e}{2m} G_1^\alpha \nabla \delta \phi_0 - \frac{1}{2} G_1^\alpha \frac{\partial H_1}{\partial x^\alpha}. \]

Given a desired form for the new second-order phase-space Lagrangian, Eq. (3.11) again represents a set of inhomogeneous equations expressing the components of the second-order Lie generating vector field \( G_2 \) in terms of the first-order generating vector field \( G_1 \), the magnetic field perturbation \( \delta B_0 \), and the second-order phase-space gauge function \( S_2 \).

Finally, we note again that in the case of a purely electrostatic perturbation, the components of the generating vector field \( G_2 \) are uniquely determined by the phase-space gauge function \( S_2 \), and the transformation generated by \( G_1 \) and \( G_2 \) is a canonical transformation.

In the next section, we shall give two different representations, which either considers vanishing components for the symplectic structure (i.e., \( \hat{\Gamma}_1 = 0 = \hat{\Gamma}_2 \)) or considers certain nonvanishing symplectic components. (From the discussion in 1.2.3.5, the invariance of the magnetic moment requires that \( \hat{\Gamma}_{n\mu} = 0 \) for \( n = 1, 2, \) etc., and we choose \( \hat{\Gamma}_{n\zeta} = 0 \) for \( n = 1, 2, \) etc.)
3.2 Unperturbed or Perturbed Symplectic Structures

The (total) extended phase-space Lagrangian, in guiding-center coordinates, is

\[
\gamma = \left[ \frac{\Omega}{B} \left( A + \epsilon \delta A_0 + \rho_\parallel B \right) - \frac{\mu B}{\Omega} W \right] \cdot dX + \epsilon \frac{\Omega}{B} \left( \delta A_0 \cdot \frac{\partial \rho_0}{\partial \mu} \right) d\mu \\
+ \left( \mu + \epsilon \frac{\Omega^2}{B^2} \delta A_0 \cdot \frac{\partial \rho_0}{\partial \zeta} \right) d\zeta - w dt - \left( \frac{1}{2} \rho_\parallel^2 \Omega^2 + \mu B - w + \epsilon \frac{e}{m} \delta \phi_0 \right) d\tau,
\]

where we have omitted for the moment the \( O(\epsilon_B) \) terms appearing in Eq. (3.6). It is clear that, through the dependence of \( (\delta \phi_0, \delta A_0) \) on \( \rho_0 \) (and therefore on \( \zeta \)), the symplectic structure and Hamiltonian have acquired gyrophase dependence. We define a gyrokinetic representation as a prescription for carrying out the operation of gyrophase averaging the perturbed phase-space Lagrangian, given by Eq. (3.12). Two generic gyrokinetic representations are possible (see section 1.3.).

The first representation is referred to as the Hamiltonian representation. It involves a transformation to gyrocenter coordinates such that the phase-space Lagrangian, Eq. (3.12), becomes

\[
\Gamma = \frac{\Omega}{B} A^* \cdot dX + \frac{\mu B}{\Omega} d\zeta - w dt - H_I d\tau,
\]

where \( A^* = A + \rho_\parallel B - \mu (B/\Omega)^2 W \), and the Hamiltonian is given as

\[
H_I = \frac{1}{2} \rho_\parallel^2 \Omega^2 + \mu B - w + \epsilon \frac{e}{m} \langle \delta \Phi_I^* \rangle.
\]

Note that the transformation has effectively transferred the symplectic-structure perturbation onto the Hamiltonian, and has produced a gyrophase-averaged effective perturbation potential \( \langle \delta \Phi_I^* \rangle \), yet to be determined (see 3.2.1.). Since the symplectic structure is unchanged, we can still use the unperturbed Poisson bracket, given by Eq. (3.2), with the new gyrocenter coordinates. The equations of motion are, therefore, given as

\[
\dot{X} = \frac{1}{\Omega B^*} \left( B \times \nabla H_I + B^* \frac{\partial H_I}{\partial \rho_\parallel} \right),
\]

\[
\dot{\rho}_\parallel = -\frac{B^*}{\Omega B^*} \cdot \nabla H_I,
\]
where $\mathbf{B}^* = \mathbf{B} + \epsilon_B (\rho_\parallel \nabla \times \mathbf{B}) + \mathcal{O}(\epsilon_B^2)$ and $B_\parallel^* = \hat{b} \cdot \mathbf{B}^*$. Note that, in this representation, the coordinate $\rho_\parallel$ is a (parallel) momentum quantity, conjugate to the parallel spatial coordinate $\hat{b} \cdot \mathbf{X}$ for a uniform magnetic field. This parallel-momentum representation was used by Hahm, Lee, and Brizard [1988] with a uniform slab geometry.

The second representation is referred to as the symplectic representation. It involves a transformation to gyrocenter coordinates such that the symplectic structure is left partially perturbed. The phase-space Lagrangian, Eq. (3.12), becomes

$$
\Gamma = \frac{\Omega}{B} A^{**} \cdot d\mathbf{X} + \frac{\mu B}{\Omega} d\zeta - w dt - H_{II} d\tau
$$

where $A^{**} = A^* + \epsilon (\delta A_0)$, and the Hamiltonian is given as

$$
H_{II} = \frac{1}{2} \rho_\parallel^2 \Omega^2 + \mu B - w + \frac{e}{m} (\delta \Phi_{II}^*)
$$

We note that, to ensure that $\mu$ is the adiabatic invariant for gyrocenter motion, the component $\Gamma_\zeta$ must always be a function of $\mu$ alone, as a consequence of Noether's theorem in phase-space Lagrangian dynamics (see I.2.3.5.). Hence, some symplectic perturbation terms will have to be transferred onto the Hamiltonian, thereby producing the effective perturbation potential $(\delta \Phi_{II}^*)$, yet to be determined (see 3.2.2.).

Now that the guiding-center symplectic structure has been perturbed, however, we must obtain new expressions for the Poisson brackets. Using these new brackets, the equations of motion take the form (see 3.3.2.2., for example)

$$
\dot{\mathbf{X}} = \frac{1}{\Omega B^*} \left[ B \mathbf{X} \left( \nabla H_{II} + \epsilon \frac{\Omega}{B} \frac{\partial (\delta A_0)}{\partial t} \right) + B^{**} \frac{\partial H_{II}}{\partial \rho_\parallel} \right],
$$

$$
\dot{\rho}_\parallel = -\frac{B^{**}}{B B^{**}} \left( \nabla H_{II} + \epsilon \frac{\Omega}{B} \frac{\partial (\delta A_0)}{\partial t} \right),
$$

where $B^{**} = B^* + \epsilon (\delta B_0)$. The inductive term $(\partial \delta A/\partial t)$ in the equation for $\dot{\rho}_\parallel$ indicates that $\rho_\parallel$ is now a parallel velocity. A parallel-velocity representation was also presented in the Appendix of Hahm, Lee, and Brizard [1988].

The presence of this inductive term in these equations of motion makes this representation useful if one wishes to have this term appear explicitly, as in chapters 5 and 6. In any case, both representations describe the same physics using different gyrocenter coordinates. In particular, an energy invariant can be found for each set of equations of motion.
3.2.1 Hamiltonian Representation

We consider the guiding-center dynamics as the unperturbed problem in our perturbation analysis. The zeroth-order gyrocenter phase-space Lagrangian is, therefore, given by the guiding-center phase-space Lagrangian, expressed in terms of the gyrocenter coordinates ($\bar{X}, \tilde{\rho}_\parallel, \tilde{\mu}, \tilde{\zeta}, \tilde{w}, \tilde{t}$). (Unless clarity of presentation requires us to use the barred notation for the gyrocenter coordinates, we shall avoid using this notation.)

The zeroth-order gyrocenter phase-space Lagrangian is, thus, given as

$$\Gamma_0 = \frac{\Omega}{B} A^* \cdot dX + \frac{\mu B}{\Omega} d\zeta - w dt - H_0 d\tau,$$

where [see Eq. (3.12)]

$$A^* = A + \rho_\parallel B - \epsilon_B \frac{\mu B^2}{\Omega^2} W \quad \text{and} \quad H_0 = \frac{1}{2} \rho_\parallel^2 \Omega^2 + \mu B - w.$$

The Hamiltonian representation transfers the symplectic perturbation onto the Hamiltonian to all orders in perturbation. Hence, the new phase-space Lagrangian possesses the original guiding-center Poisson-bracket structure, Eq. (3.2), and for $n \geq 1$, we have $\Gamma_n = -H_n d\tau$.

3.2.1.1. First-Order Analysis

The condition of vanishing symplectic-perturbation components of $\Gamma_1$ gives the following expressions for the first-order generating vector field $G_1$, from Eq. (3.7)

$$G_1 = \frac{\Omega}{B} \delta A_0 \cdot \{X + \rho_0, X\} + \{S_1, X\} - \frac{\Omega}{B} (\rho_1 \times \delta B_0) \cdot \{X + \rho_0, X\}_0,$$

$$G_1^\| = \frac{\Omega}{B} \delta A_0 \cdot \{X + \rho_0, \rho_\parallel\} + \{S_1, \rho_\parallel\} - \frac{\Omega}{B} (\rho_1 \times \delta B_0) \cdot \{X + \rho_0, \rho_\parallel\}_0,$$

$$G_1^\mu = \frac{\Omega}{B} \delta A_0 \cdot \{X + \rho_0, \mu\} + \{S_1, \mu\} - \frac{\Omega}{B} (\rho_1 \times \delta B_0) \cdot \{X + \rho_0, \mu\}_0,$$

$$G_1^\zeta = \frac{\Omega}{B} \delta A_0 \cdot \{X + \rho_0, \zeta\} + \{S_1, \zeta\} - \frac{\Omega}{B} (\rho_1 \times \delta B_0) \cdot \{X + \rho_0, \zeta\}_0,$$

$$G_1^w = \{S_1, w\} + \frac{\Omega}{B} \frac{\partial \delta A_0}{\partial t},$$

where $\{,\}$ represents the extended guiding-center Poisson bracket, Eq. (3.2), and $\{,\}_0$ is its uniform $O(\epsilon_B^0)$ limit,

$$\{F, G\}_0 = \frac{\Omega}{B} \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \zeta} \right) + \left( \frac{\partial F}{\partial w} \frac{\partial G}{\partial t} - \frac{\partial F}{\partial t} \frac{\partial G}{\partial w} \right)$$

$$- \frac{\tilde{b}}{\Omega} \cdot (\nabla F \times \nabla G) + \frac{\tilde{b}}{\Omega} \left( \nabla F \frac{\partial G}{\partial \rho_\parallel} - \nabla G \frac{\partial F}{\partial \rho_\parallel} \right).$$
The equation which determines the first-order Hamiltonian $H_1$ and the phase-space gauge function $S_1$ is obtained from Eqs. (3.7) and (3.14)

$$H_1 + \{S_1, H_0\} = \frac{e}{m} \delta \phi_0 - \frac{\Omega}{B} \delta A_0 \cdot \{X + \rho_0, H_0\} - \rho_1 \cdot \left( \frac{e}{m} \delta E_0 + \frac{\Omega}{B} \nu \times \delta B_0 \right),$$  \hspace{1cm} (3.16)

where $\{X + \rho_0, H_0\} = \nu + \nu_d + \rho_\parallel \Omega^2 \{\rho_0, \rho_\parallel\}$. The magnetic drift (grad B and curvature) velocity is denoted $\nu_d$, and

$$\Omega \{\rho_0, \rho_\parallel\} = \hat{b} \cdot \left( \nabla \rho_0 + W \frac{\partial \rho_0}{\partial \zeta} \right) = \frac{1}{2} \left[ (\nabla \cdot \hat{b}) \rho_0 + (\hat{b} \cdot \nabla \hat{b}) \frac{\partial \rho_0}{\partial \zeta} \right] - \hat{b} \cdot (\nabla \hat{b} \cdot \rho_0) \hat{b}. \hspace{1cm} (3.16)$$

The first-order Hamiltonian is defined as the gyrophase-averaged part of the right-hand side of Eq. (3.16):

$$H_1 = \frac{e}{m} \langle \delta \phi_0 \rangle - \frac{\Omega}{B} \langle \delta A_0 \cdot \{X + \rho_0, H_0\} \rangle - \frac{e}{m} \langle \rho_1 \cdot \left( \delta E_0 + \frac{\nu \times \delta B_0}{c} \right) \rangle. \hspace{1cm} (3.17)$$

The $O(\epsilon_B^3)$ term in Eq. (3.17), denoted $H_{10}$, is given by the well-known expression

$$H_{10} = \frac{e}{m} \langle \delta \phi_0 \rangle - \frac{\Omega}{B} \langle \nu \cdot \delta A_0 \rangle, \hspace{1cm} (3.18)$$

while the $O(\epsilon_B)$ term in Eq. (3.17), denoted $H_{11}$, is given by the more complicated expression

$$H_{11} = -\frac{\Omega}{B} \left\langle \left[ \nu_d + \rho_\parallel \Omega \hat{b} \cdot \left( \nabla \rho_0 + W \frac{\partial \rho_0}{\partial \zeta} \right) \right] \cdot \delta A_0 \right\rangle \hspace{1cm} (3.19)$$

$$- \left\langle \rho_1 \cdot \left( \frac{e}{m} \delta E_0 + \frac{\Omega}{B} \nu \times \delta B_0 \right) \right\rangle.$$

An expression that displays the leading-order couplings between the terms associated with the equilibrium magnetic nonuniformity (i.e., $\rho_1$, $\nabla \rho_0$, and $W$) and the lowest-order terms in the FLR ($k_\perp \rho_1$) expansion of the electromagnetic perturbation fields (i.e., $\delta \phi_0 = \delta \phi + \rho_0 \cdot \nabla \delta \phi + \cdots$, etc.) is

$$H_{11} = -\frac{\Omega}{B} \nu_d \cdot (\delta A + \rho_\parallel \delta B) - \frac{1}{2} \left( \rho_\parallel \hat{b} \cdot \nabla \times \hat{b} \right) \mu \delta B_\parallel + \left( \rho_\parallel \frac{\nabla \times B}{B} \right) \cdot \mu \delta B_\perp \hspace{1cm} (3.20)$$

$$- \frac{\partial}{\partial \mu} \left( \mu \nu_d \right) \cdot \frac{\hat{a}}{B} \times \nabla \delta \phi,$$

where $\nabla \times B$ is the unperturbed diamagnetic current density and

$$\frac{\partial}{\partial \mu} (\mu \nu_d) = \frac{\hat{b}}{\Omega} \times (2 \mu \nabla B + \rho_\parallel^2 \Omega^2 \hat{b} \cdot \nabla \hat{b}).$$
Finally, the phase-space gauge function \( S_1 \) is determined from the gyrophase-dependent part of the right-hand side of Eq. (3.16):

\[
\{S_1, H_0\} = \frac{e}{m} (\delta \tilde{\psi}_{10} + \epsilon_B \delta \tilde{\psi}_{11}),
\]

where \( \delta \tilde{\psi} = \delta \psi - \langle \delta \psi \rangle \), and \( e/m(\delta \psi_{10}) \) and \( e/m(\delta \psi_{11}) \) are the \( \mathcal{O}(\epsilon_B^0) \) and \( \mathcal{O}(\epsilon_B) \) parts of the first-order Hamiltonian \( H_1 \), respectively. The equation for \( S_1 \) will be solved in 3.2.3.

3.2.1.2. Second-Order Analysis

The condition of vanishing symplectic-perturbation components of \( \Gamma_2 \) gives the following expressions for the second-order generating vector field \( G_2 \), from Eq. (3.11):

\[

g_{20} = \{S_{20}, X\}_0 + \frac{1}{2} (G_{10}^* \times \delta B_0) \times \frac{\hat{b}}{B},
\]

\[
g_{20}^{\rho} = \{S_{20}, \rho_0\}_0 + \frac{1}{2} (G_{10}^* \times \delta B_0) \cdot \frac{\hat{b}}{B},
\]

\[
g_{20}^{\mu} = \{S_{20}, \mu\}_0 + \frac{\Omega^2}{2B^2} (G_{10}^* \times \delta B_0) \cdot \frac{\partial \rho_0}{\partial \zeta},
\]

\[
g_{20}^{\zeta} = \{S_{20}, \zeta\}_0 - \frac{\Omega^2}{2B^2} (G_{10}^* \times \delta B_0) \cdot \frac{\partial \rho_0}{\partial \mu},
\]

\[
g_{20}^{w} = \{S_{20}, w\}_0 - \frac{\Omega}{2B} G_{10}^* \cdot \frac{\partial \delta A_0}{\partial t},
\]

where \( G_{10}^* = \{S_{10}, X + \rho_0\}_0 \), and \( S_{n0} \) is the lowest-order solution for the phase-space gauge function \( S_n \). These components can also be written in general form as

\[
g_{20}^{z} = \{S_{20}, z^a\}_0 - \frac{\Omega}{2B} \{S_{10}, \delta A_0\}_0 \cdot \{X + \rho_0, z^a\}_0
\]

\[
+ \frac{\Omega}{2B} \{S_{10}, X + \rho_0\}_0 \cdot \delta A_0, z^a\}_0,
\]

where \( z = (X, \rho_0, \mu, \zeta, w, t) \), and \( G_{20}^{z} = 0 \).

The equation which determines \( H_{20} \) and \( S_{20} \) is obtained from Eqs. (3.11) and (3.23):

\[
H_{20} + \{S_{20}, H_0\}_0 =
\]

\[
- \frac{e}{2m} \{S_{10}, \delta \phi_0\}_0 - \frac{1}{2} \{S_{10}, H_{10}\}_0 - \frac{\Omega}{2B} (\delta A_0, \cdot \{X + \rho_0, H_{10}\}_0
\]

\[
+ \{S_{10}, X + \rho_0\}_0 \cdot \delta A_0, H_0\}_0 - \{S_{10}, \delta A_0\}_0 \cdot \{X + \rho_0, H_0\}_0,
\]
where the identity \( \{ \delta \phi_0, X + \rho_0 \}_0 = 0 \) was used. This identity follows from Eqs. (3.9) and (3.15), and also applies to \( \delta A_0 \) with Eqs. (3.10) and (3.15). Its use allows us to rearrange Eq. (3.24), which becomes

\[
H_{20} + \{ S_{20}, H_0 \}_0 = -\frac{e}{2m} \{ S_{10}, \delta \psi_{10} \}_0 - \frac{1}{2} \{ S_{10}, H_{10} \}_0 - \frac{e\Omega}{2mB} \delta A_0 \cdot \{ X + \rho_0, \delta \psi_{10} \}_0 \\
+ \frac{\Omega}{2B} \{ H_0, \{ S_{10}, X + \rho_0 \}_0 \cdot \delta A_0 \}_0, \tag{3.25}
\]

where we have also made use of the Jacobi identity for Poisson brackets (see I.2.3.4.)

\[
\{ f, \{ g, h \} \} + \{ h, \{ f, g \} \} + \{ g, \{ h, f \} \} = 0,
\]

which is verified, by construction, by the extended guiding-center Poisson bracket, Eq. (3.2).

The second-order Hamiltonian is defined as the gyrophase-averaged part of the right-hand side of Eq. (3.25)

\[
H_{20} = -\frac{e}{2m} \left< \{ S_{10}, \delta \bar{\psi}_{10} \}_0 \right> + \frac{\Omega^2}{2B^2} \left< | \delta A_0 |^2 \right> \equiv \frac{e}{m} \left< \delta \psi_{20} \right>, \tag{3.26}
\]

where

\[
c \{ X + \rho_0, \delta \psi_{10} \}_0 = \{ X + \rho_0, v \}_0 \cdot \delta A_0 = -\delta A_0,
\]

and

\[
\{ H_0, \{ S_{10}, X + \rho_0 \}_0 \cdot \delta A_0 \}_0 = -\Omega \frac{\partial}{\partial \zeta} (\{ S_{10}, X + \rho_0 \}_0 \cdot \delta A_0)
\]

were used. The right-hand side of the latter expression displays the lowest-order contribution in gyrokinetic ordering. Finally, the phase-space gauge function \( S_{20} \) is determined from the gyrophase-dependent part of the right-hand side of Eq. (3.25) (Brizard [1989a])

\[
\{ S_{20}, H_0 \}_0 = \frac{e}{m} \delta \bar{\psi}_{20}, \tag{3.27}
\]

and will be solved in 3.2.3.

### 3.2.2 Symplectic Representation

The zeroth-order gyrocenter phase-space Lagrangian is, again, given as

\[
\Gamma_0 = \frac{\Omega}{B} \Lambda^* \cdot dX + \frac{\mu B}{\Omega} d\zeta - \omega dt - H_0 d\tau.
\]
In the symplectic representation, the symplectic perturbation of the guiding-center phase-space Lagrangian is partially retained in the gyrocenter phase-space Lagrangian. We shall present the derivation of a phase-space Lagrangian given by the expression

$$\Gamma = \Gamma_0 + \epsilon \frac{\Omega}{B} \langle \delta A^*_0 \rangle_{\parallel} \cdot \delta X - (\epsilon H_1 + \epsilon^2 H_2) dt,$$  \hspace{1cm} (3.28)

where we have defined

$$\delta A^*_0 = \delta \cdot \delta A^*_0 = \delta \cdot (\delta A_0 - \rho_1 \times \delta B_0).$$ \hspace{1cm} (3.29)

This form is motivated by the derivation of the gyrofluid Ohm’s law in chapter 6 which requires the presence of the parallel induction term in the parallel electric field [see for instance Eq. (3.24)]. Note that the gyrocenter phase-space Lagrangian, Eq. (3.28), has a symplectic representation at first order, and a Hamiltonian representation at second order and beyond.

3.2.2.1. First-Order Analysis

Because the only symplectic component of $\Gamma_1$ is $\tilde{\Gamma}_1 X$, the components $G^\mu_1$, $G^\zeta_1$, and $G^w_1$ are not changed from the previous Hamiltonian representation, as given in Eq. (3.14). The other two components are obtained from the following expression

$$B^* \times G_1 + G^p_1 B = \delta A^*_0 - \langle \delta A^*_0 \rangle_{\parallel} \delta + \left( \nabla \rho_0 + W \frac{\partial \rho_0}{\partial \zeta} \right) \cdot \delta A_0 + \frac{B}{\Omega} \nabla S_1.$$ \hspace{1cm} (3.30)

It is obvious that $G_1$ is also unchanged from the previous treatment and, thus, one easily obtains the new $\rho_{\parallel}$ component

$$G^p_1 = G^p_1 (\text{Eq. (3.14)}) - \frac{1}{B} \langle \delta A^*_0 \rangle_{\parallel},$$ \hspace{1cm} (3.31)

where $G^p_1 (\text{Eq. (3.14)})$ is the expression for $G^p_1$ given in Eq. (3.14).

The equation which determines $H_1$ and $S_1$ is given as

$$H_1 + \{S_1, H_0\} = \frac{e}{m} \delta \phi_0 - \frac{\Omega^2}{B} \delta A_0 \cdot \frac{\partial \rho_0}{\partial \zeta} - \rho_1 \left( \frac{e}{m} \delta E_0 + \Omega \frac{\partial \rho_0}{\partial \zeta} \times \delta B_0 \right)$$ \hspace{1cm} (3.32)

$$- \rho_{\parallel} \frac{\Omega^2}{B} \delta \tilde{A}^*_0 - \frac{\Omega}{B} \delta A_0 \cdot \{X + \rho_0, H_0\} - v.$$ 

Again, the first-order Hamiltonian is defined as the gyrophase-averaged part of the right-hand side of Eq. (3.32):

$$H_1 = \frac{e}{m} \langle (\delta \phi_0 + \rho_1 \cdot \nabla \delta \phi_0) \rangle - \frac{\Omega^2}{B} \left( \langle \delta A_0 + \rho_1 \cdot \nabla \delta A_0 \rangle \cdot \frac{\partial \rho_0}{\partial \zeta} \right)$$ \hspace{1cm} (3.33)

$$- \frac{\Omega}{B} \langle \delta A_0 \cdot [(T_{GC}^{-1})^{-1} v - v] \rangle,$$
3.2. Unperturbed or Perturbed Symplectic Structures

where

\[
\epsilon_B^{-1} \left[ (T_{GC})^{-1} \nu - \nu \right] = \epsilon_B^{-1} \{ \nu_0 + \epsilon_B \rho_1 + \mathcal{O}(\epsilon_B^2), H_0 \}
\]

\[= \nu_0 + \rho_1 \Omega \cdot \mathbf{b} \cdot \left( \nabla \rho_0 + \mathbf{W} \frac{\partial \rho_0}{\partial \zeta} \right) + \Omega \frac{\partial \rho_1}{\partial \zeta} + \mathcal{O}(\epsilon_B).\]

The \(\mathcal{O}(\epsilon_B^0)\) term in Eq. (3.33) is given as

\[
H_{10} = \frac{e}{m} \langle \delta \phi_0 \rangle - \frac{\Omega^2}{B} \left\langle \delta A_0 \cdot \frac{\partial \rho_0}{\partial \zeta} \right\rangle,
\]

while the \(\mathcal{O}(\epsilon_B)\) term has an expression which, to leading order, is given as

\[
H_{11} = -\frac{\Omega}{B} \nu_0 \cdot \delta \mathbf{A} + \rho_1 \mu \delta \mathbf{B} \cdot \left( \mathbf{b} \times \mathbf{b} \times \mathbf{b} - \frac{1}{2} \mathbf{b} \cdot \nabla \times \mathbf{b} \right) - \frac{\partial}{\partial \mu} \left( \mu \nu_0 \right) \frac{\delta \phi_0}{B} \times \nabla \delta \phi_0.
\]

Finally, it is interesting to note that the equation for the first-order phase-space gauge function \(S_1\) is the same as Eq. (3.21):

\[
\{ S_1, H_0 \} = \frac{e}{m} \langle \delta \phi_0 + \epsilon_B \delta \phi_1 \rangle.
\]

3.2.2.2. Second-Order Analysis

With a non-vanishing first-order symplectic term in the gyrocenter phase-space Lagrangian, the components of the second-order generating vector field \(G_2\) will differ from the previous second-order analysis. These new components are given as follows

\[
G_{20}^\mu = \{ S_{20}, \rho_0 \}_0 + \frac{1}{2} \{ G_{10}^\ast \times \delta \mathbf{B}_0 \} \cdot \mathbf{b}B - \frac{1}{2B} \{ \mathbf{G}_{10}^\mu \} \frac{\partial \langle \delta A_{||0} \rangle}{\partial \mu},
\]

\[
G_{20}^\xi = \{ S_{20}, \xi_0 \}_0 - \frac{\Omega^2}{2B^2} \{ G_{10}^\ast \times \delta \mathbf{B}_0 \} \cdot \frac{\partial \rho_0}{\partial \mu} - \frac{\Omega^2}{2B^2} \{ \mathbf{G}_{10} - \mathbf{b} \} \frac{\partial \langle \delta A_{||0} \rangle}{\partial \mu},
\]

\[
G_{20}^w = \{ S_{20}, w_0 \}_0 - \frac{\Omega}{2B} \left( \mathbf{G}_{10}^\ast \cdot \frac{\partial \mathbf{A}_0}{\partial t} + \mathbf{G}_{10} \cdot \mathbf{b} \frac{\partial \langle \delta A_{||0} \rangle}{\partial t} \right),
\]

while the components \(G_{20}^\xi\) and \(G_{20}^w\) are the same as in Eq. (3.22), and \(G_{10}^\ast = \{ S_{10}, X + \rho_0 \}_0\).

The equation which determines \(H_{20}\) and \(S_{20}\) is given as

\[
H_{20} + \{ S_{20}, H_0 \}_0 =
-\frac{e}{2m} \{ S_{10}, \phi_0 \}_0 - \frac{1}{2} \{ S_{10}, H_{10} \}_0 - \frac{\Omega}{2B} \{ \mathbf{A}_0 \cdot \{ X + \rho_0, H_{10} \}_0 \}
+ \{ S_{10}, X + \rho_0 \}_0 \cdot \{ \delta \mathbf{A}_0, H_0 \}_0 - \{ S_{10}, \delta \mathbf{A}_0 \}_0 \cdot \{ X + \rho_0, H_0 \}_0
+ \frac{\rho_1}{2B} G_{10}^\mu \frac{\partial \langle \delta A_{||0} \rangle}{\partial \mu} + \frac{1}{2B} \left( \frac{\partial S_{10}}{\partial \rho_0} \frac{\partial \langle \delta A_{||0} \rangle}{\partial t} + \langle \delta A_{||0} \rangle \frac{\partial H_{10}}{\partial \rho_0} \right).
\]
The first term in the last bracket, on the right-hand side of Eq. (3.37), should be neglected since it is a higher-order term in gyrokinetic ordering, and the second term of the same bracket vanishes since $\partial H_{10}/\partial \rho_\parallel = 0$.

Use of the various identities given in Eq. (3.9) and the Jacobi identity produces the expression

$$\begin{align*}
H_{20} + \{ S_{20}, H_0 \} &= -\frac{e}{2m} \{ S_{10}, \delta \psi_{10} \} - \frac{1}{2} \{ S_{10}, H_{10} \} + \frac{\Omega^2}{2B^2} \left| \delta A_{\perp 0} \right|^2 \\
&\quad + \frac{\rho_\parallel \Omega^2}{2B} G_{10} \frac{\partial (\delta A_{\parallel 0})}{\partial \mu} - \frac{\Omega^2}{2B} \frac{\partial}{\partial \zeta} \left( \{ S_{10}, X + \rho_0 \} \delta A_0 \right).
\end{align*}$$

(3.38)

Again, the second-order Hamiltonian is defined as the gyrophase-averaged part of the right-hand side of this expression,

$$\begin{align*}
H_{20} &= -\frac{e}{2m} \langle \{ S_{10}, \delta \tilde{\psi}_{10} \} \rangle + \frac{\Omega^2}{2B^2} \langle \left| \delta A_{\perp 0} \right|^2 \rangle + \frac{\rho_\parallel \Omega^4}{2B^3} \left\langle \delta A_{\parallel 0} \cdot \frac{\partial \rho_0}{\partial \zeta} \right\rangle \frac{\partial \langle \delta A_{\parallel 0} \rangle}{\partial \mu},
\end{align*}$$

(3.39)

and the gauge function $S_{20}$ is determined by the gyrophase-dependent part of the right-hand side.

### 3.2.3 Phase-space Gauge Function

#### 3.2.3.1. First-order Phase-space Gauge Function

The equation determining the first-order gauge function $S_1$ is given by Eq. (3.21)

$$\{ S_1, H_0 \} = \frac{e}{m} \delta \tilde{\psi}_1,$$

and is independent of the representation (symplectic or Hamiltonian) used. If we show explicitly the gyrokinetic and magnetic nonuniformity orderings (identified with the small parameters $\epsilon$ and $\epsilon_B$, respectively), this equation becomes

$$\begin{align*}
\Omega \frac{\partial S_1}{\partial \zeta} + \epsilon \left( \frac{\partial}{\partial t} + \rho_\parallel \Omega \tilde{b} \cdot \nabla_0 \right) S_1 &= \epsilon_B \left[ \nabla_d \cdot \nabla_0 - \frac{\tilde{b} \cdot \nabla H_0}{\Omega} \frac{\partial S_1}{\partial \rho_\parallel} \right] \\
\quad + \rho_\parallel \Omega \tilde{b} \cdot \left( \nabla_B S_1 + \mathcal{W} \frac{\partial S_1}{\partial \zeta} \right) &= \delta \tilde{\psi}_{10} + \epsilon_B \delta \tilde{\psi}_{11},
\end{align*}$$

(3.40)

(3.41)

where $\nabla_0$ indicates a gradient operating on perturbation fields, and $\nabla_B$ indicates a gradient operating on quantities associated with the unperturbed magnetic field. In
addition, we have \( \delta \psi = \delta \psi - \langle \delta \psi \rangle \), and

\[
\frac{e}{m} \delta \psi_{10} = \frac{e}{m} \delta \phi - \frac{\Omega}{B} \delta A_0 \cdot \nu,
\]

\[
\frac{e}{m} \delta \psi_{11} = -\frac{\Omega}{B} \left[ \nu_d + \rho_{||} \hat{\Omega} \cdot \left( \nabla \rho_0 + W \frac{\partial \rho_0}{\partial \zeta} \right) \right] \cdot \delta A_0 - \rho_1 \cdot \left( \frac{e}{m} \delta E_0 + \frac{\Omega}{B} \nu \times \delta B_0 \right).
\]

The lowest-order equation

\[
\Omega \frac{\partial S_1}{\partial \zeta} = \frac{e}{m} \delta \psi_{10}
\]

has a solution, denoted \( S_{10} \), given by the expression

\[
S_{10} = \int \frac{d\zeta}{\Omega} \left[ \frac{e}{m} \delta \phi_0 - \frac{\Omega}{B} (\nu \cdot \delta A_0) \right].
\]

A useful limit for this solution uses the leading-order FLR terms of \( \delta \psi_{10} \), given as

\[
\frac{e}{m} \delta \psi_{10} = \rho_0 \cdot \left( \frac{e}{m} \nabla \delta \phi - \frac{\rho_{||} \Omega}{B} \nabla \delta A_{||} + \frac{\mu}{2} \nabla \delta B_{||} \right) - \frac{\Omega^2}{B} \delta A_{||} \cdot \frac{\partial \rho_0}{\partial \zeta}.
\]

When the undetermined gyrophase integration is performed, one obtains

\[
S_{10} = \frac{e}{B} \delta \psi_{10} = -\frac{\partial \rho_0}{\partial \zeta} \cdot \left( \frac{e}{B} \nabla \delta \phi - \frac{\rho_{||} \Omega}{B} \nabla \delta A_{||} + \frac{\mu}{2 \Omega} \nabla \delta B_{||} \right) - \frac{\Omega^2}{B} \delta A_{||} \cdot \rho_0.
\]

A more careful analysis requires the use of the following expansion for \( S_1 \),

\[
S_1 = \sum_{n=0} \sum_{m=0} e^n e^m S_{1nm}.
\]

For \( m = 0 \) and \( n \geq 1 \), we have

\[
\left( \frac{\partial}{\partial t} + \rho_{||} \hat{\Omega} \cdot \nabla \right) S_{1,n-1,0} + \Omega \frac{\partial S_{1,n,0}}{\partial \zeta} = 0,
\]

while for \( n = 0 \) and \( m \geq 1 \), we have

\[
\left[ \nu_d \cdot \nabla_0 S_{1,0,m-1} - \frac{\hat{b} \cdot \nabla H_0}{\Omega} \frac{\partial S_{1,0,m-1}}{\partial \rho_{||}} + \rho_{||} \hat{\Omega} \cdot \left( \nabla B S_{1,0,m-1} + W \frac{\partial S_{1,0,m-1}}{\partial \zeta} \right) \right] + \Omega \frac{\partial S_{1,0,m}}{\partial \zeta} = \delta \psi_{1,m}.
\]

For our purposes, the function \( S_1 \) is taken to be \( S_{10} \), and higher-order terms are neglected.
3.2.3.2. Second-Order Phase-space Gauge Function

The second-order gauge function $S_2$ is determined, to lowest order, by the equation

$$\Omega \frac{\partial S_2}{\partial \zeta} = \frac{e}{m} \delta \bar{\psi}_{20},$$

where the right-hand side depends on the representation used. We need not be more specific than to say that the lowest-order phase-space gauge function is given as

$$S_{20} = \frac{c}{B} \delta \bar{\psi}_{20}. \quad (3.46)$$
3.3 Gyrocenter Hamiltonian Dynamics

The characteristics of the (nonlinear) gyrokinetic Vlasov equation, which will be studied in the next chapter, are the gyrocenter equations of motion. In a self-consistent particle-field kinetic theory, the electromagnetic fields influence the dynamics of the particles (and their distribution in phase space), and in turn, the (distribution of) particles act as the sources that create the (self-consistent) electromagnetic fields. A complete description of gyrocenter dynamics is, therefore, necessary for the derivation of the gyrokinetic formalism. These gyrocenter equations of motion can also be used to study the dynamics of guiding-center particles under the influence of electromagnetic perturbation fields, which satisfy the gyrokinetic ordering.

3.3.1 Hamiltonian Representation

In this subsection, we present the gyrocenter Hamilton’s equations in the Hamiltonian representation derived in 3.2.1.

3.3.1.1. Gyrocenter Hamiltonian

The expression for the gyrocenter Hamiltonian, in the Hamiltonian representation, is given as

\[ H = \frac{1}{2} \rho_0^2 \Omega^2 + \mu B - w + \epsilon \left( \frac{e}{m} \langle \delta \phi_0 \rangle - \frac{\Omega}{B} \langle \mathbf{v} \cdot \delta \mathbf{A}_0 \rangle \right) + \epsilon \epsilon_B \left( \frac{e}{m} \langle \delta_h \psi_{11} \rangle \right) + \frac{e^2}{2 B^2} \left( \frac{\Omega^2}{B^2} \langle | \delta \mathbf{A}_0 |^2 \rangle - \frac{e^2}{m^2 \Omega} \langle \{ \delta \tilde{\psi}_10, \delta \tilde{\psi}_{10} \} \rangle \right), \]

where \( \epsilon/m \langle \delta_h \psi_{11} \rangle \) is given by Eq. (3.20).

3.3.1.2. Gyrocenter Poisson Bracket

The Poisson bracket used in the Hamiltonian representation is that of the guiding-center phase-space Lagrangian, but expressed in terms of gyrocenter phase-space coordinates. The extended Poisson bracket is, therefore, given by Eq. (3.2)

\[ \{ F, G \} = \frac{\Omega}{B} \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \zeta} \right) + \left( \frac{\partial F}{\partial \psi} \frac{\partial G}{\partial t} - \frac{\partial F}{\partial t} \frac{\partial G}{\partial \psi} \right) + \frac{B}{\Omega B_{*}} \left[ \left( \nabla F + \psi \frac{\partial F}{\partial \zeta} \right) \times \left( \nabla G + \psi \frac{\partial G}{\partial \zeta} \right) \right] + \frac{B_{*}}{\Omega B_{*}} \left[ \left( \nabla F + \psi \frac{\partial F}{\partial \rho_{||}} \right) \frac{\partial G}{\partial \rho_{||}} - \left( \nabla G + \psi \frac{\partial G}{\partial \zeta} \right) \frac{\partial F}{\partial \rho_{||}} \right]. \]
3.3.1.3. Gyrocenter Equations of Motion in Hamiltonian Representation

The gyrocenter Hamilton’s equations, in the Hamiltonian representation, are given in terms of the Hamiltonian function, Eq. (3.47), and the unperturbed guiding-center Poisson bracket, Eq. (3.2), as:

\[ \dot{X} = \{X, H\} = \frac{B}{\Omega B^*_{\parallel}} \times \nabla \widehat{H} + \frac{B^*}{\Omega B^*_{\parallel}} \frac{\partial \widehat{H}}{\partial \rho_{\parallel}}, \]

\[ \dot{\rho}_{\parallel} = \{\rho_{\parallel}, H\} = -\frac{B^*}{\Omega B^*_{\parallel}} \cdot \nabla \widehat{H}, \]

\[ \dot{\mu} = \{\mu, H\} = 0, \]

\[ \dot{\zeta} = \{\zeta, H\} = \frac{\Omega}{B} \frac{\partial \widehat{H}}{\partial \mu} + \mathbf{W} \cdot \dot{X}, \]

\[ \dot{t} = \{t, H\} = -\frac{\partial H}{\partial \omega} = 1, \]

\[ \dot{\omega} = \{\omega, H\} = \frac{\partial \widehat{H}}{\partial t}. \]

These equations can also be given in a form which explicitly exhibits the property of phase-space-volume conservation

\[ \Omega B^*_{\parallel} \dot{X} = -\nabla \times (B \widehat{H}) + \frac{\partial (B^* \widehat{H})}{\partial \rho_{\parallel}}, \]

\[ \Omega B^*_{\parallel} \dot{\rho}_{\parallel} = -\nabla \cdot (B^* \widehat{H}), \]

\[ \Omega B^*_{\parallel} \dot{\zeta} = -\frac{\partial}{\partial \mu} \left( \frac{\Omega^2}{B} B^*_{\parallel} \widehat{H} \right) + \nabla \cdot (\mathbf{W} \times B \widehat{H}) + \frac{\partial}{\partial \rho_{\parallel}} (\mathbf{W} \cdot B^* \widehat{H}), \]

\[ \Omega B^*_{\parallel} \dot{t} = \Omega B^*_{\parallel}, \]

\[ \Omega B^*_{\parallel} \dot{\omega} = \frac{\partial (\Omega B^*_{\parallel} \widehat{H})}{\partial t}, \]

where \( B_{\parallel}^* = B + \rho_{\parallel} \widehat{B} \cdot \nabla \times \mathbf{B} - \mu (B/\Omega)^2 \widehat{B} \cdot \nabla \times \mathbf{W} \) is independent of time. Since the gyrocenter equations of motion are independent of the gyrophase \( \zeta \), it is simple to show that the identity known as the Liouville theorem

\[ \frac{\partial}{\partial z^\alpha} (\Omega B^*_{\parallel} \dot{z}^\alpha) = 0 \]

is satisfied.
Finally, the linearized version of Eq. (3.48), which ignores the contribution of $H_{11}$, is given by the expressions

$$\begin{align*}
\dot{X} &= \tilde{b} \left( p_{||} \Omega - \frac{\Omega}{B} \langle \delta A_{||0} \rangle \right) + p_{||} \Omega \left( \frac{\delta B_{\perp 0}}{B} \right) + p_{||} \tilde{b} \times (\tilde{b} \cdot \nabla \delta) \left( p_{||} \Omega - \frac{\Omega}{B} \langle \delta A_{||0} \rangle \right) \\
&\quad + \frac{\tilde{b}}{\Omega} \times \left[ \frac{e}{m} \nabla \langle \delta \psi_0 \rangle + \mu \nabla (B + \delta B_{||}) \right], \\
\dot{p}_{||} &= -\frac{\tilde{b}}{\Omega} \cdot \nabla \left[ \frac{1}{2} \rho_{||}^2 \Omega^2 + \mu (B + \delta B_{||}) + \frac{e}{m} \left( \langle \delta \psi_0 \rangle - \frac{p_{||} \Omega}{c} \langle \delta A_{||0} \rangle \right) \right], \\
\dot{\psi} &= \Omega \left( 1 + \frac{\delta B_{||}}{B} \right) + \frac{c}{2B} \left( \nabla_\perp \delta \psi - \frac{p_{||} \Omega}{c} \nabla_\perp \delta A_{||} \right) + \mathbf{W} \cdot \tilde{b} \left( p_{||} \Omega - \frac{\Omega}{B} \langle \delta A_{||0} \rangle \right) \\
&\quad + \mathbf{W} \cdot \left[ p_{||} \Omega \left( \frac{\delta B_{\perp 0}}{B} \right) + \frac{\tilde{b}}{\Omega} \times \left( \frac{e}{m} \nabla \langle \delta \psi_0 \rangle + \mu \nabla \delta B_{||} \right) \right].
\end{align*}$$

Eq. (3.50)

The parallel component of the gyrocenter drift velocity, $[\tilde{b} \cdot \dot{X} = p_{||} \Omega - \Omega \langle \delta A_{||0} \rangle / B]$, identifies $p_{||} \Omega$ as a parallel momentum coordinate. The perpendicular components of the gyrocenter drift, $\dot{X}_\perp$, are respectively described as the perturbed parallel motion due to the perpendicular magnetic field perturbation (the so-called magnetic flutter term), the total curvature drift, the perturbed $E \times B$ velocity, and the total grad $B$ drift.

The parallel acceleration law, $\dot{p}_{||}$, involves the total mirror force term, and a perturbed effective $E_{||}$ term (without the parallel inductive term).

### 3.3.2 Symplectic Representation

In this subsection, we present the gyrocenter Hamilton's equations, in the symplectic representation derived in 3.2.2.

#### 3.3.2.1. Gyrocenter Hamiltonian

The expression for the gyrocenter Hamiltonian, in the symplectic representation, is given as

$$
H = \frac{1}{2} \rho_{||}^2 \Omega^2 + \mu B - w + e \left( \frac{e}{m} \langle \delta \psi_0 \rangle - \frac{\Omega^2}{B} \langle \delta A_0 \cdot \frac{\partial \rho_0}{\partial \zeta} \rangle \right) + e B \varepsilon \left( \frac{e}{m} \langle \delta \psi_{11} \rangle \right) \\
+ \frac{e^2}{2} \left( \frac{\Omega^2}{B^2} \langle \delta A_{\perp 0} \rangle^2 \right) + \frac{\rho_{||} \Omega^4}{B^3} \left( \delta A_0 \cdot \frac{\partial \rho_0}{\partial \zeta} \right) \frac{\partial \langle \delta A_{||0} \rangle}{\partial \mu} - \frac{e^2}{m^2 \Omega} \langle \{ \delta \tilde{\psi}_{10}, \delta \tilde{\psi}_{10} \} \rangle,
$$

where $e/m \langle \delta \psi_{11} \rangle$ is given by Eq. (3.35).
3.3.2.2. Lagrange and Poisson Brackets

Because the symplectic structure of the gyrocenter phase-space Lagrangian is different than the guiding-center phase-space Lagrangian, we must evaluate the new gyrocenter Poisson brackets.

Let the gyrocenter symplectic structure, in the symplectic representation, be given, up to order \(O(\epsilon)\) and \(O(\epsilon_B)\), by the expression

\[
\Gamma = \frac{\Omega}{B} A^* \cdot dX + \frac{\mu B}{\Omega} d\zeta - \omega dt,
\]

where \(A^* = A + \rho_B B - \mu(B/\Omega)^2 W + (\delta A_{||0}) \hat{b}\).

The derivation of Poisson and Lagrange brackets is described in subsection 1.2.3. The Lagrange two-form \(\omega\), using the gyrocenter phase-space coordinates, is given as

\[
\omega = d\Gamma = \hat{\omega}_{ij} dz^i \wedge dz^j,
\]

where each component \(\hat{\omega}_{ij}\) defines the Lagrange bracket \([z^i, z^j]\),

\[
\hat{\omega}_{ij} = \left[ z^i, z^j \right] = \frac{\partial \hat{\Gamma}_j}{\partial z^i} - \frac{\partial \hat{\Gamma}_i}{\partial z^j}.
\]

As explained in 1.2.3., the Lagrange two-form determines the Poisson-bracket structure, which automatically satisfies the Jacobi identity. A simple application of the inversion algorithm of Littlejohn [1981], gives the following expression for the Poisson bracket of two phase-space functions \(F\) and \(G\)

\[
\{F, G\} = \frac{\Omega}{B} \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \zeta} \right) - \frac{B}{\Omega B^*} \left[ \left( \nabla F + W \frac{\partial F}{\partial \zeta} \right) \times \left( \nabla G + W \frac{\partial G}{\partial \zeta} \right) \right]
\]

\[
+ \frac{B^*}{\Omega B^*} \left[ \left( \nabla F - \hat{b} \frac{\Omega}{B} \frac{\partial F}{\partial \omega} \frac{\partial (\delta A_{||0})}{\partial t} + W^* \frac{\partial F}{\partial \zeta} \right) \frac{\partial G}{\partial \rho} \right]
\]

\[
- \left( \nabla G - \hat{b} \frac{\Omega}{B} \frac{\partial G}{\partial \omega} \frac{\partial (\delta A_{||0})}{\partial t} + W^* \frac{\partial G}{\partial \zeta} \right) \frac{\partial F}{\partial \rho} \right]
\]

\[
+ \left( \frac{\partial F}{\partial \omega} \frac{\partial G}{\partial t} - \frac{\partial F}{\partial t} \frac{\partial G}{\partial \omega} \right),
\]

where \(B^* = \hat{b} \cdot B^*\) and

\[
B^* = B + \nabla (\delta A_{||0}) \times \hat{b} + (\rho_B \nabla \times B + (\delta A_{||0}) \nabla \times \hat{b}),
\]

\[
W^* = W - \frac{\hat{b}}{B^2} \frac{\partial (\delta A_{||0})}{\partial \mu} \simeq R + \frac{\hat{b}}{2B} (\hat{b} \nabla \times B - \nabla^2 (\delta A_{||})).
\]
3.3. Gyrocenter Hamiltonian Dynamics

Note that $W^*$ involves both the unperturbed parallel current density $(\vec{b} \cdot \nabla \times \vec{B})$ and the perturbed parallel current density $(-\nabla^2 \delta A_{||})$.

3.3.2.3. Gyrocenter Equations of Motion in the Symplectic Representation

Finally, the gyrocenter Hamilton’s equations, in the symplectic representation, are given in terms of the Hamiltonian, Eq. (3.51), and the gyrocenter Poisson bracket, Eq. (3.53), as

$$
\dot{X} = \{X, H\} = \frac{B}{\Omega B_{||}} \times \nabla \vec{H} + \frac{B^* \partial \vec{H}}{\Omega B_{||}} \partial \rho_{||},
$$

$$
\dot{\rho}_{||} = \{\rho_{||}, H\} = -\frac{B^*}{\Omega B_{||}} \left( \nabla \vec{H} + \frac{\Omega}{B} \frac{\partial (\delta A_{||0})}{\partial t} \right),
$$

$$
\dot{\mu} = \{\mu, H\} = 0,
$$

$$
\dot{\zeta} = \{\zeta, H\} = \frac{B_0}{\Omega B_{||}} \frac{\partial \vec{H}}{\partial \mu} + W^* \cdot \dot{X},
$$

$$
i = \{i, H\} = 1,
$$

$$
\dot{w} = \{w, H\} = \frac{\partial \vec{H}}{\partial t} - \rho_{||} \frac{\Omega^2 \partial (\delta A_{||0})}{B}.
$$

(3.54)

Again, these equations can be given in phase-space conserving form as

$$
\Omega B_{||}^* \dot{X} = -\nabla \times (B^* \vec{H}) + \frac{\partial (B^* \vec{H})}{\partial \rho_{||}},
$$

$$
\Omega B_{||}^* \dot{\rho}_{||} = -\nabla \cdot (B^* \vec{H}) - \frac{\Omega B_{||}^* \partial (\delta A_{||0})}{B},
$$

$$
\Omega B_{||}^* \dot{\zeta} = \frac{\partial}{\partial \mu} \left( \frac{\Omega^2}{B} B_{||} \vec{H} \right) + \nabla \cdot (W^* \times B \vec{H}) + \frac{\partial}{\partial \rho_{||}} (W^* \cdot B^* \vec{H}),
$$

$$
\Omega B_{||}^* \dot{i} = \Omega B_{||}^*,
$$

$$
\Omega B_{||}^* \dot{w} = \Omega B_{||}^* \left( \frac{\partial \vec{H}}{\partial t} - \rho_{||} \frac{\Omega^2 \partial (\delta A_{||0})}{B} \right).
$$

(3.55)

It is simple to show that these gyrocenter equations of motion satisfy the Liouville theorem

$$
\frac{\partial}{\partial x^\alpha} (\Omega B_{||}^* \dot{x}^\alpha) = 0.
$$

The linearized version of Eq. (3.54), which ignores the contribution of $H_{11}$, is given by the following expressions

$$
\dot{X} = \rho_{||} \Omega \vec{b} + \rho_{||} \delta \times (\vec{b} \cdot \nabla \vec{b}) \left( \rho_{||} \Omega + \frac{\Omega}{B} (\delta A_{||0}) \right) + \rho_{||} \Omega \left( \frac{\delta B_{10}}{B} \right)
$$
\[ + \frac{\mathbf{b}}{\Omega} \times \left[ \frac{e}{m} \nabla(\delta \phi_0) + \mu \nabla(B + \delta B) \right], \]

\[ \dot{\rho}_\parallel = - \frac{1}{\Omega} \left( \frac{\mathbf{b} + \left( \frac{\delta B_{10}}{B} \right)}{B} \right) \cdot \nabla \left[ \frac{1}{2} \rho_\parallel^2 \Omega^2 + \mu(B + \delta B) + \frac{e}{m}(\delta \phi_0) \right] - \frac{1}{B} \frac{\partial(\delta A_{10})}{\partial t}, \]

\[ \dot{\zeta} = \Omega \left( 1 + \frac{\delta B_\parallel}{B} \right) + \frac{c}{2B} \nabla^2 \delta \phi + \rho_\parallel \Omega \mathbf{b} \cdot \mathbf{W}^* \]

\[ + \mathbf{W} \cdot \left[ \rho_\parallel \Omega \left( \frac{\delta B_{10}}{B} \right) + \frac{\mathbf{b}}{\Omega} \times \left( \frac{e}{m} \nabla(\delta \phi_0) + \mu \nabla \delta B_\parallel \right) \right]. \tag{3.56} \]

We note that the parallel component of the gyrocenter drift velocity, \( \mathbf{b} \cdot \mathbf{X} = \rho_\parallel \Omega \), now has the usual interpretation of a velocity. The perpendicular components of the gyrocenter drift velocity are the same as before.

Finally, the parallel acceleration law, \( \dot{\rho}_\parallel \), contains the parallel induction term and also contains the parallel gradient along the total magnetic field \( (B + \delta B) \).
Chapter 4

Gyrokinetic Maxwell-Vlasov System

Nonlinear gyrokinetic equations were first developed by Frieman and Chen [1982] who used a gyrophase averaging method to derive a nonlinear gyrokinetic Vlasov equation (for the non-adiabatic part of the perturbed distribution function) describing (fully) electromagnetic perturbations of a plasma with general magnetic configuration. The use of nonlinear gyrokinetic (Poisson-Vlasov) equations for particle simulation studies was first proposed by Lee [1983, 1987]. A rigorous Hamiltonian formulation of the gyrokinetic Poisson-Vlasov system was later derived by Dubin et al. [1983] with the help of the Hamiltonian Lie perturbation method. These equations described electrostatic perturbations of a plasma in a slab geometry. For the first time, their work also gave the gyrokinetic Poisson equation, and derived the electrostatic gyrökinetic energy invariant for the gyrokinetic Poisson–Vlasov system. The Hamiltonian Lie perturbation method was again used by Yang and Choi [1985] to derive the gyrokinetic Poisson–Vlasov system for electrostatic perturbations of a plasma with a general magnetic configuration. Unfortunately, their equations were not gyrogauge invariant because of the appearance of the vector $R$, as discussed in 2.1.3.2. This difficulty was later eliminated by Hagan and Frieman [1985], who used a modified version of the Hamiltonian Lie perturbation method, and by Hahm [1988], who used the Phase-space Lagrangian Lie perturbation method. Electromagnetic perturbations were considered by Hahm, Lee, and Brizard [1988], who used the Hamiltonian Lie perturbation method to derive a gyrokinetic (finite-$\beta$) Maxwell–Vlasov system describing electrostatic and perpendicular magnetic perturbations of a plasma in a
slab geometry. Their work presented the gyrokinetic Poisson and parallel Ampere equations, and the (finite-$\beta$) electromagnetic gyrokinetic energy invariant. Finally, Brizard [1989a] used the Phase-space Lagrangian Lie perturbation method to derive the full gyrokinetic Maxwell–Vlasov system for electromagnetic perturbations of a plasma with general magnetic configuration. The proof of the adiabatic invariance of the gyrokinetic energy was later given by Brizard [1989b].

The nonlinear gyrokinetic Maxwell–Vlasov system, derived in Brizard [1989a], is presented. We show that this system possesses a gyrokinetic energy (adiabatic) invariant (Brizard [1989b]). In particular, we show that the truncated Maxwell–Vlasov system has an exact energy invariant, which is just the appropriately truncated expression for the gyrokinetic energy invariant.

Finally, two additional topics in the development of the gyrokinetic formalism are also presented. First, we present the general procedure involved in the derivation of a gyrokinetic collision operator. Only general forms for the original collision operator are considered, and specific examples are left for future research. Second, we present a gyrokinetic formalism for arbitrary frequencies, derived with the help of the phase-space Lagrangian Lie perturbation method. The high-frequency gyrokinetic formalism was originally derived by Chen and Tsai [1983a,b] with the help of the gyrophase average method. Our use of the phase-space Lagrangian method provides us with a simple and rigorous derivation of the high-frequency formalism, especially in nonuniform equilibrium magnetic fields.

4.1 Gyrokinetic Vlasov Equation

There are two generic approaches to deriving a reduced Vlasov equation. The first approach involves the use of the averaging principle on the original Vlasov equation. The characteristics of the resulting equation then describe the reduced single-particle motion. The second approach involves the transformation from the original phase space to a reduced phase space, where the motion along the reduced phase-space trajectory is now devoid of fast oscillations. Schematically, the averaging approach is represented by the following route

\[
\left\{ \text{Vlasov Equation} \right\} \rightarrow \left\{ \text{Reduced Vlasov Equation} \right\} \rightarrow \left\{ \text{Reduced Particle Dynamics} \right\},
\]
and the transformation approach is represented by the inverse route

\[
\{ \text{Particle Dynamics} \} \rightarrow \left\{ \begin{array}{c}
\text{Reduced} \\
\text{Particle Dynamics}
\end{array} \right\} \rightarrow \left\{ \begin{array}{c}
\text{Reduced} \\
\text{Vlasov Equation}
\end{array} \right\}.
\]

As was pointed out by Littlejohn [1984d], the averaging approach is rather awkward to carry out compared with the transformation approach. In addition, the reduction of the particle dynamics can be performed with the help of the powerful Hamiltonian perturbation techniques.

In chapter 3, we have carried the first part of the transformation approach in two steps: (1) we have eliminated from the unperturbed problem the fast oscillations due to the gyromotion by transforming to the guiding-center phase space; and (2) we have eliminated from the perturbed guiding-center problem the fast oscillations due to the perturbation fields by transforming to the gyrocenter phase space.

The derivation of the gyrokinetic Vlasov equation, the second part of the transformation approach, will similarly be carried out in two steps.

### 4.1.1 Guiding-center Vlasov Equation

The Vlasov equation can generally be written in the form

\[
\frac{\partial F}{\partial t} + X_H(F) = 0,
\]

where \(F\) is the distribution function in the original phase space, and the vector field \(X_H\) is a Hamiltonian vector field in regular phase space, i.e., \(X_H(F) = \{F, H\}\), and \(\{ , \} \) is a regular Poisson bracket. Once a phase-space transformation \(T\) has been found, which allows the expression of the desired reduced dynamics, it is our purpose to apply this transformation onto the Vlasov vector field \(X\). The mathematical operation that carries out such an induced transformation was given in I.2.1.4, and will be discussed presently.

#### 4.1.1.1 Lie Transform of Vector Fields

The induced transformation of a vector field \(X\), when the coordinate transformation is \(T_G\), is given in terms of the push-forward (or tangent map) \(T_G^*\) by the formula \(X_G = T_G^*X\). This tangent map can be represented by the exponential of a Lie derivative as follows. The Lie derivative of the vector field \(X\) with respect to the
vector field $G$ was defined in Eq. (1.30) as

$$L_G X = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (X - T^\epsilon_{G^*} X) = [G, X],$$

where $[,]$ denotes the Lie bracket (or commutator) of the two vector fields $G$ and $X$. We easily find that $T^\epsilon_{G^*} = \exp(-\epsilon L_G)$, and the expression for $X_G$ can be written as a power series of commutators.

4.1.1.2. Vlasov Vector Field

Let us consider the case where the vector field $X$ is the Vlasov vector field which, in local-magnetic phase-space (defined in 2.1.1.2.), is given as

$$X = X^i \frac{\partial}{\partial z^i},$$

(4.1)

where $z^i = (x, v_{\parallel}, \mu_0, \theta)$ with $\mu_0 = v_{\perp}^2/2B$, and

$$X^x = v, \quad X^{v_{\parallel}} = -\hat{b} \cdot \nabla H_0 + v_{\parallel} \hat{b} \cdot \nabla v_{\parallel} + \frac{v_1^2}{2} (\hat{\Omega} + \hat{\theta}) : \nabla \hat{b},$$

$$X^{\mu_0} = \frac{\Omega}{B} v_{\perp} \times \hat{b} \cdot v_d - v_{\parallel} \mu_0 (\hat{\Omega} + \hat{\theta}) : \nabla \hat{b},$$

$$X^\theta = \Omega \left(1 + \frac{v_{\parallel}}{\Omega} \hat{b} \cdot \mathbf{W} \right) + \frac{v_{\parallel}}{2} (\hat{\Omega} + \hat{\theta}) : \nabla \hat{b} + v_{\perp} (\hat{\Omega} - \frac{\hat{\theta}}{2} \nabla \ln B) - \hat{\Omega} v_d \frac{\Omega v_d}{v_{\perp}}.$$

By simply performing the gyrophase averaging operation on $X$, we do not obtain the correct expression [to order $O(\epsilon^2)$] for the guiding-center Vlasov vector field $X_{gc}$. We must, therefore, proceed with the guiding-center transformation $T^\sigma_{GC} = \cdots T_2 T_1$, where $T_n = \exp(\epsilon^n R_{n,gc})$. (The generating vector fields $G_{1gc}$ and $G_{2gc}$ were derived earlier in 2.1.3.)

When the guiding-center tangent map $T^\sigma_{GC^*}$ is applied to the vector field $X$, given by Eq. (4.1), we obtain the following power series

$$X_{gc} = X - \epsilon [G_{1gc}, X] + \epsilon^2 \left( \frac{1}{2} [G_{1gc}, [G_{1gc}, X]] - [G_{2gc}, X] \right) + O(\epsilon^3).$$

(4.2)

We refer to the scalar field $X^i$ as the $i$th component of the vector field $X$. Consequently, the $i$th component of the vector field $[G, X]$ is

$$G^k X^i_k - X^i G^i_j \equiv G(X^i) - X(G^i),$$
where the notation $\partial G^h/\partial z^i \equiv G^h_{,i}$ and the convention of summing repeated indices are used. The $i$th component of $[G, [G, X]]$ is

$$G(G(X^i)) + X(G(G^i)) - 2G(X^i).$$

Since we wish to retain terms of order $O(\epsilon_B)$ in $X_{\text{gc}}$ (i.e., terms associated with the magnetic nonuniformity), we must consider terms of order $O(\epsilon^2)$ only for the spatial component of the expression defining $X_{\text{gc}}$, given in Eq. (4.2). The other components are, in fact, obtained by simply gyrophase averaging the expressions for $X^\parallel$, $X^\mu$, and $X^\theta$. The expression for the spatial component is given as

$$v - G_{1gc}(v) - X(\rho_0) + \frac{1}{2}G^2_{1gc}(v) - \frac{1}{2}X(G_{1gc}(\rho_0)) + G_{1gc}(X(\rho_0)) - G_{2gc}(v) + X(G_{2gc}).$$

The manipulations involved here are simple, and we finally obtain the guiding-center Vlasov vector field

$$X_{\text{gc}} = (\rho_\parallel \Omega \vec{b} + \vec{v}_d) \cdot \frac{\partial}{\partial X} - \frac{\vec{b} \cdot \nabla \tilde{H}_0}{\Omega} \frac{\partial}{\partial \rho_\parallel} + \Omega(1 + \rho_\parallel \vec{b} \cdot \vec{W}) \frac{\partial}{\partial \zeta}, \quad (4.3)$$

which is correct to order $O(\epsilon_B)$, and where $(X, \rho_\parallel, \mu, \zeta)$ are the guiding-center coordinates.

In phase-space Lagrangian guiding-center theory (see 2.1.3.), the transformation from local-magnetic phase space to guiding-center phase space induces simultaneously a transformation of the Hamiltonian function and the Poisson-bracket structure. Hence, we have

$$T_{\text{GC}} \{ , H \} = \{ , H_{\text{gc}} \}_{\text{gc}},$$

where the guiding-center Hamiltonian is $T_{\text{GC}} \star H = (T_{\text{GC}}^*)^{-1} H = H_{\text{gc}}$, and the guiding-center Poisson bracket $\{ , \}_{\text{gc}}$ is given in Eq. (2.59). It is easy to recognize that $X_{\text{gc}} = T_{\text{GC}} \star X = \{ , H_{\text{gc}} \}_{\text{gc}}$, as expected.

4.1.1.3. Guiding-center Vlasov Equation

Finally, the guiding-center Vlasov equation is obtained from the original Vlasov equation as follows. Let the physical phase-space distribution function $f(r, v, t)$ evolve according to the Vlasov equation

$$\frac{\partial f}{\partial t} + \{ f, H \} = 0.$$
The guiding-center transformation $T_{GC}$ induces a transformation (pull-back) on the distribution function $f$, so that the guiding-center phase-space distribution function $F$ is given by the (pull-back) formula

$$F = (T_{GC}^*)^{-1} f = T_{GC}^* f.$$ 

The guiding-center Vlasov equation, therefore, becomes

$$T_{GC}^* \left( \frac{\partial f}{\partial t} + \{f, H\} \right) = \frac{\partial F}{\partial t} + \{F, H_{gv}\}_{gv} = 0. \quad (4.4)$$

It is simple to show by induction (Littlejohn [1984d]) that the guiding-center distribution function $F$ is independent of the gyrophase. This is because the guiding-center Hamiltonian and Poisson bracket are gyrophase independent. The advantage of expressing the Vlasov equation in terms of guiding-center phase-space coordinates was first appreciated by Catto [1978], who used the expression for the magnetic moment $\mu = \mu_0 + \varepsilon_B \mu_1$ as one of the dynamical variables. Later, Catto et al. [1981] also took into account the Baños parallel drift correction in their set of guiding-center coordinates.

### 4.1.2 Linear Gyrokinetic Vlasov Equation

The application of the gyrocenter transformation $T_{GY}$ leads to similar results. Let the gyrocenter distribution function $\hat{F}$ be obtained from the guiding-center distribution function $F$ by the (pull-back) formula $\hat{F} = (T_{GY}^*)^{-1} F = T_{GY}^* F$. The gyrokinetic Vlasov equation is then given by the expression

$$T_{GY}^* \{F, H_0\} = \{\hat{F}, H\}_{gy} = 0 = \frac{\partial \hat{F}}{\partial t} + \{\hat{F}, \hat{H}\}_{gy} \quad (4.5)$$

where $H_0$ denotes the extended guiding-center Hamiltonian, and the last expression on the right-hand side of Eq. (4.5) depends on the representation used. In this chapter, the Hamiltonian representation will be used, for simplicity.

The linear gyrokinetic equation is given, in bracket form, by the expression

$$\frac{\partial \hat{F}_1}{\partial t} + \{\hat{F}_1, \hat{H}_0\} = - \{\hat{F}_0, \hat{H}_1\}, \quad (4.6)$$

where the unperturbed (perturbed) distribution function is denoted $F_0$ ($F_1$) and the Poisson bracket is the guiding-center Poisson bracket expressed in terms of gyrocenter
coordinates. (It is also assumed that the unperturbed distribution function $F_0$ satisfies the equilibrium equation $\{F_0, H_0\} = 0$.) The perturbed Hamiltonian is given by Eq. (3.18)

$$\tilde{H}_1 = \frac{e}{m} (\delta \phi_0) - \frac{\Omega}{B} (\mathbf{v} \cdot \delta \mathbf{A}_0),$$

and the unperturbed Hamiltonian $H_0$ is given by the guiding-center Hamiltonian, expressed in gyrocenter coordinates.

Instead of using the parallel-momentum coordinate $\rho_\parallel$, a more conventional notation makes use of the (unperturbed) kinetic energy $\epsilon = \tilde{H}_0$. The linear gyrokinetic equation then becomes

$$\frac{\partial F_1}{\partial t} + (\rho_\parallel \tilde{\Omega} + \mathbf{v}_d) \cdot \nabla F_1 = \frac{\tilde{b}}{\Omega} \times \nabla F_0 \cdot \nabla \tilde{H}_1 + \frac{\partial F_0}{\partial \epsilon} (\rho_\parallel \tilde{\Omega} + \mathbf{v}_d) \cdot \nabla \tilde{H}_1,$$

where we have taken into account the equilibrium condition $\{F_0, H_0\} = (\rho_\parallel \tilde{\Omega} + \mathbf{v}_d) \cdot \nabla F_0$.

Finally, the linear gyrokinetic equation for the non-adiabatic part of $F_1$ is obtained as follows. The non-adiabatic part $\delta f$ is defined as

$$\delta f = F_1 - H_1 \frac{\partial F_0}{\partial \epsilon}.$$

When substituted in the equation above, we easily obtain

$$\left[ \frac{\partial}{\partial t} + (\rho_\parallel \tilde{\Omega} + \mathbf{v}_d) \cdot \nabla \right] \delta f = \left( \frac{\tilde{b}}{\Omega} \times \nabla \perp F_0 \cdot \nabla \perp - \frac{\partial F_0}{\partial \epsilon} \frac{\partial}{\partial t} \right) H_1. \quad (4.7)$$

If we were to use the eikonal representation for the perturbation fields in Eq. (4.7), the resulting equation would be the equation derived by Catto, Tang, and Baldwin [1981] (who used the gyrophase averaging method), Antonsen and Lane [1980], and Littlejohn [1984a] (who used the Hamiltonian Lie perturbation method).

### 4.1.3 Nonlinear Gyrokinetic Vlasov Equation

In the Hamiltonian representation, the nonlinear gyrokinetic Vlasov equation is given by the expression

$$\frac{\partial F}{\partial t} + (V\tilde{b} + \dot{X}_\perp) \cdot \nabla F + \dot{\rho}_\parallel \frac{\partial F}{\partial \rho_\parallel} = 0,$$  \quad (4.8)
where the parallel drift velocity is denoted $V$, the perpendicular drift velocity is denoted $\dot{X}_\perp$, and the parallel acceleration law is denoted $\dot{\rho}_\parallel$.

The parallel drift velocity $V$ is defined as

$$V = \dot{b} \cdot \dot{X} = \frac{1}{\Omega} \frac{\partial \hat{H}}{\partial \rho_\parallel} = \rho_\parallel \Omega - \frac{\Omega}{B} \langle \delta A_{||0} \rangle + \frac{e}{mB} \langle \{\delta \tilde{\Psi}_{10}, \delta \tilde{A}_{||0}) \rangle_0,$$

which displays the linear perturbation given in Eq. (3.49) and its nonlinear correction [obtained from Eq. (3.44)]

$$-\frac{\Omega}{B} \delta A_{||0} \cdot \frac{\delta B_{\perp}}{B} - \frac{\delta B_{||0}}{B} \left( \frac{c_0}{B} \times \nabla_\perp \delta \phi + \frac{\mu_0}{\Omega} \times \nabla \delta B_\parallel + \rho_\parallel \Omega \frac{\delta B_{||0}}{B} \right).$$

The perpendicular drift velocity is defined as

$$\dot{X}_\perp = \frac{B_\parallel}{B} \left[ v_d + (V - \rho_\parallel \Omega) \left( \frac{\nabla \times B}{B} \right) \right] + \frac{\dot{b}}{\Omega} \times \nabla (\tilde{H} - \tilde{H}_0),$$

where $(\nabla \times B)_\perp$ is the unperturbed diamagnetic current density, and the last term

$$\frac{\dot{b}}{\Omega} \times \nabla \left( \frac{e}{m} \langle \delta \phi_0 \rangle - \frac{\Omega}{B} \langle \nabla \cdot \delta A_0 \rangle - \frac{e^2}{2m^2\Omega} \langle \{\delta \tilde{\Psi}_{10}, \delta \tilde{\Psi}_{10} \} \rangle \right)$$

includes the perturbed $E \times B$ drift velocity, the perturbed magnetic flutter term $(\delta B_{||0})$, and the perturbed grad$B$ drift, with their associated nonlinear corrections.

The parallel acceleration law is defined as

$$\dot{\rho}_\parallel = -\frac{B_\parallel^*}{\Omega B_{||0}^*} \cdot \nabla \tilde{H},$$

which again identifies $\rho_\parallel$ as a momentum quantity.

Finally, due to the Liouville phase-space volume-preserving property of the gyro-center Hamiltonian flow

$$\frac{\partial (\Omega B_{||0}^*)}{\partial t} + \nabla \cdot (\Omega B_{||0}^* \dot{X}) + \frac{\partial}{\partial \rho_\parallel} (\Omega B_{||0}^* \dot{\rho}_\parallel) = 0,$$

the gyrokinetic Vlasov equation, Eq. (4.8), can also be given by the expression

$$\frac{\partial}{\partial t} (\Omega B_{||0}^* F) + \nabla \cdot (\Omega B_{||0}^* \dot{X} F) + \frac{\partial}{\partial \rho_\parallel} (\Omega B_{||0}^* \dot{\rho}_\parallel F) = 0. \quad (4.9)$$

As was shown in Eq. (3.54), the gyrokinetic Vlasov equation, in the symplectic representation, would have a similar form. This expression will be particularly useful in chapter VI where one considers moments of Eq. (4.9) [i.e., $\int 2\pi d\mu d\rho_\parallel (\cdot \cdot \cdot)$] since integration by parts is facilitated by the divergence form of Eq. (4.9).
4.2 Gyrokinetic Maxwell’s Equations

Once the successive transformations from physical phase space to guiding-center phase space and from guiding-center phase space to gyrocenter phase space have been derived, it is possible to express the density (current) integrals present in the set of Maxwell’s equations in terms of the gyrocenter distribution function and the gyrocenter coordinates.

This procedure was used by Lee [1983, 1987] and Dubin et al. [1983] for the case of Poisson’s equation in a slab geometry, Yang and Choi [1985] and Hahm [1988] for the case of Poisson’s equation in a general geometry, and Hahm, Lee, and Brizard [1988] for the case of the finite-β Maxwell’s equations in a slab geometry. The procedure was finally fully generalized for the case of the complete set of Maxwell’s equations in a general geometry by Brizard [1989a,b].

4.2.1 Guiding-center and Gyrocenter Transformations

4.2.1.1. Maxwell’s Equations in Gyrokinetic Limit

The set of Maxwell’s equations relate perturbation potentials ($\delta\phi, \delta A$) to density (current) integrals involving the distribution function alone, as in the case of Poisson’s equation

$$-\nabla^2 \delta \phi(r, t) = 4\pi \sum e \int d^3v \, f(r, v, t) + \frac{1}{c} \frac{\partial}{\partial t} \left[ \nabla \cdot \delta A(r, t) \right],$$

or with some component of the particle velocity, as in the case of Ampere’s law

$$-\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \delta A(r, t) =$$

$$-\frac{4\pi}{c} \left[ \mathbf{J}(r) - \sum e \int d^3v \, v f(r, v, t) \right] - \nabla \left[ \frac{1}{c} \frac{\partial \delta \phi(r, t)}{\partial t} + \nabla \cdot \delta A(r, t) \right].$$

Here we have assumed equilibrium quasi-neutrality in Poisson’s equation, and the equilibrium current density in Ampere’s law is denoted $\mathbf{J}(r)$.

The substitution of the Lorentz gauge condition

$$\frac{1}{c} \frac{\partial \delta \phi}{\partial t} + \nabla \cdot \delta A = 0$$
into Maxwell's equations leads to the well-known inhomogeneous wave equations for \( \delta \Phi \) and \( \delta A \). For our purposes, however, we need to apply the gyrokinetic ordering (1.1.2.) to the set of Maxwell's equations. Hence, the gauge condition becomes

\[
\nabla_\perp \cdot \delta A_\perp = \mathcal{O}(\varepsilon),
\]

and Maxwell's equations, in the gyrokinetic limit, are given as

\[
\begin{align*}
\nabla_\perp^2 \delta \Phi(r,t) &= -4\pi \sum e \int d^3 v \ f(r,v,t), \\
\nabla_\perp^2 \delta A(r,t) &= \frac{4\pi}{c} \left[ J(r) - \sum e \int d^3 v \ v \ f(r,v,t) \right].
\end{align*}
\]

(4.10)

(4.11)

4.2.1.2. General Transformation Procedure

Consider a representative Maxwell's equation to be given as

\[
\nabla_\perp^2 \chi(r) = \int d^3 v \ g(r,v) \ f(r,v) = \int d^5 z \ \delta^3(x-r) \ g(z) f(z),
\]

(4.12)

where \( z = (x,v) \) is a point in physical phase space, and \( d^5 z = d^3 x d^3 v \) is the volume element in that phase space. The expression \( f(z) d^5 z \) represents the phase-space density distribution, with \( f \) being the distribution function, and \( \delta^3(x-r) g(z) \) is a particular kernel associated with a member of the set of Maxwell's equations. The use of the delta function facilitates the phase-space transformation.

The transformation from physical phase-space coordinates \( z \) to guiding-center phase-space coordinates \( Z = T_{GC} z \) induces the following transformation on Eq. (4.12):

\[
\nabla_\perp^2 \chi(r) = \int \Omega B^*_\parallel d^6 Z \ F(Z) \ [(T_{GC}^*)^{-1} g](Z) \ \delta^3(T_{GC}^{-1} X - r),
\]

(4.13)

where \( \Omega B\parallel^* \) is the Jacobian of the transformation \( T_{GC} \) (we indicated earlier that \( T_{GC} \) is not canonical), the function \( F = (T_{GC}^*)^{-1} f \) is the guiding-center distribution function, \( d^6 Z = d^3 X \ d\rho_\parallel d\mu d\zeta \), and \( T_{GC}^{-1} X = X + \rho_0 + \mathcal{O}(\varepsilon_B) \).

We note that the variable \( Z \) appearing on the right-hand side of Eq. (4.13) acts as a dummy variable. We can therefore substitute the gyrocenter coordinates \( \bar{Z} = T_{GY} Z \), and use the relation \( F = T_{GY}^* \bar{F} \) to finally obtain

\[
\nabla_\perp^2 \chi(r) = \int \Omega B^*_\parallel d^6 \bar{Z} \ [(T_{GC}^*)^{-1} g](\bar{Z}) \ [(T_{GY}^* \bar{F})(\bar{Z})] \ \delta^3(T_{GC}^{-1} \bar{X} - r).
\]

(4.14)

Because the transformation \( T_{GY} \) is canonical, the gyrocenter phase-space volume element remains \( \Omega B^*_\parallel d^6 \bar{Z} \).
4.2.2 Gyrokinetic Maxwell’s Equations

4.2.2.1. Gyrokinetic Poisson’s Equation

The density integral for Poisson’s equation involves the zeroth velocity moment, i.e., $g(z) \equiv 1$ in Eq. (4.12). Its transformation under the operation $(T^{*}_{GC})^{-1}$ is trivial, and we obtain the gyrokinetic Poisson’s equation

$$\nabla_{\perp}^{2} \delta \phi(r,t) = -4\pi \sum e \int \Omega B_{\parallel} d^{6}Z \left[ T^{*}_{GY} F(Z,t) \right] \delta^{3}(T^{-1}_{GC}X - r), \quad (4.15)$$

where we have omitted the bar notation.

The term $T^{*}_{GY} F$ appearing on the right-hand side of Eq. (4.15) is given by the following expression (in the Hamiltonian representation)

$$T^{*}_{GY} F = F + \epsilon \left[ \left( \frac{\delta A_{0}}{B} + \frac{c}{\Omega B} \nabla \delta \tilde{\psi}_{10} \right) \cdot \hat{B} \times \nabla F + \frac{\delta A_{\parallel 0}}{B} \frac{\partial F}{\partial \rho_{\parallel}} \right. \quad (4.16)$$

$$+ \left( \frac{\Omega^{2}}{B} \frac{\partial \rho_{0}}{\partial \zeta} + \frac{e}{m} \delta \tilde{\psi}_{10} \right) \frac{1}{B} \frac{\partial F}{\partial \mu} \right] + \mathcal{O}(\epsilon^{2}).$$

A similar expression for the symplectic representation replaces $\delta A_{\parallel 0}$ by $\delta \tilde{A}_{\parallel 0}$ in Eq. (4.16).

4.2.2.2. Gyrokinetic Ampere’s Law

The current integral for Ampere’s law involves the first velocity moment, i.e., $g(z) \equiv \nu$ in Eq. (4.12). Its transformation under the operation $(T^{*}_{GC})^{-1}$ gives

$$(T^{-1}_{GC}) v \dot{=} v_{gc} = v + v_{d} + \tilde{v}_{d} + \mathcal{O}(\epsilon_{B}^{2}),$$

where $v_{d}$ is the well-known magnetic drift velocity, and

$$\tilde{v}_{d} = \rho_{\parallel} \Omega \hat{b}_{\parallel} \left( \nabla \rho_{0} + W \frac{\partial \rho_{0}}{\partial \zeta} \right) + \Omega \frac{\partial \rho_{1}}{\partial \zeta}.$$

The gyrokinetic Ampere’s law is, therefore, given as

$$\nabla_{\perp}^{2} \delta A(r,t) = \frac{4\pi}{c} \left\{ J - \sum e \int \Omega B_{\parallel} d^{6}Z \left[ v_{gc}(Z) \right] \left[ T^{*}_{GY} F(Z,t) \right] \delta^{3}(T^{-1}_{GC}X - r) \right\}. \quad (4.17)$$

The expressions given by Eqs. (4.15)–(4.17) are necessary for a self-consistent description of gyrokinetics. In addition, in chapter 6, these expressions will play an important role in the derivation of a closed set of energy-conserving gyrofluid equations.
4.3 Gyrokinetic Energy Conservation

An integral expression is presented for the gyrokinetic total energy of a magnetized plasma, with a general magnetic field configuration perturbed by fully electromagnetic fields. It is shown that the gyrokinetic energy is conserved by the gyrokinetic Hamiltonian flow to all orders in perturbed fields. An explicit demonstration that a gyrokinetic Hamiltonian containing quadratic nonlinearities preserves the gyrokinetic energy up to third order is given. The Poisson-bracket formulation greatly facilitates this demonstration with the help of the Jacobi identity and other properties of the Poisson brackets. (The work presented here is an expanded version of a paper by Brizard [1989b].)

The issue of the preservation of the gyrokinetic total energy by the gyrokinetic Hamiltonian flow was discussed by various authors, but due to the tediousness of the calculations involved mixed results have been obtained. See Dubin et al. [1983], Hagan and Frieman [1985], Hagan [1986], and Hahm et al. [1988]. By using the algebraic simplicity associated with the Poisson-bracket formulation, we show conclusively that the total gyrokinetic energy is conserved by the gyrokinetic Hamiltonian flow to all orders in our asymptotic analysis.

In fact, this result is not surprising at all due to the Hamiltonian nature of the method used (Littlejohn [1979, 1981, 1983]). The necessity, however, of keeping all the relevant nonlinear terms in our gyrokinetic Maxwell-Vlasov system makes the explicit demonstration of this result valuable and represents an important test on the correctness of our nonlinear gyrokinetic system.

Finally, we remark that, for practical purposes, a truncation (at order $n$, say) must be imposed on our asymptotic expansion and, thus, the gyrokinetic total energy should be considered as an adiabatic invariant. We show for $n = 1$ and $2$ that an appropriately truncated Maxwell-Vlasov system has an exact energy invariant. Our nonlinear gyrokinetic energy invariant generalizes an earlier result of Dubin et al. [1983] for the case of electrostatic perturbations.

4.3.1 Derivation of the Gyrokinetic Energy Invariant

Using the transformation procedure outlined in 4.2.1.2., we derive an expression for the gyrokinetic energy invariant. First, the guiding-center kinetic energy integral
is given by the expression

\[ E_{\text{gk}} = \sum \int \Omega B_{\parallel}^* d^\mathbb{Z} \bar{H}_0 \, F(Z,t), \quad (4.18) \]

where \( \bar{H}_0 = (1/2)e B^2 \mu + \mu B \) is the unperturbed guiding-center. Since time and energy clearly have different roles from the other (regular) phase-space coordinates, we shall not use the extended phase-space formulation and omit the use of the notation ~ on the Hamiltonian. Second, the gyrocenter kinetic energy integral is given by the expression

\[ E_{\text{Gk}}(t) = \sum \int \Omega B_{\parallel}^* d^\mathbb{Z} \left\langle (T_{\text{GY}}^*)^{-1} \bar{H}_0 \right\rangle (\mathbb{Z},t) \, \bar{F}(\mathbb{Z},t), \quad (4.19) \]

where \( d^\mathbb{Z} = 2\pi d^3 X d^3 \mu \), and

\[ \left\langle (T_{\text{GY}}^*)^{-1} \bar{H}_0 \right\rangle = \bar{H} - \frac{e}{m} \left\langle \delta \phi_0 - \frac{e}{B} \{ \delta \bar{V}_{10}, \delta \phi_0 \}_0 \right\rangle \approx \bar{H} - \frac{e}{m} \left\langle (T_{\text{GY}}^*)^{-1} \delta \phi_0 \right\rangle. \quad (4.20) \]

The expressions for \( \bar{H}_0 \) and \( \bar{H} \) are given in Eq. (3.46).

The total gyrokinetic energy integral can finally be given by the expression

\[ E_{\text{GY}}(t) = \sum \int \Omega B_{\parallel}^* d^\mathbb{Z} \left\langle (T_{\text{GY}}^*)^{-1} H_0 \right\rangle (Z,t) \, F(Z,t) \quad (4.21) \]

\[ + \frac{1}{8\pi} \int \left[ |\delta \mathbf{E}(r,t)|^2 + |\mathbf{B}(r) + \delta \mathbf{B}(r,t)|^2 \right]. \]

The proof of gyrokinetic (adiabatic) invariance will be made under the assumption of a uniform equilibrium magnetic field, and comments about the nonuniform case will be made at the end of this section. (For simplicity, we use units for which \( e = m = c = 1 \).

### 4.3.2 Linear Gyrokinetic Maxwell–Vlasov System

#### 4.3.2.1 Linear Gyrokinetic Maxwell–Vlasov System

The linear gyrokinetic Vlasov equation is given in bracket form as

\[ \frac{\partial F}{\partial t} + \{F, H\} = 0, \quad (4.22) \]

where \( F \) is the total gyrocenter distribution function and \( H \) is the total (linear) gyrokinetic Hamiltonian:

\[ H = H_0 + \epsilon H_1 = \frac{1}{2} e B^2 + \mu B + \epsilon \left( (\delta \phi_0) - \langle \mathbf{v} \cdot \delta \mathbf{A}_0 \rangle \right). \quad (4.23) \]
The (linear) gyrokinetic Maxwell's equations, in gyrocenter phase-space, are obtained from the linear version of Eqs. (4.15) and (4.17), respectively given as

\[
\nabla^2 \delta \phi(x, t) = -4\pi \int B^2 d^5 Z \delta^3(X + \rho_0 - x) F(Z, t), \quad (4.24)
\]

\[
\nabla^2 \delta A(x, t) = -4\pi \int B^2 d^5 Z \delta^3(X + \rho_0 - x) [v \cdot F(Z, t)], \quad (4.25)
\]

where the zeroth-order expression for \( T_{GY}^c \) was used [see Eq. (4.16)] and summation over species is implied.

This set of equations (22)–(4.25) represents the linear gyrokinetic Maxwell–Vlasov system.

4.3.2.2. Linear Gyrokinetic Energy Invariant

We now show that the linear system, Eqs. (4.22)–(4.25), has an exact energy invariant. The expression for the linear total energy integral is obtained from Eq. (4.21), and is given as

\[
E^{(1)}_{GY}(t) = \int B^2 d^5 Z F(Z, t) \left[ H_0 - \langle G_{1gy}(H_0) \rangle \right] + \frac{1}{8\pi} \int d^3 r \left( |\nabla \delta \phi|^2 + |\nabla \times \delta A|^2 \right), \quad (4.26)
\]

where \( G_{1gy}(H_0) = \delta \phi_0 - \langle v \cdot \delta A_0 \rangle \) so that \( \langle G_{1gy}(H_0) \rangle = -\langle v \cdot \delta A_0 \rangle \). The time derivative of \( E^{(1)}_{GY}(t) \) is given by the expression

\[
\frac{d}{dt} E^{(1)}_{GY} = \int B^2 d^5 Z \left[ \frac{\partial F}{\partial t} \left( H_0 - \langle v \cdot \delta A_0 \rangle - F \left( v \cdot \frac{\partial \delta A_0}{\partial t} \right) \right) \right] + \frac{1}{4\pi} \int d^3 r \left[ -\delta \phi \frac{\partial}{\partial t} (\nabla^2 \delta \phi) + \frac{\partial \delta A}{\partial t} \cdot \nabla \times (\nabla \times \delta A) \right], \quad (4.27)
\]

where integration by parts was performed in the second (field energy) integral. If we now substitute the expressions for the gyrokinetic Maxwell's equations, Eqs. (4.24)–(4.25), we obtain

\[
\frac{d}{dt} E^{(1)}_{GY} = \int B^2 d^5 Z \frac{\partial F}{\partial t} \left( H_0 + \langle \delta \phi_0 \rangle - \langle v \cdot \delta A_0 \rangle \right), \quad (4.28)
\]

where the terms involving \( \langle v \cdot \delta A_0 / \partial t \rangle \) have cancelled each other.

Finally, using the linear gyrokinetic Vlasov equation, Eq. (4.22), we obtain

\[
\frac{d}{dt} E^{(1)}_{GY} = \int B^2 d^5 Z \frac{\partial F}{\partial t} H = -\int B^2 d^5 Z \{F, H\} H = \int B^2 d^5 Z \{H, HF\} = 0, \quad (4.29)
\]
where the last result is a property of the Poisson bracket. The expression for $E_{\text{GY}}^{(1)}$ given by Eq. (4.26) is, therefore, an exact invariant for the linear gyrokinetic Maxwell–Vlasov system, represented by Eqs. (4.22)–(4.25).

### 4.3.3 Nonlinear Gyrokinetic Maxwell–Vlasov System

#### 4.3.3.1. Gyrokinetic Energy Conservation

The expression for the gyrokinetic total energy was given in Eq. (4.21) as

$$E_{\text{GY}}(t) = \int B^2 ((T_{\text{GY}}^*)^{-1} \mathbf{H}_0) F d^5 \mathbf{z} + \frac{1}{8\pi} \int (|\mathbf{\delta E}|^2 + |\mathbf{B} + \mathbf{\delta B}|^2) d^3 \mathbf{x}.$$

Its time derivative can be shown to be given as

$$\frac{dE_{\text{GY}}}{dt} = \int B^2 d^5 Z F \left[ \left( \frac{\partial H}{\partial t} - \frac{d_{\text{gy}}}{dt} ((T_{\text{GY}}^*)^{-1} \mathbf{\delta \phi}_0) \right) - ((T_{\text{GY}}^*)^{-1} (\mathbf{v} \cdot \mathbf{\delta E}_0)) \right],$$

where the Maxwell–Vlasov system [given by Eqs. (4.5), (15), and (17)] was used, and $d_{\text{gy}}/dt = \partial / \partial t + \{ , H \}$ represents the total time derivative along the gyrocenter phase-space trajectory. From this expression, the energy conservation equation is defined as

$$\frac{dE_{\text{GY}}}{dt} = \int B^2 d^5 Z F \frac{\delta E}{\delta t},$$

where

$$\frac{\delta E}{\delta t} = \frac{\partial H}{\partial t} - \frac{d_{\text{gy}}}{dt} ((T_{\text{GY}}^*)^{-1} \mathbf{\delta \phi}_0) - ((T_{\text{GY}}^*)^{-1} (\mathbf{v} \cdot \mathbf{\delta E}_0)). \quad (4.30)$$

Note that Eq. (4.30) depends entirely on the gyrocenter dynamics and, therefore, does not involve the gyrocenter distribution function (as opposed to the formulation used by Hagan [1986]).

Using the expression for the gyrokinetic Hamiltonian, Eq. (3.47),

$$H = \frac{1}{2} \rho^2 \mathbf{B}^2 + \mu \mathbf{B} + ((\mathbf{v} \cdot \mathbf{\delta \phi}_0) - (\mathbf{v} \cdot \mathbf{\delta A}_0)) + \frac{1}{2} (|\mathbf{\delta A}_0|^2) - \frac{1}{2B} \langle \{\delta \tilde{\psi}_{10}, \delta \tilde{\psi}_{10}\} \rangle,$$

it is easy to show that

$$\frac{\partial H}{\partial t} = \left( T_{\text{GY}}^{-1} \frac{\partial \delta \psi_{10}}{\partial t} \right),$$

where

$$\frac{\partial \delta \psi_{10}}{\partial t} = \left( \frac{\partial \delta \phi_0}{\partial t} + \{\delta \phi_0, H_0\} + \mathbf{v} \cdot \mathbf{\delta E}_0. \right)$$
The energy conservation equation is, therefore, given by the exact expression

$$
\frac{\delta E}{\delta t} = \left\langle (T_{GY}^*)^{-1} \frac{d_{gc}}{dt} \delta \phi_0 \right\rangle - \frac{d_{gy}}{dt} \left\langle (T_{GY}^*)^{-1} \delta \phi_0 \right\rangle,
$$

where $d_{gc}/dt = \partial/\partial t + \{ , H_0 \}$ represents the total time derivative along the unperturbed (guiding-center) phase-space trajectory. Note that Eq. (4.31) vanishes at the lowest (first) order since in this case we have $T_{GY}^* \equiv 1$, $d_{gy}/dt = d_{gc}/dt$, and

$$
\left\langle \frac{d_{gc}}{dt} \delta \phi_0 \right\rangle = \frac{d_{gc}}{dt} \left( \delta \phi_0 \right).
$$

Using the $O(\epsilon^2)$ expressions for $G_{1gy}$ and $G_{2gy}$, obtained from Eqs. (3.14) and (3.23), the explicit expression for Eq. (4.31) becomes

$$
\frac{\delta E}{\delta t} = \{\delta \psi_0, \{\delta \phi_0\}_0\} - \{\delta \psi_{10}, \delta \phi_0\}_0

+ \{\delta \bar{\psi}_{10} + \{S_{10}, \delta \psi_{10}\}_0, \delta \phi_0\}_0

+ \{\delta \psi_{20}, \delta \phi_0\}_0 - \left\langle \left( G_{20} - \frac{1}{2} G_{10}^2 \right) B \frac{\partial \delta \phi_0}{\partial \zeta} \right\rangle + O(\epsilon^4).
$$

Finally, using the Jacobi identity for Poisson brackets

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

the $O(\epsilon^2)$ terms cancel out and Eq. (4.32) reduces to

$$
\frac{\delta E}{\delta t} = \{\delta \psi_{20}, \{\delta \phi_0\}_0\} - \left\langle \left( G_{20} - \frac{1}{2} G_{10}^2 \right) B \frac{\partial \delta \phi_0}{\partial \zeta} \right\rangle + \{\{S_{10}, \delta \phi_0\}_0, \{\delta \psi_{10}\}_0\},
$$

where we have ignored terms of order $O(\epsilon^4)$. We now wish to show that Eq. (4.34) vanishes at order $O(\epsilon^3)$.

The second term on the right-hand side of Eq. (4.34) is given explicitly by the expression

$$
- \left\langle \left( G_{20} - \frac{1}{2} G_{10}^2 \right) B \frac{\partial \delta \phi_0}{\partial \zeta} \right\rangle = \{\delta \bar{\psi}_{20}, \delta \phi_0\}_0 + \left\langle \left\{ \delta \bar{\psi}_{10}, \delta A_0 \cdot P \left( \frac{\partial \delta \phi_0}{\partial \zeta} \right) \right\}_0 \right\rangle

+ \frac{1}{2B} \left\langle \left\{ \delta \bar{\psi}_{10}, \left\{ \delta \bar{\psi}_{10}, \frac{\partial \delta \phi_0}{\partial \zeta} \right\}_0 \right\}_0 \right\rangle

- \frac{1}{2B} \left\langle \{\delta A_0 \cdot P(\delta \bar{\psi}_{10}), H_0\}_0, \delta \phi_0\}_0 \right\rangle

+ \frac{1}{2} \left\langle \delta A_0 \cdot P \left( P \left( \frac{\partial \delta \phi_0}{\partial \zeta} \right) \right) \cdot \delta A_0 \right\rangle,
$$
where \( P(\delta x) = \{ X + \rho_0, \delta x \} \). It can be further simplified by using the following identities. In the second and fifth terms on the right-hand side of Eq. (4.35), we use the Jacobi identity, Eq. (4.33), to show that (keeping only the relevant terms)

\[
\left\{ X + \rho_0, B \frac{\partial \delta \phi_0}{\partial \zeta} \right\}_0 = \{ X + \rho_0, \{ \delta \phi_0, H_0 \} \}_0 = \{ \delta \phi_0, v \}_0,
\]

so that

\[
\langle \{ \delta \tilde{\Psi}_{10}, \delta A_0 \cdot \{ \delta \phi_0, v \} \} \rangle_0 = \langle \{ \delta \tilde{\Psi}_{10}, \{ \delta \psi_{10}, \delta \phi_0 \} \} \rangle_0,
\]

and

\[
\left\{ X + \rho_0, \left\{ X + \rho_0, B \frac{\partial \delta \phi_0}{\partial \zeta} \right\} \right\}_0 = \{ \delta \phi_0, \{ X + \rho_0, v \} \}_0 = \{ \delta \phi_0, I \}_0 = 0,
\]

where \( I \) is the 3-D unit dyadic. Using again the Jacobi identity and integrating the gyrophase by parts, the third term on the right-hand side of Eq. (4.35) becomes

\[
\frac{1}{2} \left\langle \left\{ \delta \tilde{\Psi}_{10}, \left\{ \delta \tilde{\Psi}_{10}, \frac{\partial \delta \phi_0}{\partial \zeta} \right\} \right\} \right\rangle_0 = -\langle \{ \delta \tilde{\Psi}_{10}, \{ \delta \psi_{10}, \delta \phi_0 \} \} \rangle_0 + \frac{1}{2} \langle \{ \{ \delta \tilde{\Psi}_{10}, \delta \tilde{\psi}_{10} \}, \delta \phi_0 \} \rangle_0.
\]

With these simplifications, the energy conservation equation, Eq. (4.34), is given by the expression

\[
\frac{\delta E}{\delta t} = \left\langle \left\{ \left[ \delta \psi_{20} + \{ S_{10}, \delta \psi_{10} \} \right]_0 + \frac{1}{2} \{ S_{10}, \delta \tilde{\psi}_{10} \}_0 \right. \right. \\
\left. \left. - \frac{1}{2} \{ \delta A_0 \cdot \{ X + \rho_0, S_{10} \}_0, H_0 \} \right\} \right\rangle.
\]

Finally, using the Poisson-bracket expression for \( \delta \psi_{20} \), given on the right-hand side of Eq. (3.25),

\[
\delta \psi_{20} = \frac{1}{2} \left| \delta A_0 \right|^2 - \frac{1}{2} \{ S_{10}, \delta \tilde{\psi}_{10} \}_0 - \{ S_{10}, \{ \delta \psi_{10} \} \}_0 \\
+ \frac{1}{2} \{ \delta A_0 \cdot \{ X + \rho_0, S_{10} \}_0, H_0 \}_0,
\]

where \( S_{10} = \frac{1}{\epsilon} \delta \tilde{\psi}_{10} \) is used, we obtain

\[
\frac{\delta E}{\delta t} = \langle \delta A_0 \cdot \{ \delta A_0, \delta \phi_0 \} \rangle_0 = 0,
\]

since \( \{ \delta A_0, \delta \phi_0 \}_0 = 0 \). The gyrokinetic total energy is, therefore, conserved at order \( \epsilon^3 \).
4.3.3.2. Exact Gyrokinetic Energy Invariant

The following expression for the gyrokinetic total energy (the unperturbed magnetic field is assumed uniform),

\[ E_{\text{GY}}(t) = \sum \int B^2 \, d^\text{3}Z \, F \langle (T_{\text{GY}})^{-1} H_0 \rangle + \frac{1}{8\pi} \int (|\nabla \delta \phi|^2 + |\nabla \times \delta A|^2) \, d^\text{3}x, \]  
\[ (4.36) \]

is an exact gyrokinetic invariant for the gyrokinetic Maxwell–Vlasov system given by the Maxwellian equations

\[ \nabla^2 \delta \phi = -4\pi \sum \int B^2 d^\text{3}Z \, \delta^3(X + \rho_0 - r) \left[ F + \delta A_0 \cdot \{X + \rho_0, F\}_0 + \frac{1}{B} \{\delta \bar{\Psi}_{10}, F\}_0 \right], \]
\[ (4.37) \]

and

\[ \nabla^2 \delta A = -4\pi \sum \int B^2 d^\text{3}Z \, \delta^3(X + \rho_0 - r) \left[ F + \delta A_0 \cdot \{X + \rho_0, F\}_0 + \frac{1}{B} \{\delta \bar{\Psi}_{10}, F\}_0 \right], \]
\[ (4.38) \]

and by the Hamiltonian given by the expression

\[ H = H_0 + \langle \delta \phi_0 \rangle - \langle \mathbf{v} \cdot \delta A_0 \rangle + \frac{1}{2} \langle |\delta A_0|^2 \rangle - \frac{1}{2B} \langle \{\delta \bar{\Psi}_{10}, \delta \bar{\Psi}_{10}\}_0 \rangle. \]
\[ (4.39) \]

The demonstration of the exactness of Eq. (4.36) is simplified through the use of the following Poisson-bracket identities

\[ \int \Omega B_{||} d^\text{3}Z \{F, G\} = 0 \text{ and } \int \Omega B_{||} d^\text{3}Z \delta A_0 \cdot \{X + \rho_0, G\} = 0, \]

where \( F \) and \( G \) are arbitrary phase-space functions.

We remark that the gyrokinetic Hamiltonian, Eq. (4.39), is truncated at second order, whereas the gyrokinetic Maxwellian equations, Eqs. (4.37)–(4.38), are truncated at first order because the field energy integral is already quadratic in perturbation field amplitudes. The gyrokinetic energy invariant, Eq. (4.36), constitutes a generalization of the exact energy invariant found by Dubin et al. [1983] in the case of electrostatic perturbations. (An expression for the exact gyrokinetic energy invariant can also be derived when the unperturbed magnetic field is nonuniform.)

4.3.3.3. Discussion

Looking at the gyrokinetic ordering of the two terms on the right-hand side of Eq. (4.31), we see that the first term contains all terms up to order \( O(\epsilon^3) \), and the second term contains all terms of \( O(\epsilon^4) \). Showing that the gyrokinetic total energy
is preserved up to order $O(\epsilon^3)$ is, therefore, the best that can be achieved with a gyrokinetic Hamiltonian containing quadratic nonlinearities.

The claim that the gyrokinetic total energy is preserved to all orders of perturbation by the gyrokinetic Hamiltonian flow is consequently made in the sense of adiabatic invariance: the gyrokinetic dynamics generated by the Hamiltonian

$$H = H_0 + \epsilon H_1 + \cdots + \epsilon^n H_n,$$

where $n = 1, 2, \ldots$, preserves the (gyrokinetic) magnetic moment $\mu$ up to order $O(\epsilon^n)$, i.e.,

$$\dot{\mu} = O(\epsilon^{n+1}),$$

and preserves the gyrokinetic total energy up to order $O(\epsilon^{n+1})$, i.e.,

$$\frac{\delta E}{\delta t} = O(\epsilon^{n+2}).$$

The explicit proof of this statement was made for $n = 2$ in 4.3.3.1 [see Eq. (4.34)].

The energy conservation equation for the case of a general magnetic field configuration has again the same form as Eq. (4.31) and leads to the same conclusion as the one arrived at above. For instance, it is easy to show that the gyrokinetic total energy is preserved at orders $O(\epsilon \epsilon_B)$ and $O(\epsilon^2 \epsilon_B)$ [provided the first-order generating vector field contains all terms of order $O(\epsilon \epsilon_B)$], where $\epsilon_B$ is a small parameter used to describe the magnetic nonuniformity, introduced in Brizard [1989a].

The adiabatic invariance of the gyrokinetic total energy ensures the existence of an exact gyrokinetic energy invariant for every truncated gyrokinetic Maxwell–Vlasov system. In the work by Dubin et al. [1983], the gyrokinetic total energy for the gyrokinetic Poisson–Vlasov system is claimed not to be conserved at order $O(\epsilon^3)$. The difficulty comes from the treatment of the phase-space gauge function $S_{10}$, which is known to be the solution of the equation

$$\frac{\partial S_{10}}{\partial t} + \{S_{10}, H_0\}_0 = \delta \bar{\psi}_{10}.$$

Following the discussion in 3.2.3., this solution is given as

$$S_{10} = \frac{1}{\tilde{B}} \delta \bar{\Psi}_{10}^{(1)} - \frac{\epsilon}{\tilde{B}^2} \left( \frac{\partial}{\partial t} + \rho \parallel \mathbf{B} \cdot \nabla_0 \right) \delta \bar{\Psi}_{10}^{(2)} + O(\epsilon^2),$$
where $\partial \tilde{\Psi}_{10}^{(n)}/\partial \zeta \doteq \delta \tilde{\Psi}_{10}^{(n-1)}$ and $\delta \tilde{\Psi}_{10}^{(0)} \doteq \delta \tilde{\Psi}_{10}$. The neglect of the higher-order terms in $S_{10}$ causes the energy conservation equation, Eq. (4.31), not to vanish at order $O(\varepsilon^3)$.

More recently, Hagan [1986] gave a proof of the energy conservation property, up to $O(\varepsilon^4)$, for the gyrokinetic Poisson–Vlasov system with a Hamiltonian containing cubic nonlinearities. The proof is claimed to be tedious and is also made difficult through the introduction of the gyrocenter distribution function (more precisely, its unperturbed and perturbed parts).

In this work, we have shown that the energy conservation equation, Eq. (4.30), can be entirely written in terms of the gyrocenter dynamics. In addition, we have shown that the gyrokinetic total energy for the gyrokinetic Maxwell–Vlasov system is preserved by the gyrokinetic Hamiltonian flow. The demonstration was simplified by the use of the Poisson-bracket formulation. Finally, we have also given an exact gyrokinetic energy invariant, Eq. (4.36), for the gyrokinetic Hamiltonian given by Eq. (4.39) and the gyrokinetic Maxwell’s equations, Eqs. (4.37)–(4.38).
4.4 Further Developments in Gyrokinetic Formalism

We present two additional topics in the development of the gyrokinetic formalism. The first topic deals with the procedure used in the derivation of a gyrokinetic collision operator. Only general forms for the original collision operator are considered, and specific examples will be dealt with in future research.

The second topic deals with the so-called arbitrary-frequency gyrokinetic formalism, which allows the consideration of ion-cyclotron physics. Although such formalism already exists in the literature, we present here a simplified derivation based on the use of the extended Phase-space Lie perturbation method.

4.4.1 Gyrokinetic Collision Operator

4.4.1.1. Transformation of Boltzmann's Equation

We wish to consider the transformation of Boltzmann's equation induced by the extended phase-space transformation $T_G$. Boltzmann's equation has the form

$$\{f, h\} = \frac{\partial f}{\partial t} + \{f, \hat{h}\} = C[f],$$

(4.40)

where $h = \hat{h} - w$ is the extended phase-space Hamiltonian, $f(z, t)$ is the phase-space distribution function, and $C$ is a collision operator (treated as a functional of $f$).

Since phase-space scalar fields transform according to the pull-back formula

$$F = (T_G^*)^{-1} f, \quad H = (T_G^*)^{-1} h,$$

we then have

$$T_G\cdot(\{f, h\}) = \{(T_G^*)^{-1} f, (T_G^*)^{-1} h\}_g = \{F, H\}_g,$$

where we have indicated the possibility that $T_G$ may not be canonical by using the transformed Poisson bracket $\{ \, , \}_g$. The transformed Boltzmann's equation is, therefore, given by the expression

$$\{F, H\}_g = \frac{\partial F}{\partial t} + \{F, \tilde{H}\}_g = T_G\cdot C[T_G^* F] \doteq C_G[F].$$

(4.41)

The derivation of an expression for the collision operator $C_G$ is the goal of this subsection. In fact, only the procedure involved in the derivation, similar to the procedure used by Auerbach [1985], will be presented.
In the spirit of Lie perturbation theory, we expand the operators $T_G$ and $T_G^*$ in powers of $\epsilon$. Consequently, the transformed collision operator, Eq. (4.41), acquires the following expansion

$$C_G[F] = C_0[F] + \epsilon C_1[F] + \epsilon^2 C_2[F] + \cdots,$$

(4.42)

where

$$C_0[F] = C[F],$$

$$C_1[F] = C[G_1(F)] - G_1(C[F]),$$

$$C_2[F] = \frac{1}{2} G_2^2(C[F]) + \frac{1}{2} C[G_2^2(F)] - G_1(C[G_1(F)]) + C[G_2(F)] - G_2(C[F]).$$

(4.43)

4.4.1.2. Krook Collision Operator

A simple application of this procedure is shown by considering the Krook collision operator

$$C_K[f](z) = \nu(z) [f_0(z) - f(z)],$$

(4.44)

where the collision frequency $\nu(z)$ may depend on phase-space coordinates and $f_0$ is a prescribed phase-space (Maxwellian distribution) function. If we use the identities

$$T^*F G = (T^*F) (T^*G) \quad \text{and} \quad (T^*)^{-1} F T^*G = [(T^*)^{-1} F] G,$$

(4.45)

where $F$ and $G$ are arbitrary phase-space scalar fields, we easily obtain

$$T_G C_K T_G^* F = (T_G^*)^{-1} \nu \left[ (T_G^*)^{-1} f_0 - F \right] \equiv C_{GK}[F].$$

(4.46)

The gyrokinetic Krook collision operator has obviously retained the properties of the original operator.

4.4.1.3. Fokker–Planck Collision Operator

An important class of collision operators can be represented by the operator

$$C[f](z,t) = -\frac{\partial}{\partial \nu} \left[ K(z,t) f(z,t) - \mathcal{D}(z,t) \frac{\partial f}{\partial \nu}(z,t) \right].$$

(4.47)

This collision operator shows explicitly the property of particle density conservation, while the properties of momentum and energy conservation require a specification of the (dynamic friction) vector field $K$ and (diffusion) tensor field $\mathcal{D}$. An example
for such a collision operator is given by the Fokker–Planck collision operator $C_{ss'}$ for colliding species $s$ and $s'$:

$$C_{ss'}[f_s] = -\Gamma_s \frac{z_{s'}}{z_s} \frac{\partial}{\partial v} \left( \frac{m_s}{m_{s'}} \frac{\partial h_{s'}}{\partial v} f_s - \frac{1}{2} \frac{\partial^2 g_{s'}}{\partial v \partial v} \frac{\partial f_s}{\partial v} \right),$$  \hspace{1cm} (4.48)

where $\Gamma_s = 4\pi(z_s/m_s)^2e^4 \ln \Lambda$, and the Rosenbluth potentials are given by the expressions

$$h_{s'} = \int d^3v' \frac{f_{s'}}{|v-v'|}, \quad g_{s'} = \int d^3v' \frac{f_{s'}}{|v-v'|}.$$

We simply note that the diffusion tensor $D$ is normally symmetric, as in Eq. (4.48), and refer the reader to Hinton [1983] for more details about the Fokker–Planck collision operator.

A natural interpretation of Boltzmann's equation, Eq. (4.40), is that it describes the evolution of the extended phase-space density $f \omega^6$, where $\omega^6$ is the extended phase-space volume element. For simplicity, we shall describe the transformation of Eq. (4.40) into its guiding-center equivalent, and consider the (regular) phase-space volume element $\omega^6 (\omega^6 \equiv \omega^6 dw dt)$. The original Boltzmann's equation is therefore given as

$$\left( \frac{\partial f}{\partial t} + \{f, \hat{h}\} \right) \omega^6 = -\left( \text{div} \ A \right) \omega^6,$$  \hspace{1cm} (4.49)

where $\text{div} \ A = \partial A^i / \partial x^i$, and the vector field $A$ is given in Eq. (4.47). (Note that only its velocity components are non-vanishing in the original phase-space formulation.)

The transformation to guiding-center phase space is represented by the expression

$$(T_{GC}^*)^{-1} \left[ \left( \frac{\partial f}{\partial t} + \{f, \hat{h}\} \right) \omega^6 \right] = -\left( T_{GC}^* \right)^{-1} \left[ \left( \text{div} \ A \right) \omega^6 \right].$$  \hspace{1cm} (4.50)

The left-hand side of Eq. (4.50) is given by the well-known expression

$$\left( \frac{\partial F}{\partial t} + \{F, \hat{H}_{GC}\}_{GC} \right) J_{GC} \omega^6_{GC},$$

where the Jacobian of the guiding-center transformation is defined as

$$J_{GC} = 1 - \epsilon \frac{\partial C^i_{GC}}{\partial x^i} + \mathcal{O}(\epsilon^2).$$

The particle conservation property of the Fokker–Planck collision operator depends on the divergence form displayed in Eqs. (4.47) and (4.48). We would now like
to show that the right-hand side of Eq. (4.50) will retain its divergence form. First, we use the simple formula
\[
d \omega^{x} = d(i_{A} \omega^{x}) = L_{A} \omega^{x},
\]
which associates the divergence of a vector field \(A\) to the application of the Lie derivative \(L_{A}\) to \(\omega^{x}\). The right-hand side of Eq. (4.50), at first order, involves the expression \(L_{G} L_{A} \omega^{x}\). Using the definition \(\omega^{x} = d(z^{k}) \wedge \omega^{k}_{x}\) (no summation), we easily obtain
\[
i_{G}(d \omega^{x}) = (d \omega^{x}) G_{1gc}^{i} \omega^{5}_{i},
\]
(4.51)
where the summation rule is implied on the right-hand side of Eq. (4.51). The expression \(L_{G} L_{A} \omega^{x}\) is finally obtained if we evaluate the exterior derivative of Eq. (4.51), which gives
\[
d \left[ i_{G}(d \omega^{x}) \right] = d \left[ (d \omega^{x}) G_{1gc}^{i} \right] \wedge \omega^{5}_{i} = \frac{\partial}{\partial z^{i}} \left[ G_{1gc}^{i} (d \omega^{x}) \right] \omega^{x}.
\]
(4.52)
Boltzmann's equation, therefore, becomes
\[
J_{gc} \{F, H_{gc}\}_{gc} = -d A^{i} + \epsilon \frac{\partial}{\partial z^{i}} \left[ G_{1gc}^{i} (d \omega^{x}) \right] + \mathcal{O}(\epsilon^{2}),
\]
(4.53)
and obviously displays the desired divergence form. The original particle conservation property of the collision operator is therefore preserved. From the definition of the Jacobian \(J_{gc}\), Eq. (4.53) is also given by the expression
\[
J_{gc} \{F, H_{gc}\}_{gc} = -J_{gc}(d A^{i}) + \epsilon G_{1gc}^{i} \frac{\partial}{\partial z^{i}}(d A^{i}).
\]
Finally, we note that the new guiding-center vector field \(A_{gc}\), defined as
\[
A_{gc}^{i} = A^{i} - \epsilon \frac{\partial A^{k}}{\partial z^{k}} + \mathcal{O}(\epsilon^{2}),
\]
has acquired non-vanishing spatial components through the vector field \(G_{1gc}\). The guiding-center version of the Fokker–Planck collision operator will, therefore, contain spatial divergence terms. This in turn implies that the moments of the transformed Boltzmann's equation will immediately display the so-called transport coefficients associated with the spatial gradients.

Future work will deal with the conservation properties of the guiding-center and gyrokinetic Fokker–Planck collision operators, and the rederivation of the Braginskii transport coefficients (Braginskii [1965]). The reader is also referred to the work of Mynick [1988] and Mynick and Duvall [1989] for the use of the action-angle formalism of Kaufman [1972] in the generalization of the Balescu–Lenard collision operator.
4.4.2 Gyrokinetic Formalism for Arbitrary Frequencies

The dynamical reduction based on the elimination of the gyrophase from the equations of motion can still be carried out if the perturbation fields do not obey the low-frequency ordering. This high-frequency gyrokinetic formalism is especially important in the study of ion-cyclotron physics (e.g., the interaction of ICRF waves with a nonuniform plasma, see Chiu [1985] and Lashmore-Davies and Dendy [1989]).

4.4.2.1. Modified Linear Gyrokinetic Ordering

The high-frequency gyrokinetic ordering, as introduced by Chen and Tsai [1983a,b], uses two small parameters: \( \lambda \simeq \varepsilon \simeq \varepsilon \), and \( \varepsilon_B \) (although not used explicitly). It assumes that the equilibrium distribution function \( F_0 \) satisfies the condition

\[
| \rho_i \nabla F_0 | / F_0 \sim \mathcal{O}(\lambda),
\]

as in the low-frequency ordering.

The high-frequency ordering, however, assumes that the perturbed quantities (i.e., the perturbation electromagnetic fields and the perturbed distribution function) obey the following ordering

\[
| \frac{1}{\Omega} \frac{\partial}{\partial t} | \sim | \rho_i \nabla | \sim \mathcal{O}(1),
\]

in marked contrast with the low-frequency ordering. Because the perturbed quantities are still ordered \( \mathcal{O}(\lambda) \) smaller than the equilibrium ones, a perturbation analysis can be carried out.

4.4.2.2. Linear Gyrokinetic Vlasov Equation

We present the derivation of the linear gyrokinetic Vlasov equation for arbitrary frequencies. In the Hamiltonian representation of gyrokinetics given in 3.2.1., the Poisson-bracket structure is that of the guiding-center problem. In our perturbation analysis of high-frequency gyrokinetics, we choose the unperturbed problem to be the guiding-center problem, in order to avoid coupling between the unperturbed gyro-motion and the perturbation fields. (More comments will be made below about this choice.)

The linear gyrokinetic Vlasov equation is, therefore, given by the expression

\[
\frac{\partial F_i}{\partial t} + \{ F_i, H_0 \} + \{ F_0, H_1 \} = 0, \quad (4.54)
\]
where the unperturbed (guiding-center) Hamiltonian vector field gives

\[ \{ F_1, H_0 \} = (U\hat{b} + v_d) \cdot \nabla F_1 + \mu B(\nabla \cdot \hat{b}) \frac{\partial F_1}{\partial U} + \Omega \left( 1 + \frac{U}{\Omega} \hat{b} \cdot W \right) \frac{\partial F_1}{\partial \zeta}, \]

and the perturbed Hamiltonian vector field gives

\[ \{ F_0, H_1 \} = -\frac{\hat{b}}{\Omega} \times \nabla F_0 \cdot \nabla H_1 - \frac{\partial F_0}{\partial U} \frac{B^*}{B_\parallel} \cdot \nabla H_1 - \left( \frac{\Omega}{B} \frac{\partial F_0}{\partial \mu} + \frac{\partial F_0}{\partial \mu} \hat{b} \cdot W \right) \frac{\partial H_1}{\partial \zeta}. \]

The definitions of the guiding-center quantities were defined previously, and we shall omit the \( \sim \) notation on the Hamiltonian functions.

We now adopt the following expressions for \( F_1 \) and \( H_1 \)

\[ (F_1, H_1) = \sum_{l=-\infty}^{\infty} (\langle F_1 \rangle_l, \langle H_1 \rangle_l) \exp -il\zeta, \quad (4.55) \]

where the \( l \)th harmonic is defined as

\[ \langle G \rangle_l = \int_0^{2\pi} \frac{d\zeta}{2\pi} G \exp il\zeta. \]

Because the guiding-center Poisson-bracket structure is gyrophase independent, the harmonics are decoupled and each harmonic satisfies the following gyrokinetic equation

\[ \left[ \frac{\partial}{\partial t} + (U\hat{b} + v_d) \cdot \nabla \right] \langle F_1 \rangle_l \cdot il\Omega \left( 1 + \frac{U}{\Omega} \hat{b} \cdot W \right) \langle F_1 \rangle_l + \mu B(\nabla \cdot \hat{b}) \frac{\partial \langle F_1 \rangle_l}{\partial U} \langle F_1 \rangle_l \quad (4.56) \]

\[ = \frac{\hat{b}}{\Omega} \times \nabla F_0 \cdot \nabla \langle H_1 \rangle_l + \frac{\partial F_0}{\partial U} \frac{B^*}{B_\parallel} \cdot \nabla \langle H_1 \rangle_l - il \left( \frac{\Omega}{B} \frac{\partial F_0}{\partial \mu} + \frac{\partial F_0}{\partial \mu} \hat{b} \cdot W \right) \langle H_1 \rangle_l. \]

A more conventional notation makes use of the unperturbed guiding-center (kinetic) energy \( \varepsilon = U^2/2 + \mu B \) as one of the phase-space coordinates. The left-hand side of the linear gyrokinetic equation, Eq. (4.56), becomes

\[ \left[ \frac{\partial}{\partial t} + (U\hat{b} + v_d) \cdot \nabla \right] - il\Omega \left( 1 + \frac{U}{\Omega} \hat{b} \cdot W \right) \langle F_1 \rangle_l \equiv \langle L_\varepsilon \rangle_l \langle F_1 \rangle_l, \quad (4.57) \]

where \( U = (2\varepsilon - \mu B)^{1/2} \), and the right-hand side of Eq. (4.56) becomes

\[ \left[ \frac{\hat{b}}{\Omega} \times \nabla F_0 + (U\hat{b} + v_d) \frac{\partial F_0}{\partial \varepsilon} \right] \cdot \nabla \langle H_1 \rangle_l - il\Omega \left[ \frac{1}{B} \frac{\partial F_0}{\partial \mu} + \left( 1 + \frac{U}{\Omega} \hat{b} \cdot W \right) \frac{\partial F_0}{\partial \varepsilon} \right] \langle H_1 \rangle_l. \quad (4.58) \]
The formal solution of the linear gyrokinetic Vlasov equation is, therefore, given by the following expression

\[ \langle F_1 \rangle_l = -\langle L_g \rangle_l^{-1} \langle K_g \rangle_l \]

\[ = \langle L_g \rangle_l^{-1} \left\{ \left[ \frac{\hat{b}}{\Omega} \times \nabla F_0 + (\tilde{U} \hat{b} + \nu_d) \frac{\partial F_0}{\partial \epsilon} \right] \cdot \nabla \right. \]

\[ - il\Omega \left[ \frac{1}{B} \frac{\partial F_0}{\partial \mu} + \left( 1 + \frac{U}{\Omega} \hat{b} \cdot W \right) \frac{\partial F_0}{\partial \epsilon} \right] \right\} \langle H_1 \rangle_l, \tag{4.59} \]

where \( \langle L_g \rangle_l^{-1} \) is the inverse operator associated with the operator \( \langle L_g \rangle_l \) defined in Eq. (4.57). (See Lee, Myra, and Catto [1983] for a similar expression for the nonadiabatic part of the perturbed distribution function.)

The equation actually derived by Chen and Tsai [1983a,b] has the form

\[ \langle F_1 \rangle_l = -\langle L_g \rangle_l^{-1} \left( \langle K_g \rangle_l + \sum_{\nu \neq l} L_{g\nu} \langle F_1 \rangle_{\nu} \right), \tag{4.60} \]

where the operators \( L_{g\nu} \) correspond to cyclotron-harmonic coupling due to magnetic nonuniformity. Following the discussion of Chen and Tsai [1983a,b], these harmonic-coupling operators turn out to be always negligible (in the lowest-order approximation) compared to the operator \( \langle L_g \rangle_l \), provided the condition \( k_{1\rho_i} \ll \epsilon_B^{-1} \approx \ell_B / \rho_i \) holds. Since we are normally interested in wavelengths for which \( k_{1\rho_i} \leq 1 \) and \( \epsilon_B \ll 1 \), this condition is well satisfied, and we were quite justified in taking the guiding-center problem as the starting point of our perturbation analysis.

4.2.3. First-order Gyrokinetic Hamiltonian

We now wish to give an expression for the perturbed Hamiltonian \( \langle H_1 \rangle_l \) appearing in the equation given in Eq. (4.59). We begin with Eq. (3.16),

\[ H_1 + \left( \frac{\partial S_1}{\partial t} + \{ S_1, H_0 \} \right) = \frac{e}{m} \delta \phi_0 - \frac{\Omega}{B} \nu \cdot \delta A_0 - \frac{\Omega}{B} \nu_d \cdot \delta A_0 \]

\[ - \frac{U\Omega}{B} \hat{b} \cdot \left( \nabla \rho_0 + W \frac{\partial \rho_0}{\partial \zeta} \right) \cdot \delta A_0 - \rho_1 \cdot \left( \frac{e}{m} \delta E_0 + \frac{\Omega}{B} \nu \times \delta B_0 \right). \tag{4.61} \]

Using the expression given in Eq. (4.55),

\[ H_1 = \sum_l \langle H_1 \rangle_l \exp -il\zeta, \]
we easily obtain from Eq. (4.61)

\[
\langle H_1 \rangle_t = \frac{e}{m} \langle \delta \phi_0 \rangle_t - \frac{\Omega}{B} \langle \mathbf{v} \cdot \delta \mathbf{A}_0 \rangle_t - \frac{\Omega}{B} \mathbf{v}_d \cdot \langle \delta \mathbf{A}_0 \rangle_t \\
- \frac{\Omega}{B} \mathbf{U} \mathbf{b} \cdot \left( \left( \nabla \rho_0 + \mathbf{W} \frac{\partial \rho_0}{\partial \zeta} \right) \cdot \delta \mathbf{A}_0 \right)_t - \langle \mathbf{v}_1 \cdot \left( \frac{e}{m} \delta \mathbf{E}_0 + \frac{\Omega}{B} \mathbf{v} \times \delta \mathbf{B}_0 \right) \rangle_t,
\]

provided the gauge phase-space function satisfies the homogeneous linear equation

\[
\frac{\partial \langle S_1 \rangle_t}{\partial t} + (\mathbf{U} \mathbf{b} + \mathbf{v}_d) \cdot \nabla \langle S_1 \rangle_t - i l \Omega \left( 1 + \frac{U}{\Omega} \mathbf{b} \cdot \mathbf{W} \right) \langle S_1 \rangle_t = 0,
\]

for each integer value \( l = 0, \pm 1, \pm 2 \), etc.

The first three terms on the right-hand side of Eq. (4.62) are well-known, although the last of these is usually added in \textit{ad hoc} fashion. (See remarks about this term in Chiu [1985].) The last two angular brackets are directly related to the consistent derivation of guiding-center coordinates \((X, U, \mu, \zeta)\) in nonuniform magnetic fields, where \(\mu\) represents the true adiabatic invariant.
PART III : REDUCED FLUID DESCRIPTIONS

We present three different generations of reduced fluid equations used to investigate low-frequency \((\omega \ll \omega_i)\), small-perpendicular-wavelength \((k_{\perp}a > 1)\) nonlinear tokamak dynamics.

The first generation of reduced fluid equations, the so-called reduced MHD (RMHD) equations, is obtained as a result of the application of a reduction procedure, based on the small parameter known as the inverse-aspect-ratio in toroidal geometry, to the ideal (or resistive) MHD equations. The low-\(\beta\) incompressible RMHD equations (Strauss [1976]) and the high-\(\beta\) compressible RMHD equations (Strauss [1983], and Izzo et al. [1983]) are presented.

Because the original MHD equations do not describe FLR physics (such as FLR stabilization of various unstable fluid modes, and gyroviscous effects), the MHD equations need to be modified in order to include FLR effects. The second generation of reduced fluid equations, the so-called reduced FLR-MHD equations, is obtained as a result of the application of the same reduction procedure to the FLR-MHD equations. We present three sets of reduced FLR-MHD equations: (1) the four-field model of Hasegawa and Wakatani [1983]; (2) the three-field model of Hazeltine [1983]; and (3) the four-field model of Hazeltine, Kotschenreuther, and Morrison [1985].

Finally, the third generation of reduced fluid equations, the so-called gyrofluid equations (Bernstein and Catto [1985], and Lee [1989]), is derived as a closed set of moments of the gyrokinetic Maxwell–Vlasov equations. We present a thorough derivation of a closed set of energy-conserving gyrofluid equations, with a simple moment-hierarchy closure scheme (based on the assumption of long perpendicular wavelengths with \(k_{\perp}\rho_i < 1\)) which allows us to compare reduced equations of the second and third generations.
Chapter 5

Reduced Fluid Description of Tokamak Plasmas

We present a reduced description of nonlinear tokamak dynamics based on the first two generations of reduced fluid equations: the reduced MHD equations; and the reduced FLR-MHD equations. The reduced equations presented in this chapter are the following nonlinear dynamics of tokamak plasmas: shear-Alfvén dynamics (the low-β RMHD equations of Strauss [1976] and the three-field model of Hazelton [1983]); ballooning-Alfvén dynamics (the high-β RMHD equations of Strauss [1977, 1983] and Izzo et al. [1983]); and drift-ballooning-Alfvén dynamics (the four-field model of Hasegawa and Wakatani [1983] and the four-field model of Hazelton et al. [1985]).

5.1 Nonlinear Reduced MHD Equations

The low-β, incompressible reduced ideal MHD (RMHD) equations were first derived by Strauss [1976] for the study of nonlinear low-β MHD modes in large-aspect-ratio tokamak plasmas. These reduced equations were later generalized by Strauss [1977, 1983] and Izzo et al. [1983], who gave high-β, compressible (resistive) RMHD equations, suitable for the study of ballooning-mode physics in tokamak plasmas. Other first-generation reduced fluid equations were derived by Shmalz [1981], and Drake and Antonsen [1984] (who also considered the reduction of Braginskii's equations).

The second-generation reduced fluid equations presented in this work are the phenomenological models of Hasegawa and Wakatani [1983] and Hazelton [1983]. We also
present the reduced FLR-MHD equations (four-field model) of Hazeltine, Kotschenreuther, and Morrison [1985], which contain the first-generation RMHD models as limiting models.

We briefly review the derivation of the low-\(\beta\), incompressible ideal RMHD equations of Strauss [1976], and the high-\(\beta\), compressible ideal (resistive) RMHD equations of Strauss [1977, 1983] and Izzo et al. [1983]. It is known (White [1989]) that these reduced MHD equations can provide an adequate basis for the analysis of nonlinear tokamak phenomena as diverse as vacuum bubble formation, internal and external kink dynamics, magnetic island dynamics, and ballooning-mode physics. Because they involve a relatively small number of independent fields, RMHD equations often provide a simpler model than the full set of MHD equations.

5.1.1 Strauss’ Low-\(\beta\) RMHD Equations

The reduced MHD equations, developed by Strauss [1976], were derived for studying the nonlinear (fixed-boundary) ideal kink modes in a low-\(\beta\), large-aspect-ratio tokamak plasma (with non-circular cross section). The nonlinear evolution of these modes, as governed by the reduced equations, revealed that the magnetic field was only slightly perturbed, while large convective plasma motion was (numerically) observed. (These observations were confirmed by exact MHD numerical codes.)

We begin the derivation by giving the ideal, incompressible, low-\(\beta\) (single-fluid) MHD equations

\[
\begin{align*}
\frac{d\mathbf{V}}{dt} &= \mathbf{J} \times \mathbf{B} - \nabla P, \\
\nabla \times \mathbf{B} &= \mathbf{J}, \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}), \\
\nabla \cdot \mathbf{B} &= 0,
\end{align*}
\] (5.1)

where \(\rho\) is the fluid mass density, \(\mathbf{V}\) is the fluid velocity, \(P\) is the fluid pressure, \(\mathbf{B}\) is the total magnetic field (i.e., the sum of the equilibrium and perturbed fields), and \(\mathbf{J}\) is the total current density flowing through the plasma.

The reduction of this set of equations (involving the two independent fields \(\mathbf{V}\) and \(\mathbf{B}\)) is based on the small parameter known as the inverse aspect ratio \(\epsilon_0 \equiv a/R\), where \(a\) and \(R\) are the minor and major radii of the tokamak plasma, respectively.
Using this small parameter, the low-\(\beta\) \((\beta \sim \epsilon_0^2)\), large-aspect-ratio tokamak ordering is given as

\[
\mathcal{O}(1) : \varrho, \nabla_\perp, B_z, \\
\mathcal{O}(\epsilon_0) : B_\perp, V_\perp, J_z, \partial/\partial t, \partial/\partial z, \\
\mathcal{O}(\epsilon_0^2) : P, (B_z - B_0), J_\perp, V_\parallel,
\]

where \(B_0\) is a constant value for the toroidal magnetic field. The derivation of the reduced equations systematically ignores terms of order \(\mathcal{O}(\epsilon_0^2)\). This ordering is consistent with the shear-Alfvén time scale with \(v_A \sim \mathcal{O}(1)\) and \(\omega \sim k_\parallel \ll k_\perp\) and the compressional-Alfvén mode is automatically excluded from the analysis.

From the incompressibility constraint on the fluid velocity, \(\nabla \cdot \mathbf{V} = 0\), we give the expression for the perpendicular fluid velocity as

\[
V_\perp = \nabla_\perp U \times \hat{z},
\]

where \(U\) is the so-called stream function for the velocity, and we have ignored the parallel velocity component [assumed to be of order \(\mathcal{O}(\epsilon_0^2)\)]. Similarly, the expression for the magnetic field is given as

\[
\mathbf{B} = B_z \hat{z} + \nabla_\perp A \times \hat{z},
\]

where \(B_z\) is assumed to be large compared to the transverse component \(B_\perp = \nabla_\perp A \times \hat{z}\) and \(A\) is the poloidal stream function. In addition, up to order \(\mathcal{O}(\epsilon_0^2)\), \(B_z\) is assumed to be uniform in order to satisfy the divergenceless condition \(\nabla \cdot \mathbf{B} = 0\).

The application of the reduction process to the ideal MHD equations, based on Eqs. (5.3)–(5.4) and the low-\(\beta\) tokamak ordering, is rather simple and will not be presented here. (See Strauss [1976] for details.) The resulting reduced MHD equations, up to order \(\mathcal{O}(\epsilon_0^2)\), are given by the following expressions

\[
\frac{\partial}{\partial t} \nabla_\perp^2 U + V_\perp \cdot \nabla_\perp \nabla_\perp^2 U = \mathbf{B} \cdot \nabla_\perp \nabla_\perp^2 A, \\
\frac{\partial A}{\partial t} = \mathbf{B} \cdot \nabla U,
\]

where the parallel component of the curl of the equation of motion was used to derive Eq. (5.5), and the mass density \(\varrho\) was taken to be unity. This procedure, which annihilates the pressure-gradient term, leads to the so-called vorticity equation.
5.1. Nonlinear Reduced MHD Equations

Substituting $U = -\phi/B_z$ in Eq. (5.6), we obtain the ideal MHD constraint: $E_\parallel = \mathbf{E} \cdot \mathbf{B}/B = 0$. As we indicated before the linear time scale associated with the two-field model [Eqs. (5.5)–(5.6)] is related to the shear-Alfvén frequency $\omega = \pm k_\parallel v_A$ and consequently the model is concerned with the nonlinear shear-Alfvén dynamics.

Finally, it is simple to show that this low-$\beta$ two-field $(U, A)$ RMHD model, given by Eqs. (5.5)–(5.6), has the energy invariant

$$\frac{1}{2} \int d^3r \left( |\nabla \times U|^2 + |\nabla \times A|^2 \right). \tag{5.7}$$

The first term represents the fluid kinetic energy, while the second term represents the (magnetic) potential energy. We note that due to the incompressibility constraint the fields $P$ and $B_z$ are absent.

5.1.2 Strauss’ High-$\beta$ RMHD Equations

An obvious generalization of the low-$\beta$ (two-field) RMHD model takes into account the effects of high $\beta$ values, plasma diamagnetism, finite resistivity, finite resistivity, and/or toroidal and poloidal curvature. Strauss [1977, 1983], Shmalz [1981], and Izzo et al. [1983], among others, have considered such a generalization, and have applied their models to the study of the nonlinear evolution of ideal (or resistive) high-$\beta$ ballooning modes in a finite-aspect-ratio tokamak plasma. Numerical studies based on the exact MHD equations, performed by Izzo et al. [1985] for example, have confirmed the predictions of the set of reduced MHD equations.

We briefly review the derivation of a high-$\beta$, ideal RMHD (four-field) model, based on the works of Strauss [1977, 1983] and Izzo et al. [1983]. The ordering used here is the same as the one presented above, with the exception that the pressure now satisfies the high-$\beta$ ordering ($\beta \sim \epsilon_0$) and its evolution is governed by the equation of state

$$\frac{dP}{dt} = -\gamma_s P \nabla \cdot \mathbf{V}. \tag{5.8}$$

Because of the ordering $d/dt \sim \mathcal{O}(\epsilon_0) \sim P$ (with $\omega \simeq v_A/qR$), we only need to retain terms of order $\mathcal{O}(\epsilon_0)$ in the compressibility term $\nabla \cdot \mathbf{V}$ which requires us to retain toroidal-field nonuniformity (i.e., $\nabla B$).

The four-field model begins with the introduction of an expression for the magnetic field

$$\mathbf{B} = I_0 \nabla \xi + \nabla \times \mathbf{A} = I \nabla \xi + \mathbf{H} \times \nabla \xi, \tag{5.9}$$
where $I_0$ represents the (uniform) contribution of the vacuum toroidal magnetic field, $A$ is the perturbed magnetic vector potential, and $\xi$ is the toroidal angle. To ensure the divergenceless property of the magnetic field, we also have

$$
\mathbf{H} = \nabla \psi - \frac{\partial A}{\partial \xi}, \quad (5.10)
$$

$$
I = I_0 + R^2 \nabla \xi \cdot \nabla \times \mathbf{A}, \quad (5.11)
$$

where $\psi = R^2 \nabla \xi \cdot \mathbf{A}$ is the poloidal flux function and $R^{-2} = |\nabla \xi|^2$.

The leading-order terms obtained from the equation of motion appear in the form of a gradient of the sum of fluid pressure and magnetic pressure terms,

$$
\nabla \left( P + \frac{1}{2} B^2 \right) = \mathbf{B} \cdot \nabla \mathbf{B}, \quad (5.12)
$$

where the right-hand side corresponds to the magnetic curvature term. Up to order $O(\epsilon_0)$, the pressure balance equation becomes

$$
P + \frac{B_0 I_1}{R_0} = \text{constant}, \quad (5.13)
$$

where $I_0 = B_0 R_0$. We note that Eq. (5.13) is often used to eliminate the perturbed diamagnetic component $I_1$ in favor of the perturbed pressure $P$.

Finally, the fluid velocity is given by the expression

$$
\mathbf{V} = V_{||} \mathbf{\hat{b}} + \frac{R}{R_0} \nabla \times \mathbf{U} \times \mathbf{\hat{b}}, \quad (5.14)
$$

and the leading-order expression for the compressibility term ($\nabla \cdot \mathbf{V}$) gives

$$
\nabla \cdot \mathbf{V}_\perp = \frac{1}{R_0} (\nabla R^2 \times \nabla U) \cdot \nabla \xi, \quad (5.15)
$$

where the term $\nabla R/R_0$ is related to the toroidal curvature term $\kappa = -\nabla R/R_0 = \nabla \ln B$. Once again, the pressure balance terms are eliminated from the equation of motion by evaluating the parallel component of the curl of the equation of motion, multiplied by $R^2$. This operation produces the so-called vorticity,

$$
W \doteq \epsilon_0 \nabla \xi \cdot \nabla \times (R^2 \mathbf{V})
$$

which, to leading order, becomes $W = -\epsilon_0 / R_0 \nabla \cdot (R^2 \nabla \perp U)$. 

5.1. Nonlinear Reduced MHD Equations

The high-$\beta$ reduced MHD equations of Strauss [1977, 1983] and Izzo et al. [1983] are summarized as follows. The reduced vorticity equation is

$$\frac{D W}{D t} = B \cdot \nabla J_\xi - J \cdot \nabla I - (\nabla R^2 \times \nabla P) \cdot \nabla \xi,$$  \hspace{1cm} (5.16)

where $D/Dt = \partial/\partial t + R_0 \nabla U \times \nabla \xi \cdot \nabla$ and $I$ can be replaced with $-PR_0/B_0$. The term $J \cdot \nabla I$ is given by the expression $J \cdot \nabla I = R^{-2}(J_\xi \partial I/\partial \xi + \partial H/\partial \xi \cdot \nabla I)$, where $J_\xi = -R^2 \nabla \cdot (R^{-2}H)$. (The pressure-gradient terms appearing on the right-hand side of the vorticity equation represent the ballooning driving force.) The reduced parallel acceleration law is

$$e_0 \frac{DV_\parallel}{Dt} = -\hat{b}_0 \cdot \nabla P,$$ \hspace{1cm} (5.17)

where $\hat{b}_0 = R_0 \nabla \xi + (\nabla \psi \times \nabla \xi)/B_0$. The reduced equation for the transverse magnetic component $H$ is

$$\frac{\partial H}{\partial t} = \nabla \left( \frac{R^2}{R_0} B \cdot \nabla U \right) + \frac{1}{B_0} \left( \nabla P \frac{\partial U}{\partial \xi} - \nabla U \frac{\partial P}{\partial \xi} \right),$$ \hspace{1cm} (5.18)

and finally the reduced pressure equation is

$$\frac{dP}{dt} = -\gamma_* P \left( \hat{b}_0 \cdot \nabla V_\parallel + \nabla \cdot V_\perp \right),$$ \hspace{1cm} (5.19)

where $d/dt = \partial/\partial t + (V_\parallel \hat{b}_0 + R^2/R_0 \nabla U \times \nabla \xi) \cdot \nabla$.

This four-field ($U, V_\parallel, \psi, P$) model, given by Eqs. (5.16)–(5.19), can be shown to have the energy invariant

$$\int d^3 r \left\{ \frac{\rho_0}{2} \left[ V_\parallel^2 + (R/R_0)^2 |\nabla U|^2 \right] + \frac{1}{2} |H|^2 + \frac{P}{\gamma_* - 1} \right\}.$$ \hspace{1cm} (5.20)

We note that the incompressibility constraint, $\nabla \cdot V = 0$, implies that the parallel flow $V_\parallel$ becomes decoupled from $W$ and $H$. Finally, the low-$\beta$ two-field model, given by Eqs. (5.5)–(5.7), is recovered with the assumptions of plasma incompressibility and low $\beta$. 


5.2 Nonlinear Reduced FLR–MHD Equations

In the previous section, we presented elementary reduced fluid models derived from the ideal MHD equations. In this section we will present the derivation of three reduced fluid models based on the FLR-MHD equations. (See Appendix B for more details on these FLR-MHD equations.)

5.2.1 Model of Hasegawa and Wakatani

The first set of reduced equations we wish to consider is the set derived by Hasegawa and Wakatani [1983]. These reduced equations were derived to describe microturbulence in nonuniform magnetized plasmas of general geometry. The assumed features of this microturbulent behavior followed experimental observations: (1) \( k_\perp \rho_s \sim \mathcal{O}(1) \), so that mode-coupling effects required a nonlinear treatment; (2) \( k_\parallel R \sim \mathcal{O}(1) \), which displayed the usual anisotropy in the wavenumber spectrum; and (3) the relative fluctuation levels were assumed to be \( n_1/n_0 \sim \mathcal{O}(\rho_s/\ell_n) \) and \( B_1/B_0 \sim \mathcal{O}(\rho_s/R) \). This last feature justified the introduction of two small parameters, \( \epsilon_1 \equiv \rho_i/L_\perp \) and \( \epsilon_2 \equiv L_\perp/L_\parallel \). For a large-aspect-ratio tokamak configuration, we find \( L_\perp \approx \ell_n \approx a \) and \( L_\parallel \approx qR \), so that \( \epsilon_1 \ll \epsilon_2 \). (In our notation, we have \( \epsilon = \epsilon_1 \) and \( \epsilon_B = \epsilon_1 \epsilon_2 \).)

The nonlinear effects considered in the Hasegawa–Wakatani model are the nonlinear advection of a perturbed fluid element by the perturbed \( E \times B \) fluid velocity, and the magnetic field perturbation due to the bending of the magnetic field lines. Using the small parameters \( (\epsilon_1, \epsilon_2) \), the perturbed electromagnetic field, represented by the potentials \( (\phi, A_\parallel) \), obeys the ordering

\[
\hat{b} = B/B = \hat{z} + \mathcal{O}(\epsilon_1 \epsilon_2),
\]

\[
J_\parallel = J_z + \mathcal{O}(\epsilon_1 \epsilon_2),
\]

\[
J_\perp/J_z = \mathcal{O}(\epsilon_1 \epsilon_2),
\]

\[
e\phi/T_e = \mathcal{O}(\epsilon_1),
\]

\[
A_\parallel = A_z + \mathcal{O}(\epsilon_1 \epsilon_2).\]

Due to the ordering \( J_\perp \ll J_\parallel \), the perturbed parallel current density \( J_\parallel \) is assumed to be the source of the perturbed electromagnetic field. We note that the neglect
5.2. Nonlinear Reduced FLR-MHD Equations

of the perpendicular component of the perturbed magnetic vector potential, the so-called compressional shear-Alfvén component \( A_\perp \), is made with the assumption that \( \beta \) satisfies \( \beta < \epsilon_2 \).

The equations of Hasegawa and Wakatani [1983] are represented by a four-field model, \( (\phi, A_z, P_e, P_i) \), based on the FLR-corrected MHD equations. The one-fluid velocity is taken to be the perturbed \( E \times B \) velocity, while the electron parallel velocity is given as \( v_{||e} = -J_{||}/en_0 \) and the ion parallel velocity is neglected (\( \omega \gg k_{||} V_i \)). Finally, FLR effects are included in the following manner: due to the finiteness of its gyration radius, the ion samples the spatial variation of the perturbed electric field during its gyromotion and therefore sees an electric field which is different than what the electron sees at the ion’s guiding-center position. Consequently, the perpendicular \( E \times B \) drift velocity is different for the ions and the electrons, and a perturbed current density is created. (An expression for this current due to Rosenbluth, Krall, and Rostoker [1962] will be given below.)

5.2.1.1. Vorticity Equation

First, we give the equation governing the time evolution of the perturbed electrostatic potential \( \phi \). In RMHD theory, this equation is given by the parallel component of the curl of the MHD equation of motion, i.e., the reduced version of the vorticity equation. Considering the equivalence between the vorticity equation and the quasineutrality equation \( \nabla \cdot J = 0 \) (c.f., Tang [1978]), Hasegawa and Wakatani [1983] give the evolution equation for \( \phi \) as

\[
\nabla_\perp \cdot J_\perp + B \cdot \nabla \left( \frac{J_{||}}{B} \right) = 0,
\]

where \( J_{||} = J \cdot \hat{b} \), and \( J_\perp \) is the sum of the guiding-center drift current density (curvature, grad \( B \), and polarization currents) and the current density resulting from the difference between the ion and electron \( E \times B \) velocities. The latter current was first derived by Rosenbluth, Krall, and Rostoker [1962], and is given by the expression

\[
J_{\perp E} = \frac{en_0 c p_i^2}{B_0} \hat{z} \times \nabla_\perp \nabla_\perp^2 \phi,
\]

while the other current densities, derived from guiding-center drift theory, are given as

\[
J_{\text{curv}} + J_{\nabla B} = \sum_s \frac{c \hat{z}}{B_0} \times (P_{\perp s} \nabla \ln B + P_{|| s} \hat{z} \cdot \nabla \hat{b})
\]
and

\[ \mathbf{J}_{\text{pol}} = -\frac{c^2}{4\pi \nu_{A0}^2} \frac{d \nabla \phi}{dt}, \]

where \( d/dt = \partial/\partial t + (c/B_0) \mathbf{\hat{z}} \times \nabla \phi \cdot \nabla. \)

Up to order \( \mathcal{O}(\epsilon_1^2 \epsilon_2), \) Eq. (5.21) becomes

\[ \mathbf{\hat{b}} \cdot \nabla J_{\parallel} = -\nabla_{\perp} \cdot \mathbf{J}_{\perp} = \frac{m_i n_0 c^2}{B_0^2} \frac{d}{dt} \nabla^2 \phi + \frac{c^2}{B_0^2 \Omega_{A0}} \mathbf{\hat{z}} \times \nabla P_{\perp s} \cdot \nabla \nabla^2 \phi \]

\[ + \sum \frac{c^2}{B_0} (\nabla P_{\perp s} \times \nabla \ln B + \nabla P_{\parallel s} \times \mathbf{\hat{z}} \cdot \nabla \mathbf{b}). \]  

On the left-hand side of Eq. (5.22), we have defined

\[ \mathbf{\hat{b}} \cdot \nabla = \left( \frac{\partial}{\partial z} + \frac{B_0}{B_0} \cdot \nabla \right), \]

is the perturbed magnetic field. As a further simplification, Hasegawa and Wakatani [1983] assume that the plasma pressure is isotropic, so that \( P_{\parallel s} = P_{\perp s} = P_s \) \((s = i, e)\).

Finally, we note that it is possible to eliminate \( J_{\parallel} \) in favor of \( A_{\parallel} \) by using the parallel component of Ampere's law, appropriately given as

\[ \mathbf{\hat{b}} \cdot \nabla (\nabla^2 A_z) = -\frac{4\pi}{c} \mathbf{\hat{b}} \cdot \nabla J_{\parallel} = \frac{4\pi}{c} \nabla \nabla \cdot \mathbf{J}_{\perp}, \]

where \( \nabla_{\perp} \cdot \mathbf{J}_{\perp} \) is given by the right-hand side of Eq. (5.22). The evolution of \( \phi \) is, therefore, coupled to the evolution of \( A_z, P_i, \) and \( P_e. \)

5.2.1.2. Parallel Ohm's Law

The evolution equation for \( A_z \) is given by the parallel component of Ohm's law, obtained from the first moment of the electron drift kinetic equation. Up to order \( \mathcal{O}(\epsilon_1^2 \epsilon_2), \) it is given as

\[ c \eta J_z = -\frac{\partial A_z}{\partial t} - e \mathbf{\hat{b}} \cdot \nabla \phi + \frac{c}{e n_0} \mathbf{\hat{b}} \cdot \nabla P_e, \]

where \( \eta \) is the resistivity. The distinction between parallel and perpendicular resistivities as well as the electron-inertia terms have been neglected.

5.2.1.3. Equations of State
Finally, the equations of state, governing the evolution of the pressure of each fluid species, are given as follows. On the one hand, the ion fluid is assumed to be incompressible \( \nabla \cdot \mathbf{v}_i = \nabla \cdot \mathbf{v}_E = 0 \), so that we have
\[
\frac{dP_i}{dt} = 0.
\tag{5.25}
\]
On the other hand, the electron fluid is assumed to be isothermal, so that we use the electron continuity equation with \( \mathbf{v}_e = \mathbf{v}_E - \left( J_{\parallel}/e n_0 \right) \mathbf{\hat{b}} \), to obtain
\[
\frac{dP_e}{dt} = \frac{c T_e}{4\pi e} \mathbf{\hat{b}} \cdot \nabla \nabla_A^2 A_z.
\tag{5.26}
\]

### 5.2.1.4. Hasegawa–Wakatani Model

Equations (5.23)–(5.26) constitute the Hasegawa–Wakatani model. It has been used by various authors to study the nonlinear drift-ballooning modes in tokamak plasmas (see for example Shukla [1985]). Unfortunately, the Hasegawa–Wakatani equations do not conserve energy and consequently additional (FLR) physics must be introduced.

We note that the two-field RMHD model of Strauss [1976] is recovered from Eqs. (5.22) and (5.24) if we assume a low-\( \beta \) tokamak ordering [see Eqs. (5.5) and (5.6)]. We also note that the Hasegawa–Mima equation (Hasegawa and Mima [1977]) can be recovered from the Hasegawa–Wakatani model as follows. In Eq. (5.24), if we assume that the potential \( \phi \) is related to the electron pressure \( P_e \) according to the Boltzmann relation \( \phi = P_e/e n_0 \) (which corresponds to the electrons responding adiabatically to electrostatic perturbations) then Eq. (5.24) represents the well-known magnetic field diffusion equation and decouples from the other equations. Next, using the Boltzmann relation in Eq. (5.26) we find
\[
\frac{n_0 e^2}{T_e} \frac{\partial \phi}{\partial t} = \frac{c}{4\pi} \mathbf{\hat{b}} \cdot \nabla \nabla_A^2 A_z,
\]
which when substituted in Eq. (5.22) gives the (low-\( \beta \)) Hasegawa–Mima equation
\[
\frac{\partial \phi}{\partial t} = \rho_s^2 \left( \frac{\partial}{\partial t} + \frac{c\mathbf{\hat{b}}}{B} \times \nabla_A \phi \cdot \nabla_A \right) \nabla_A^2 \phi.
\tag{5.27}
\]
This equation has been studied extensively in plasma physics and also appears in other branches of physics [e.g., geophysics (Charney [1948])]. It is known to have nonlinear solutions which display several properties that are found in more complex models (e.g., the vector nonlinearities leading to vortices). (See Mikhailovskii [1986] for a review on vortex dynamics.)
5.2.2 Model of Hazeltine

The purpose of Hazeltine's three-field model (Hazeltine [1983]) is to combine the low-\(\beta\) two-field (RMHD) model of Strauss, given by Eqs. (5.5)–(5.6), and the Hasegawa-Mima equation, Eq. (5.27), into an energy-conserving inclusive model where the two sets of equations are coupled by FLR effects. This model also includes dissipation effects through the introduction of parallel resistivity in the parallel Ohm's law, but does not consider temperature gradients.

5.2.2.1 Hazeltine's Normalization Procedure – I

First, we give an expression for the magnetic field, normalized to the toroidal magnetic field strength at the magnetic axis \((B_T)\) and consistent with the low-\(\beta\) ordering:

\[
\frac{B}{B_T} = \tilde{z}(1 + \epsilon x)^{-1} - \epsilon \tilde{z} \nabla \psi,
\]

(5.28)

where \(\psi = A_z / \epsilon B_T a\) and \(\epsilon = a / R_0 \ll 1\). We also use the dimensionless coordinates

\[
x = (R - R_0) / a, \quad y = Z / a, \quad z = -\zeta.
\]

In terms of these coordinates, we find the vector identities (Hazeltine and Meiss [1985])

\[
a \nabla S = \nabla S + \frac{\epsilon \tilde{z}}{(1 + \epsilon x)} \frac{\partial S}{\partial z},
\]

\[
a \nabla \cdot A = \nabla \cdot A + \epsilon (1 + \epsilon x)^{-1} (\partial A_z / \partial z + A_x),
\]

\[
a \nabla \times A = \frac{\tilde{z}}{(1 + \epsilon x)} \nabla [(1 + \epsilon x) A_z] + \tilde{z} \tilde{z} \nabla \times A + \frac{\epsilon \tilde{z}}{(1 + \epsilon x)} \frac{\partial A_z}{\partial z},
\]

(5.29)

where \(S\) is an arbitrary scalar field and \(A = A_z \tilde{z} + A_\perp\) is an arbitrary vector field.

Also, the perpendicular gradient is given as \(\nabla \perp = \tilde{z} \partial / \partial x + \tilde{y} \partial / \partial y\).

Another useful expression is obtained from considering \(B \cdot \nabla\) which, using the normalization given by Eqs. (5.27)–(5.28), is given as

\[
a \frac{B}{B_T} \cdot \nabla S = \epsilon \nabla \parallel S,
\]

where \(\nabla \parallel = \partial / \partial z - \tilde{z} \times \nabla \psi \cdot \nabla\). Using the notation

\[
[f, g] = \tilde{z} \cdot (\nabla \perp f \times \nabla \perp g),
\]

(5.30)

we obtain

\[
\nabla \parallel S = \frac{\partial S}{\partial z} - [\psi, S].
\]

(5.31)
Second, we introduce the quantities

\[
\tau_A = a/v_A, \quad v_A^2 = B_T^2/4\pi n_e m_i, \\
\beta_e = 8\pi n_e T_e/B_T^2, \\
\Omega_i = eB_T/m_i c, \\
\delta = c/2\omega_p a = v_A/2a\Omega_i,
\]

where \( n_e \) represents the constant part of the fluid density and \( \delta \) gives a measure of FLR effects (\( \delta^2 \beta_e = T_e/2m_e a^2 \Omega_i^2 \)). The parameter \( \tau_A \) is used to define the time scale of interest, the so-called shear-Alfvén time scale \( \tau = \epsilon t/\tau_A \), so that the time derivative takes on the form

\[
\frac{\partial}{\partial t} = \frac{\epsilon}{\tau_A} \frac{\partial}{\partial \tau}.
\]

(5.32)

Finally, we introduce the normalized field quantities

\[
\phi = \frac{c}{v_A eB_T a} \Phi, \quad \text{and} \quad \hat{\eta} = \left( \frac{\eta c^2}{4\pi a^2} \right) \frac{a}{v_A \epsilon}.
\]

(5.33)

5.2.2.2. Component Models

The respective component models, without their couplings, are given as follows. The low-\( \beta \) resistive RMHD equations are given as

\[
\frac{\partial U}{\partial \tau} + [\phi, U] + \nabla_\parallel J = 0,
\]

(5.34)

\[
\frac{\partial \psi}{\partial \tau} + \nabla_\parallel \phi = \hat{\eta} J,
\]

(5.35)

with the notation \( U = \nabla_\parallel^2 \phi, \quad J = \nabla_\parallel^2 \psi \).

The Hasegawa–Mima equation [Eq. (5.27)] is given as

\[
\frac{\partial U}{\partial \tau} + [\phi, U] = \frac{\partial \phi}{\partial \tau},
\]

(5.36)

with the same notation as the one used above. The term on the right-hand side of Eq. (5.36) is due to the assumption that the electrons behave adiabatically.

5.2.2.3. Inclusive Nonlinear Model

We begin the derivation of the inclusive model by considering the Hall-term modification \( -B \cdot \nabla \mathcal{E}/en_e \) to the parallel Ohm's law [see Eq. (5.24)]

\[
\frac{\partial \psi}{\partial \tau} + \nabla_\parallel \phi = \hat{\eta} J + \frac{\Delta}{2c} \nabla_\parallel \left( \frac{\tilde{n}}{n_e} \right),
\]

(5.37)
where the electron density has the general form \( n = n_c + \tilde{n} \) and \( \Delta = 2\beta_e \delta \).

Next, we consider the electron equation of continuity

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0. \tag{5.38}
\]

The perpendicular velocity is assumed to be the \( E \times B \) velocity, while the parallel velocity (neglecting ion parallel flow) is \( v_{\parallel e} = -J_{\parallel}/en_c \). Using the normalization given above, Eq. (5.38) becomes

\[
\frac{\partial}{\partial \tau} \left( \frac{\tilde{n}}{n_c} \right) + \left[ \phi, \frac{\tilde{n}}{n_c} \right] + \frac{2e\alpha}{\Delta} \nabla_{\parallel} J = 0,
\]

where \( \alpha = \delta \Delta = \rho_s^2 / a^2 \). If we replace \( \tilde{n}/n_c \) by the quantity \( \chi = (2e\delta)^{-1} \tilde{n}/n_c \), we finally obtain

\[
\frac{\partial \chi}{\partial \tau} + [\phi, \chi] + \nabla_{\parallel} J = 0. \tag{5.39}
\]

Making use of the same substitution in the parallel Ohm's law, Eq. (5.37) becomes

\[
\frac{\partial \phi}{\partial \tau} + \nabla_{\parallel} \phi = \tilde{n}J + \alpha \nabla_{\parallel} \chi. \tag{5.40}
\]

Finally, with the second RMHD equation [the vorticity equation given by Eq. (5.34)]

\[
\frac{\partial U}{\partial \tau} + [\phi, U] + \nabla_{\parallel} J = 0. \tag{5.41}
\]

we obtain a three-field model which contains the two-field RMHD model and the Hasegawa–Mima equation. It represents the simplest model with such a feature. In addition, by neglecting dissipation effects (\( \tilde{n} = 0 \)) in Eq. (5.40), it is simple to show that the three-field model has the energy invariant

\[
\frac{1}{2} \int d^3 r \left( |\nabla_{\perp} \phi|^2 + |\nabla_{\perp} \psi|^2 + \alpha \chi^2 \right).
\]

The first two terms in the expression for the energy invariant are well known from low-\( \beta \) RMHD theory. The last term \( \alpha \chi^2 / 2 \) appears as a result of the isothermal assumption, which allows the replacement of the pressure perturbation with the density perturbation.

In the limit where the FLR coefficient \( \alpha \) vanishes, the continuity equation, Eq. (5.39), is decoupled from the other two (RMHD) equations. On the other hand, when the electrons respond adiabatically to the electrostatic perturbation (\( \phi = \alpha \chi \)), Eq. (5.39)
becomes $\partial X/\partial \tau = -\nabla \| J$ and when it is substituted in the vorticity equation we obtain the Hasegawa–Mima equation, Eq. (5.36). Furthermore, Ohm’s law, Eq. (5.40), gives the ordinary resistive diffusion equation for the magnetic field $\partial \psi / \partial \tau = n \nabla^2 \psi$.

In order to appreciate the nature of the nonlinear dynamics described by the three-field model, it is useful to consider the linear version of the model in which the effect of an equilibrium density gradient is introduced. In this case we obtain the well-known dispersion relation

$$\left(1 - \frac{\omega_{SA}^2}{\omega^2}\right) + \frac{\omega_D}{\omega} \left(1 - \frac{\omega_{SA}^2}{\omega^2}\right) = 0,$$

where $\omega_{SA} = \pm k_\parallel v_A$ is the shear-Alfvén frequency, $\omega_{KSA} = \omega_{SA}(1 + k_\parallel^2 \rho_A^2)^{1/2}$ is the kinetic shear-Alfvén frequency, and the diamagnetic drift frequency is defined as

$$\omega_D \equiv -i \frac{cT_e}{eB} \vec{b} \times \nabla \ln n \cdot \nabla \perp.$$

It is simple to see that when $\omega_D$ vanishes (which is the case considered by the three-field model) one is left with the kinetic shear-Alfvén linear waves. The three-field model therefore investigates the nonlinear dynamics of shear-Alfvén waves. Because it does not describe linear drift modes the three-field model has limited practical interest since we are interested in a nonlinear model which enables us to study nonlinear drift-Alfvén dynamics while still having the appropriate linear physics.

### 5.2.3 Model of Hazeltine, Kotschenreuther, and Morrison

The following four-field model, derived by Hazeltine, Kotschenreuther, and Morrison [1985], was designed to generalize the RMHD equations by consistently include FLR effects. Its general purpose is to study nonlinear drift–shear–Alfvén dynamics in nonuniform magnetized plasmas.

The Hazeltine–Kotschenreuther–Morrison (HKM) four-field model considers the following FLR corrections to the continuity and momentum equations: (1) the fluid velocity is given by the expression $\vec{V} = V_\parallel \vec{b} + \vec{V}_E + \vec{V}_D$; and (2) the gyroviscous stress tensor $\Pi_{\text{g},ij}$, which provides the well-known gyroviscous cancellations. The HKM model was subsequently generalized by Hsu, Hazeltine, and Morrison [1986] (referred to as HsHM), by considering all terms of order $\mathcal{O}(\rho_e^2/a^2)$ through the equation for the pressure tensor equation (third moment of the Boltzmann equation).
The presentation given here will concentrate on the dissipationless version of the first derivation (HKM) of the model.

5.2.3.1. Hazeltine’s Normalization Procedure – II

Using the Hazeltine normalization introduced earlier in 5.2.2.1, we give the following expression for the magnetic field

$$
\frac{B}{B_T} = \hat{z}[1 + \epsilon x]^{-1} + eb - \epsilon \hat{z} \times \nabla \psi + \mathcal{O}(\epsilon^2),
$$

(5.42)

where we have included the effect of plasma diamagnetism, through the introduction of the term $b$. We complete our normalization procedure by introducing two fields, absent from the three-field model: (1) the parallel fluid velocity

$$
V_\parallel = \epsilon v_A v, 
$$

(5.43)

and (2) the electron pressure (with uniform temperature $T_e$),

$$
P_e = n_e T_e + \epsilon \frac{B_T^2}{8\pi} p, 
$$

(5.44)

with the ion pressure given as $P_i = (T_i/T_e) P_e$. Finally, using Eq. (5.42), we give a useful expression for the current density [correct up to order $\mathcal{O}(\epsilon^2)$]

$$
J = -\frac{cB_T}{4\pi a} \left[ \epsilon \left( \hat{z} \times \nabla b + \hat{z} \nabla^2 \psi \right) + \epsilon^2 \left( \hat{y} b - \nabla_\perp \frac{\partial \psi}{\partial z} \right) \right], 
$$

(5.45)

so that

$$
\frac{J \times B}{c} = \frac{B_T^2}{4\pi a} \left\{ -\epsilon \nabla_\perp \left[ b \left( 1 + eb/2 \right) \right] + \epsilon^2 \left[ 2\pi \nabla_\perp b - \nabla_\perp (zb) - \hat{z} \times \nabla_\perp \frac{\partial \psi}{\partial z} \right. 
\right.
\left. - \nabla_\perp \psi \nabla^2_\perp \psi - \hat{z} \cdot (\nabla_\perp \psi \times \nabla_\perp b) \right\}. 
$$

(5.46)

Furthermore, the electric field is given as

$$
E_\perp = -\epsilon \frac{B_T v_A}{c} \nabla_\perp \phi + \mathcal{O}(\epsilon^2),
$$

(5.47)

$$
E_\parallel = -\epsilon^2 \frac{B_T v_A}{c} \left( \frac{\partial \psi}{\partial \tau} + \nabla_\parallel \phi \right) + \mathcal{O}(\epsilon^3).
$$

(5.48)

We now proceed with the derivation of three $\mathcal{O}(\epsilon^2)$ equations of motion, describing shear-Alfvén physics, obtained from the momentum equation. The first equation
we shall derive is the parallel acceleration law, the second equation is the so-called vorticity equation, and the third equation is Ohm's law, derived from the electron momentum equation. The fourth equation — the equation of state — is obtained from the electron continuity equation and will complete the closed set of equations describing the evolution of the four fields $\phi, \psi, v$, and $p$.

5.2.3.2. Momentum Equation

First, we consider the momentum equation for fluid species $s$,

$$m_s n_s \frac{d}{dt} V_s = m_s n_s \left( \frac{\partial}{\partial t} + V_s \cdot \nabla \right) V_s$$

$$= -\nabla \cdot P_s + e_s n_s \left( E + \frac{V_s \times B}{c} \right),$$

(5.49)

where we have neglected dissipative effects (e.g., the collisional friction force). The pressure tensor $P$ is assumed to be of the form

$$P_s = P_s I + \Pi_{gs},$$

(5.50)

where $P_s = n_s T_s$ is the scalar pressure, and $\Pi_{gs}$ is the (collisionless) gyroviscous part of the pressure tensor. (We note again that the HKM model neglects temperature-gradient effects, so that the pressure evolution is governed by the continuity equation, which will be dealt with later.) The model also assumes quasineutrality, so that the relation $n_i = n_e = n$ is used in what follows.

Finally, the fluid velocity is given by the expression

$$V_s = V_{\parallel s} + V_E + V_{Ds},$$

(5.51)

where

$$V_E = \frac{c}{B^2} E \times B \quad \text{and} \quad V_{Ds} = \frac{1}{n m_s \Omega_s B} B \times \nabla P_s.$$  

(5.52)

The gyroviscous tensor $\Pi_{gs}$ [see Appendix B, Eq. (B.24)], was given in terms of the (incompressible) fluid velocity by the expression

$$\Pi_{gs} = \frac{P_s}{4 \Omega_s} \left[ (\vec{z} \times \nabla V_{\perp s} - \nabla V_{\perp s} \times \vec{z}) + \text{(transpose)} \right]$$

$$+ \frac{P_s}{\Omega_s} \left[ \vec{z} \times \nabla V_{\parallel s} + \text{(transpose)} \right].$$


This expression, when combined with the total (convective) time derivative of the ion diamagnetic drift, gives the so-called gyroviscous cancellation term (see for example Hazeltine and Meiss [1985])

$$m_i n \frac{d}{dt} V_{Di} + \nabla \cdot \Pi_{gi} = -\nabla \left[ \frac{P_i}{2 \Omega_i} (\hat{z} \cdot \nabla \times V_{\perp i}) \right] - V_{Di} \cdot \nabla V_{\parallel i}. \quad (5.53)$$

The second term on the right-hand side of Eq. (5.53) will effectively cancel the diamagnetic contribution to the convective derivative of the parallel velocity. Up to order $\mathcal{O}(\varepsilon^2)$, the momentum equation, Eq. (5.49), therefore becomes

$$m_i n \left[ \left( \frac{\partial}{\partial t} + V_e \cdot \nabla \right) V_E + \left( \frac{\partial}{\partial t} + V_e \cdot \nabla \right) V_{\parallel e} \right]$$

$$+ \nabla \left[ P_e \left( 1 - \frac{B}{2 \Omega_i B} \cdot \nabla \times V_e \right) \right] = n_e \left( \frac{1}{c^2} \frac{V_e \times B}{2} \right).$$

By summing this equation over species, we get rid of the electric field (as a result of the quasineutrality condition), and we obtain

$$m_i n \left( \frac{d}{dt} V_e + \frac{d}{dt} V_{\parallel i} \right) + \nabla P - \nabla \left( \frac{P_i}{2 \Omega_i B} B \cdot \nabla \times V_i \right) = \frac{J \times B}{c}, \quad (5.55)$$

where $dE/dt = \partial/\partial t + V_e \cdot \nabla$. (In this work, the electron FLR terms are completely neglected by taking $m_e = 0$.)

The first two terms on the left-hand side of Eq. (5.55) are of order $\mathcal{O}(\varepsilon^2)$, while the last two terms are of order $\mathcal{O}(\varepsilon)$. Hence, using the Hazeltine normalization, we have the following $\mathcal{O}(\varepsilon)$ terms

$$\frac{J \times B}{c} = -\frac{B_i^2}{4 \pi a} \nabla_{\perp} b,$$

$$\nabla P = \frac{B_i^2}{8 \pi a} (1 + T_i/T_e) \nabla_{\perp} P,$$

$$\nabla \left( \frac{P_i}{2 \Omega_i B} B \cdot \nabla \times V \right) = \frac{B_i^2}{8 \pi a} \frac{T_i}{T_e} \delta \beta_e \nabla_{\perp} \left( \nabla_{\perp} \phi^* \right).$$

In the last equation, we have used the lowest-order incompressible part of the perpendicular ion fluid velocity,

$$V_{\perp i} = \epsilon v_A \hat{z} \times \nabla_{\perp} \phi^* + \mathcal{O}(\varepsilon^2), \quad (5.56)$$
where $\phi^* = \phi + (T_i/T_e)\delta p$. To order $O(\epsilon)$, Eq. (5.55) gives the so-called pressure balance equation

$$\nabla_\perp b = -\frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) \nabla_\perp p + \frac{\delta \beta_e}{2} \frac{T_i}{T_e} \nabla_\perp (\nabla_\perp^2 \phi^*),$$

which represents the (lowest-order) compressional Alfvén equilibration [compare with Eq. (5.13)]. (We note that Eq. (5.57) contains a term of order $O(\beta_e)$, which is missing from the equation given by HKM, but reappears in HsHM.)

Each species $s$ has a parallel component associated with its velocity field, $V_{||s} = \vec{b} \cdot \vec{V}_s$. From these components, one can define the parallel mass flow velocity

$$MV_{||} = m_i V_{||i} + m_e V_{||e},$$

where $M = m_i + m_e$, and the parallel current density

$$J_{||} = e n (V_{||i} - V_{||e}).$$

If we neglect the electron mass ($m_e = 0$), the ion parallel velocity, therefore, corresponds to the parallel mass flow velocity. Using the Hazeltine normalization, the scalar product of the magnetic field $B$ with Eq. (5.55) gives, up to order $O(\epsilon^2)$, the parallel acceleration law

$$\frac{\partial v}{\partial \tau} + [\phi, v] + \frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) \nabla_\parallel p = \frac{\delta \beta_e}{2} \frac{T_i}{T_e} \nabla_\parallel (\nabla_\parallel^2 \phi^*).$$

Once again, the HKM parallel acceleration law does not contain the term on the right-hand side of the expression shown here. However, to ensure energy conservation, HKM use the parallel-velocity moment of the ion drift-kinetic equation (which contains the total magnetic drift velocity) to obtain the (modified) reduced parallel acceleration law given as

$$\frac{\partial v}{\partial \tau} + [\phi, v] + \frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) \nabla_\parallel p = \delta \beta_e \frac{T_i}{T_e} \left\{ \frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) [p, v] + 4[x, v] \right\},$$

where the pressure balance equation, Eq. (5.57), was used to eliminate $b$ in favor of $p$. (The $O(\beta_e)$ term in the pressure balance equation is totally ignored by HKM, reappears in HsHM, and is also present in the Hamiltonian formulation of HKM by Hazeltine, Hsu, and Morrison [1987].)

The vorticity equation is obtained by taking the parallel component of the curl of the momentum equation, Eq. (5.55). As is well known, the curl effectively takes the
\( \mathcal{O}(\epsilon) \) terms out of the momentum equation, since they appear in the form of gradients. The remaining terms are, therefore, of order \( \mathcal{O}(\epsilon^2) \) and are given as follows

\[
B \cdot \nabla \times \left( \frac{J \times B}{c} \right) = \epsilon^2 \frac{B_T^2}{4\pi a^2} (\nabla_\parallel \nabla_\perp^2 \psi + 2[b, x]), \quad (5.61)
\]

\[
B \cdot \nabla \times \left[ m_e n \left( \frac{\partial}{\partial t} + V_e \cdot \nabla \right) V_E \right] = \epsilon^2 \frac{B_T^2}{4\pi a^2} \left( \frac{\partial \nabla_\perp^2 \phi}{\partial \tau} + \tilde{\varepsilon} \times \nabla \phi^* \cdot \nabla_\perp \nabla_\perp^2 \phi \right) + \delta \frac{T_i}{T_e} \tilde{\varepsilon} \times \nabla \nabla p : \nabla \nabla \phi. \quad (5.62)
\]

Using the notation \( U = \nabla_\parallel^2 \phi \) and \( J = \nabla_\perp^2 \psi \), and ignoring terms of order \( \mathcal{O}(\beta_e) \), we obtain the reduced vorticity equation, given by HKM as

\[
\frac{\partial U}{\partial \tau} + \left[ \phi + \delta \frac{T_i}{T_e} p, U \right] + \nabla_\parallel J + \left( 1 + \frac{T_i}{T_e} \right) [x, p] = \delta \frac{T_i}{T_e} [\nabla_\perp \phi; \nabla_\perp p], \quad (5.63)
\]

where \( [A; B] \equiv \sum_i [A_i, B_i] \).

Finally, the electron momentum equation, Eq. (5.54) with \( m_e = 0 \), is used to obtain the reduced generalized Ohm's law. By considering the parallel component of the electron momentum equation, with the electron-inertia terms neglected, we easily obtain to order \( \mathcal{O}(\epsilon^2) \)

\[
\frac{\partial \phi}{\partial \tau} + \nabla_\parallel \phi = \delta \nabla_\parallel p. \quad (5.64)
\]

### 5.2.3.3. Continuity Equation

The fourth reduced equation is concerned with the evolution of the perturbed pressure. Since the temperature is assumed to be constant, we find that the perturbed electron pressure \( \tilde{P}_e \) is related to the perturbed density \( \tilde{n} \) according to the expression \( \tilde{P}_e = \tilde{n} T_e \). The perturbed pressure, therefore, evolves according to the density evolution, described by the continuity equation

\[
\frac{\partial n}{\partial t} + V_e \cdot \nabla n = -n \nabla \cdot V_e. \quad (5.65)
\]

A quick look at the normalized form of this equation

\[
\frac{\partial p}{\partial \tau} + [\phi, p] = -\left( \frac{\beta_e}{\epsilon} \right) a \nabla \cdot u_e,
\]

where \( V_e = \epsilon v_A u_e \), reveals that the consideration of high \( \beta \) values (i.e., \( \beta_e \sim \epsilon \)) requires the consideration of plasma compressibility. The low-\( \beta \) case \( (\beta_e \sim \epsilon^2) \) normally reduces to

\[
\frac{\partial p}{\partial \tau} + [\phi, p] = 0.
\]
The evaluation of the finite-compressibility term, \( \nabla \cdot \mathbf{V}_e \), is done as follows. The electron fluid velocity is given as \( \mathbf{V}_e = \mathbf{V}_{\parallel e} + \mathbf{V}_E + \mathbf{V}_{De} \), where

\[
\mathbf{V}_E = c \frac{\mathbf{E} \times \mathbf{B}}{B^2} \quad \text{and} \quad \mathbf{V}_{De} = \frac{c}{n_e e B^2} \mathbf{B} \times \nabla \mathbf{P}_e.
\]

The divergence of \( \mathbf{V}_E \) is given by the expression

\[
\nabla \cdot \mathbf{V}_E = c \mathbf{E} \times \mathbf{B} \cdot \nabla \left( \frac{1}{B^2} \right) - \frac{1}{B} \frac{\partial B}{\partial t} - \frac{4\pi}{B^2} \mathbf{J} \cdot \mathbf{E},
\]

where Maxwell's equations were used. The reduced, normalized form of this expression is given, to order \( \mathcal{O}(\epsilon^2) \), as

\[
\nabla \cdot \mathbf{V}_E = \epsilon^2 \frac{v_A}{a} \left( 2[\phi, x] - \frac{\partial b}{\partial r} - [\phi, b] \right). \tag{5.66}
\]

Likewise, the divergence of \( \mathbf{V}_{De} \) is given by the expression

\[
\nabla \cdot \mathbf{V}_{De} = -\frac{c}{n_e e} \nabla \mathbf{P}_e \cdot \left[ \frac{\nabla \times \mathbf{B}}{B^2} + \nabla \left( \frac{1}{B^2} \right) \times \mathbf{B} \right],
\]

which acquires the \( \mathcal{O}(\epsilon^2) \) reduced, normalized form

\[
\nabla \cdot \mathbf{V}_{De} = -\epsilon^2 \frac{\delta v_A}{a} ([p, b] + 2[p, x]). \tag{5.67}
\]

Finally, the electron parallel velocity is given as \( \mathbf{V}_{\parallel e} = \mathbf{V}_{\parallel} + J_{\parallel}/en_e \) (where electron inertia is neglected). The divergence of \( \mathbf{V}_{\parallel e} \) gives

\[
\nabla \cdot \left( \frac{\mathbf{V}_{\parallel e}}{B} \right) = \mathbf{B} \cdot \nabla \left( \frac{\mathbf{V}_{\parallel e}}{B} \right),
\]

which acquires the \( \mathcal{O}(\epsilon^2) \) reduced, normalized form

\[
\mathbf{B} \cdot \nabla \left( \frac{\mathbf{V}_{\parallel e}}{B} \right) = \epsilon^2 \frac{v_A}{a} \nabla_{\parallel} (v + 2\delta J). \tag{5.68}
\]

Upon using the pressure balance equation of HKM [without the \( \mathcal{O}(\beta_e) \) term in Eq. (5.57)], and upon redefining the plasma \( \beta \) as \( \beta = \beta_e [1 + (1 + T_i/T_e)\beta_e/2]^{-1} \), we obtain the reduced pressure equation

\[
\frac{\partial p}{\partial r} + [\phi, p] = \beta \{ 2[x, \phi - \delta p] - \nabla_{\parallel}(v + 2\delta J) \}. \tag{5.69}
\]
We note that since $\beta = \beta_e + \mathcal{O}(\beta_e^2)$, $\beta_e$ is replaced by $\beta$ whenever it appears in the first three equations, Eqs. (5.60), (63), and (64).

5.2.3.4. Four-field Reduced Equations

The HKM equations are summarized as follows. The reduced vorticity equation, involving $U = \nabla^2_\perp \phi$ and $J = \nabla^2_\parallel \psi$, is

$$\frac{\partial U}{\partial \tau} + \left[ \phi + \delta \frac{T_i}{T_e} p, U \right] + \nabla_\parallel J + \left( 1 + \frac{T_i}{T_e} \right) [h, p] = \delta \frac{T_i}{T_e} \left[ \nabla_\perp \phi, \nabla_\perp p \right], \quad (5.70)$$

the reduced generalized Ohm's law is

$$\frac{\partial \psi}{\partial \tau} + \nabla_\parallel \phi = \delta \nabla_\parallel p, \quad (5.71)$$

the reduced parallel acceleration law is

$$\frac{\partial v}{\partial \tau} + [\phi, v] + \frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) \nabla_\parallel p = \delta \beta \frac{T_i}{T_e} \left\{ \frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) [p, v] + 4[h, v] \right\}, \quad (5.72)$$

and, finally, the internal energy conservation law is

$$\frac{\partial p}{\partial \tau} + [\phi, p] = \beta \{ 2[h, \phi - \delta p] - \nabla_\parallel (v + 2\delta J) \}, \quad (5.73)$$

where $h \equiv x$ is used.

This four-field model clearly reduces to the three-field model of Hazeltine [given by Eqs. (5.34), (40), and (41)] if we neglect magnetic nonuniformity ($h \equiv 0$), ion parallel flow ($v \equiv 0$), and the ion diamagnetic convective derivative term appearing in the vorticity equation. In addition, we substitute $p = \Delta \chi$ and $\delta p = \alpha \chi$.

The HKM vorticity equation, Eq. (5.70), also reduces to the vorticity equation, Eq. (5.22), in the Hasegawa-Wakatani model, except for the term on the right-hand side of the HKM equation, which is absent from the Hasegawa-Wakatani equation. This term, as will be shown later, is necessary for energy conservation. (We note that this energy-conserving term also appears in other models such as the electrostatic model of Horton, Estes, and Biskamp [1983] and the ion-pressure-gradient model of Connor [1986].) We, therefore, conclude that the Hasegawa-Wakatani model is not energy conserving. In addition, the Hasegawa-Wakatani model does not consider ion parallel motion and plasma compressibility.

5.2.3.5. Limiting Models of the HKM Model
We shall consider here various limits of the HKM model, with the lowest model being the RMHD two-field model of Strauss.

The two-field RMHD energy invariant is obtained by multiplying the vorticity equation and Ohm's law by $(-\phi)$ and $(-\nabla_1^2 \psi)$, respectively. Similarly, if we multiply the HKM vorticity equation, Eq. (5.70), and Ohm's law, Eq. (5.71), by $\phi$ and $J$, respectively, and integrate over the plasma volume, denoted as $\int d^3 x f \equiv \langle f \rangle$, we obtain

\[
- \left\langle \phi \frac{\partial U}{\partial \tau} \right\rangle = \langle \phi \nabla_\parallel J \rangle + \left( 1 + \frac{T_i}{T_e} \right) \langle \phi [h, p] \rangle, \tag{5.74}
\]
\[
- \left\langle J \frac{\partial \phi}{\partial \tau} \right\rangle = \langle J \nabla_\parallel \phi \rangle - \delta \langle J \nabla_\parallel p \rangle, \tag{5.75}
\]

where surface terms are assumed to vanish. (The model does not consider externally-imposed fluctuating fields.) The low-$\beta$ incompressible two-field RMHD model of Strauss [1976], given by Eqs. (5.5)--(5.6), is regained when we set $h = p = v = 0$ in Eqs. (5.70) and (5.71), and the energy invariant is given by Eq. (5.7). As was indicated earlier, the HKM model also reduces to the inclusive three-field model in the appropriate limit.

Next, HKM considers additional quadratic expressions in the energy invariant and whose time derivative are given as

\[
A_1 \left\langle \frac{\partial p}{\partial \tau} \right\rangle = 2\beta A_1 \langle p [h, \phi] \rangle - \beta A_1 \langle p \nabla_\parallel (v + 2\delta J) \rangle, \tag{5.76}
\]
\[
B_1 \left\langle \frac{\partial v}{\partial \tau} \right\rangle = -\frac{B_1}{2} \left( 1 + \frac{T_i}{T_e} \right) \langle v \nabla_\parallel p \rangle,
\]
\[
C_1 \left\langle \frac{\partial \psi}{\partial \tau} \right\rangle = C_1 \langle h [p, \phi] \rangle - \beta C_1 \langle h \nabla_\parallel (v + 2\delta J) \rangle,
\]

where $(A_1, B_1, C_1)$ are to be determined from the requirement of energy conservation.

The high-$\beta$ RMHD limit ($\delta = 0$) requires that $A_1 = B_1 = 0$ and $C_1 = -(1 + T_i/T_e)$, so that the energy invariant, for this limit, is given by the expression

\[
E_{\text{RMHD}}^{(\text{high})} = \frac{1}{2} \left\langle | \nabla_\perp \psi |^2 + | \nabla_\perp \phi |^2 - 2 \left( 1 + \frac{T_i}{T_e} \right) h p \right\rangle, \tag{5.77}
\]

where the last term represents the expression for the internal energy, and the first two terms represent kinetic and potential (magnetic) energies. The compressible reduced
MHD limit ($\delta = 0, T_i = T_e$) requires that $A_1 = \beta^{-1}$, $B_1 = 1$, $C_1 = 0$, so that the energy invariant for this limit is given by the expression

$$E_{CRMHD} = \frac{1}{2} \left( |\nabla \phi|^2 + |\nabla \psi|^2 + \frac{p^2}{\beta} + v^2 \right).$$  

(5.78)

Finally, the cold-ion limit ($T_i = 0$) requires that $A_1 = \beta/2$, $B_1 = 1$, and $C_1 = 0$, so that the energy invariant for this limit is given by the expression

$$E_{CRMHD}^{(cold)} = \frac{1}{2} \left( |\nabla \phi|^2 + |\nabla \psi|^2 + \frac{p^2}{2\beta} + v^2 \right).$$  

(5.79)

We note that the proofs of these conservation laws depend on the identity

$$\langle K[L,M] \rangle = \langle L[M,K] \rangle = \langle M[K,L] \rangle,$$

(5.80)

where $K$, $L$, and $M$ are arbitrary functions in real space. In particular, since we have $\nabla_\perp^2 h = 0$ (i.e., $h = x$), we have the identity

$$\langle h[K,\nabla_\perp^2 K] \rangle = 0.$$

(5.81)

5.2.3.6. Energy Conservation Property of the HKM Model

Before we proceed with the derivation of the HKM energy invariant, we comment on the energy-conserving term in the vorticity equation, Eq. (5.70). Consider the expression $\langle \phi[p,\nabla_\perp^2 \phi] - [\nabla_\perp \phi;\nabla_\perp p] \rangle$, which can easily be integrated by parts to give

$$-\langle \phi \nabla_\perp [\nabla_\perp \phi, p] \rangle = \langle \nabla_\perp \phi \cdot [\nabla_\perp \phi, p] \rangle.$$  

The term on the right-hand side obviously vanishes, as a result of the identity given by Eq. (5.80). This shows that the term $[\nabla_\perp \phi;\nabla_\perp p]$ appearing in Eq. (5.70) is needed to ensure energy conservation.

The expression for the energy invariant of the HKM model involves the use of a sixth expression, involving the evolution of $\psi$ and $v$ and a fourth parameter $D_1$, given as

$$D_1 \left\{ \frac{\partial \psi}{\partial t} + \psi \frac{\partial v}{\partial t} \right\} = -D_1 \left\{ v \frac{\partial \phi}{\partial z} \right\} - D_1 \left( \frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) \right) \left\{ \frac{\partial p}{\partial z} \right\}$$

$$+ \delta D_1 \langle v \nabla_\parallel p \rangle + D_1 \delta \beta \frac{T_i}{T_e} \left\{ \frac{1}{2} \left( 1 + \frac{T_i}{T_e} \right) \langle \psi [p,v] \rangle + 4 \langle \psi [h,v] \rangle \right\}.$$  

(5.82)
The HKM energy invariant is thus given in the following form

$$
E_{\text{HKM}} = \left\langle \frac{1}{2} \left\{ | \nabla_\perp \phi |^2 + | \nabla_\perp \psi |^2 + A_1 p^2 + B_1 v^2 \right\} + C_1 h \rho + D_1 \psi v \right\rangle. \tag{5.83}
$$

The expression for the time derivative of $E_{\text{HKM}}$ is given as

$$
\frac{dE_{\text{HKM}}}{dt} = -\delta (\langle J \nabla_\| p \rangle + 2\beta A_1 \langle p \nabla_\| J \rangle) + \left\{ (1 + T_i/T_e) \langle \phi [h, p] \rangle + 2\beta A_1 \langle p [h, \phi] \rangle + C_1 \langle h [p, \phi] \rangle \right\} \\
- \beta (C_1 \langle h \nabla_\| v \rangle - 4D_1 \delta T_i/T_e \langle \psi [h, v] \rangle) - 2C_1 \delta \beta \langle h \nabla_\| J \rangle - D_1 \left\langle v \frac{\partial \phi}{\partial z} \right\rangle \\
- \frac{D_1}{2} \left( 1 + \frac{T_i}{T_e} \right) \left\langle \psi \frac{\partial p}{\partial z} \right\rangle + \left\{ -\frac{B_1}{2} \left( 1 + \frac{T_i}{T_e} \right) \langle v \nabla_\| p \rangle - \beta A_1 \langle p \nabla_\| v \rangle + \delta D_1 \langle v \nabla_\| p \rangle \\
+ \frac{D_1}{2} \delta \beta \frac{T_i}{T_e} \left( 1 + \frac{T_i}{T_e} \right) \langle \psi [p, v] \rangle \right\},
$$

where we have used the expressions given by Eqs. (5.74)-(5.76), and (82).

The choice made by HKM for the parameters $(A_1, B_1, C_1, D_1)$ is

$$A_1 = (2\beta)^{-1}, \quad B_1 = \beta T_i/4T_e + \frac{1}{2} (1 + T_i/T_e)^{-1}, \quad C_1 = -T_i/T_e, \quad D_1 = -1/4\delta, \tag{5.84}$$

which, when substituted in the expression for $dE_{\text{HKM}}/dt$, gives

$$
\frac{dE_{\text{HKM}}}{dt} = \frac{(1 + T_i/T_e)}{8\delta} \left\langle \left( \psi - \delta \beta \frac{T_i}{T_e} v \right) \frac{\partial p}{\partial z} \right\rangle + \frac{1}{4\delta} \left\langle v \frac{\partial \phi}{\partial z} \right\rangle. \tag{5.85}
$$

In deriving Eq. (5.85), we have used the expression $\partial h/\partial z = 0$, representing the equilibrium (vacuum) expression for $\nabla \cdot B = 0$. It is obvious that Eq. (5.85) does not vanish, unless we are dealing with axisymmetric perturbations (i.e., $\partial/\partial z \equiv 0$). (See erratum by Hazeltine, Kotschenreuther, and Morrison [1986].) In this case, the expression for the energy invariant for the axisymmetric HKM equations is given as

$$
E_{\text{HKM}} = \frac{1}{2} \left\langle \left| \nabla_\perp \phi \right|^2 + \left| \nabla_\perp \psi \right|^2 + \frac{p^2}{\beta} + \left( \frac{T_e}{(T_e + T_i)} + \frac{\beta T_i}{2T_e} \right) v^2 - 2 \frac{T_i}{T_e} h \rho - \frac{\psi v}{2\delta} \right\rangle. \tag{5.86}
$$
Chapter 6

Nonlinear Gyrokinetic Moment Equations

Nonlinear reduced equations suitable for the study of finite-\(\beta\) nonlinear tokamak physics are presented. These third-generation reduced equations are derived as moments of the gyrokinetic Maxwell–Vlasov equations, obtained through the use of the phase-space Lagrangian Lie perturbation method. Fully electromagnetic perturbative effects, up to order \(O(\epsilon_B^2)\), as well as toroidal and FLR effects, up to order \(O(\epsilon_B \epsilon_s)\), are retained.

In order to close the moment-equation hierarchy, we use a simple closure scheme based on the small \((k_\perp \rho_i)\) ordering, in which terms up to order \(O(k_\perp^2 \rho_i^2)\) are retained. The resulting closed set of reduced gyrofluid (Lee [1989]) equations is then compared with two sets of FLR-MHD reduced equations: (1) the four-field model of Hasegawa and Wakatani [1983]; and (2) the four-field model of Hazeltine, Kotschenreuther, and Morrison [1985].

6.1 Nonlinear Gyrofluid Equations for Tokamak Plasmas

The reduced equations presented in this chapter are obtained from the moment equations of the gyrokinetic Maxwell–Vlasov equations, which were derived in chapters 3 and 4, through the use of the phase-space Lagrangian Lie perturbation method (with the symplectic representation). The gyrokinetic description provides a simple
and rigorous reduction scheme for the derivation of reduced fluid equations, based on the gyrokinetic ordering for tokamak plasmas (see 1.1.2.), and thus allows a more accurate description of low-frequency ($\omega \ll \omega_{ci}$), small-perpendicular-wavelength ($k_{\perp} < \rho_{i}^{-1} \gg a^{-1}$) nonlinear tokamak fluid dynamics (such as drift waves and high-$n$ ballooning modes).

The gyrofluid equations are presented here in their dissipationless form. The collisional version awaits further developments in the derivation of a gyrokinetic collision operator, as presented in 4.4.1. Finally, a simple closure scheme is used to obtain a closed set of gyrofluid equations.

6.1.1 Velocity Moments in Physical and Gyrocenter Phase Spaces

6.1.1.1 Linear Gyrokinetic Maxwell–Vlasov Equations

The gyrokinetic Vlasov equation was given in Eq. (4.5) as

$$\{F, H\} = 0 = \frac{\partial F}{\partial t} + \{F, \dot{H}\} - \frac{1}{B} \frac{\partial (\delta A_{\parallel})}{\partial t} \frac{\partial F}{\partial \rho_{\parallel}},$$

where the extended Poisson bracket, Eq. (3.5), was used. The symplectic representation is used because the parallel inductive term $\partial \delta A_{\parallel}/\partial t$ appears explicitly, which will be needed for the reduced form of the generalized Ohm's law. The gyrokinetic Vlasov equation was also given in the more conventional form, Eq. (4.9), by the expression

$$\frac{\partial}{\partial t}(\Omega B_{\parallel}^* F) + \nabla \cdot (\Omega B_{\parallel}^* \dot{X} F) + \frac{\partial}{\partial \rho_{\parallel}}(\Omega B_{\parallel}^* \dot{\rho}_{\parallel} F) = 0,$$

(6.1)

where we have used the well-known Liouville property of phase-space conservation by the Hamiltonian flow, Eq. (3.53),

$$\frac{\partial}{\partial t}(\Omega B_{\parallel}^*) + \nabla \cdot (\Omega B_{\parallel}^* \dot{X}) + \frac{\partial}{\partial \rho_{\parallel}}(\Omega B_{\parallel}^* \dot{\rho}_{\parallel}) = 0.$$

To complete the self-consistent gyrokinetic description of nonlinear tokamak physics, the gyrokinetic Vlasov equation was supplemented with the gyrokinetic Maxwell’s equations, given by Eqs. (4.15) and (4.17), as

$$\nabla_{\perp}^2 \delta \phi(r, t) = -4\pi \sum_{s} e_{s} \int \Omega B_{\parallel}^* d\mathbf{\delta} \delta^{3}(T_{GC}^{-1} X - r) T_{GY}^* F(Z, t),$$

(6.2)

$$\nabla_{\perp}^2 \delta A(r, t) = \frac{4\pi}{c} \left[ \mathbf{J} - \sum_{s} e_{s} \int \Omega B_{\parallel}^* d\mathbf{\delta} \nu_{gs} \delta^{3}(T_{GC}^{-1} X - r) T_{GY}^* F(Z, t) \right],$$

(6.3)
where $\nabla \times \mathbf{B} = 4\pi \mathbf{J} / c$ is the equilibrium Ampere's law, and the expression for $T^*_{GY} F$ is given by Eq. (4.16).

6.1.1.2. General Gyrokinetic Moment Equation.

Let $\chi(X, \rho, \mu, \zeta)$ be a gyrocenter phase-space function, with a possible gyophysical dependence. We define the velocity-space moment of $\chi$ over the gyrocenter distribution function $F$ as

$$\|\chi\| \doteq \int 2\pi \Omega B \, d\mu \, d\rho \langle \chi \rangle \, F,$$

(6.4)

where $F$ is gyrophase-independent, and $\langle \chi \rangle$ represents the gyrophase-averaged part of $\chi$. If we multiply the gyrokinetic Vlasov equation, Eq. (6.1), with $\chi$ and operate with $\| \ldots \|$, we obtain the general gyrokinetic moment equation

$$\frac{\partial}{\partial t} \| h\chi \| + \nabla \cdot \| h\dot{X} \chi \| = \| h\frac{d\chi}{dt} \|,$$

(6.5)

where integration by parts was performed, and we use the notation

$$h = \frac{B^*}{B} = 1 + \rho \hat{b} \cdot \nabla \hat{b}, \quad \text{and} \quad \frac{d\chi}{dt} = \frac{\partial \chi}{\partial t} + \dot{X} \cdot \nabla \chi + \dot{\rho} \frac{\partial \chi}{\partial \rho},$$

In the linear symplectic representation, the gyrocenter equations of motion for $\dot{X}$ and $\dot{\rho}$ were given by Eq. (3.55) as

$$h\dot{X} = \rho \Omega \hat{b}_0 + \hat{b} \times (\mu B \nabla \ln B + \rho \Omega^2 \hat{b} \cdot \nabla \hat{b}) + \frac{c}{B} \hat{b} \times \nabla \langle \phi_0 \rangle$$

(6.6)

$$- \frac{\mu B}{\Omega} \nabla \delta B_{\|},
$$

$$h\dot{\rho} = -\frac{h}{B} \frac{\partial}{\partial t} \langle \delta A_{\|} \rangle - \left( \delta B + \rho \Omega^2 \hat{b} \cdot \nabla \hat{b} \right) \cdot \left( \frac{c}{B} \nabla \langle \phi_0 \rangle + \frac{\mu B}{\Omega} \nabla \delta B_{\|} \right)$$

(6.7)

where the notation $\hat{b}_0 = \hat{b} + \langle \delta B_{\|} \rangle / B$ is used. We note that terms of order $O(\epsilon B \epsilon)$ and $O(\epsilon^2)$ have been omitted from Eq. (6.6) since they only contribute higher-order terms in our gyrofluid equations. Similar omissions have been made in Eq. (6.7).

6.1.1.3. Basic Gyrocenter-velocity-space Moments

The simple closure scheme used in this chapter is based on the small $(k_\perp \rho_i)$ ordering. In particular, we wish to retain all terms up to order $O(k_\perp^2 \rho_i^2)$ in the gyrofluid continuity equation since we use the gyrofluid continuity equation to describe the
nonlinear evolution of $\nabla_1^2 \delta \phi$ (related to the ion polarization drift). It turns out that in order to consistently keep terms up to order $O(k_1^2 \rho_i^2)$ in our gyrofluid equations, we only need to consider the four lowest exact moments of the hierarchy

$$
\|1\|_s = N_0 s,
$$
$$
\|\rho\|_s = N_0 s \frac{\delta \nu_s}{\Omega_s},
$$
$$
\|\rho_\parallel^2\|_s = \frac{1}{m_s \Omega_s^2} (P_\parallel + m_s N_0 \delta \nu_s^2),
$$

and $\|\mu\|_s = \frac{P_\parallel}{m_s B}$,

where we have used the definition $(\rho_\parallel)_s = (\tilde{\rho}_\parallel)_s + \delta \nu_s/\Omega_s$ and $\|\tilde{\rho}\|_s = 0$. We shall also make use of the truncated moments

$$
\|\tilde{\rho}_\parallel \mu\|_s \simeq \frac{\delta \nu_s P_\parallel}{m_s \Omega_s B}, \quad \|\rho_\parallel^2\|_s \simeq 3 \frac{\delta \nu_s P_\parallel}{m_s \Omega_s^3}, \quad \text{and} \quad \|\mu^2\|_s \simeq \frac{P_\parallel^2}{N_0 m_s B^2},
$$

where higher order moments were neglected.

Upon using the moments given in Eqs. (6.8) and (6.9) in the general gyrokinetic moment equation, Eq. (6.5), we will be able to derive a set of energy-conserving reduced equations by considering the following moments of the gyrokinetic Vlasov equation: (1) the gyrofluid equation of continuity is obtained by setting $\chi = 1$; (2) the gyrofluid parallel acceleration law is obtained by setting $\chi = \rho_\parallel$; (3) the gyrofluid perpendicular pressure equation is obtained by setting $\chi = \mu$; and (4) the gyrofluid parallel pressure equation is obtained by setting $\chi = \rho_\parallel^2$.

6.1.1.4. Gyrocenter phase-space and physical phase-space moments.

It is useful to establish the relationship between the moment of $\chi$ in physical phase space, $\|\chi\|^P$, and the moment of $\chi$ in gyrocenter phase space, $\|\chi\|^S$. For this purpose, using the identity derived in Eqs. (4.12)–(4.14)

$$
\int d^3 z \, \delta^3 (x - r) \, g(z) \, f(z, t) = \int d^3 Z \Omega B_\parallel^* \delta^3 (T_{GC}^{-1} X - r) \, [(T_{GC}^*)^{-1} g](Z) \, T_{GY}^* F(Z, t),
$$

where $z = (x, v)$ and $T_{GC}^{-1} X = X + \rho_0 + O(\epsilon_B)$, we find

$$
\|\chi\|^P = \|\chi^S\|^S + \int d^3 X \, \|\chi^S\|^S \, (h \delta_0^3 F^{-1} T_{GY}^* F - \delta^3) \|S,
$$

where $h = B_\parallel^*/B$, and $\delta_0^3 = \delta^3 (X + \rho_0 - r)$. 

Since the difference between \( ||\chi||^p \) and \( ||(T_{GC})^{-1}\chi||^8 \) enters our derivation of gyrofluid equations at order \( \mathcal{O}(\varepsilon^6) \), we therefore only need to consider the linear, uniform version of this difference. Consequently, if we assume that the unperturbed magnetic field is spatially uniform, then we have \( h = 1 \) and \( (T_{GC})^{-1}\chi = \exp(\rho_0 \nabla)\chi = \chi \). (We have assumed that the spatial dependence of \( \chi \) is directly associated with that of the unperturbed magnetic field.) Furthermore, the expression for the transformed phase-space function \( T_{\text{GY}}F \) is given as \( T_{\text{GY}}F = F + \varepsilon_6 G_1(F) + \mathcal{O}(\varepsilon_8^2) \), where

\[
G_1(F) = \frac{\delta A_{||}}{B} \left( \frac{\partial F}{\partial \rho_{||}} - \frac{\rho_{||} \Omega^2}{B} \frac{\partial F}{\partial \mu} \right) + \left( \frac{\Omega^2}{B^2} \left( \delta A_{0} \cdot \frac{\partial \rho_0}{\partial \zeta} \right) + \frac{e}{mB} \delta \phi_0 \right) \frac{\partial F}{\partial \mu}
\]

and \( F \) is gyrophase-independent and is also assumed to be spatially uniform under the operation \( T_{\text{GY}} \). The final expression relating gyrocenter and physical phase-space moments is, therefore, given as

\[
||\chi||^p = ||\chi||^8 + \int d^3X \ ||\chi||^8 (\delta_6^3 F^{-1} T_{\text{GY}} F - \delta^3)||^8.
\] (6.10)

The gyrocenter residue \( (\delta_6^3 F^{-1} T_{\text{GY}} F - \delta^3) \), appearing in Eq. (6.10), is given by the expression

\[
[\delta_6^3 F^{-1} T_{\text{GY}} F - \delta^3] = (\delta_6^3 - \delta^3) + \delta_6^3 \left( \frac{\Omega^2}{B^2} \left( \delta A_{0} \cdot \frac{\partial \rho_0}{\partial \zeta} \right) + \frac{e}{mB} \delta \phi_0 \right) \frac{\partial \ln F}{\partial \mu},
\] (6.11)

where to lowest order \( F \) is assumed to be Maxwellian. We note that the gyrocenter residue involves the guiding-center residue \( (\delta_6^3 - \delta^3) \), which will provide terms related to the pressure gradient, and FLR corrections due to the gyrokinetic perturbation potentials \( \delta \phi_0 \) and \( \delta A_{0,\perp} \). The gyrocenter residue, Eq. (6.11), represents the most important quantity in the derivation of gyrofluid equations.

Finally, we would like to consider three examples of the use of the gyrocenter residue, Eq. (6.11), which will be useful later on when we consider the gyrofluid version of Maxwell’s equations. First, we consider \( \chi = 1 \) in Eq. (6.10), for which we obtain

\[
||\delta_6^3 F^{-1} T_{\text{GY}} F - \delta^3||^8 = \frac{P_{1s}}{2m_s \Omega_s^2} \nabla_{\perp}^2 \delta^3
\] (6.12)

\[
= \left[ \left( N_0 \delta B_{||} - \frac{P_{1s}}{2m_s \Omega_s^2 B} \nabla_{\perp}^2 \delta B_{||} \right) \delta^3 + \frac{P_{1s}}{m_s \Omega_s^2 B} \nabla_{\perp}^2 \delta^3 \right]
\]

\[
- \frac{c}{B \Omega_s} \left[ N_0 \nabla_{\parallel} \delta^3 \cdot \nabla_{\perp} \delta \phi + \frac{P_{1s}}{2m_s \Omega_s^2} \left( \nabla_{\perp} \delta^3 \cdot \nabla_{\perp} \delta \phi + \nabla_{\perp} \delta \phi \cdot \nabla_{\perp} \nabla_{\perp} \delta^3 \right) \right],
\]
where the exact moments of eq. (8) were used. Next, we consider \( \chi = \rho_\parallel \Omega \) in Eq. (6.10), for which we obtain

\[
\| \rho_\parallel \Omega (\delta^3 F^{-1} T_{GY}^* F - \delta^3) \|_\star = \delta v_* \| \delta^3 F^{-1} T_{GY}^* F - \delta^3 \|_\star, \tag{6.13}
\]

where the truncated moments of Eq. (6.9) were used, and \((\| \delta^3 F^{-1} T_{GY}^* F - \delta^3 \|_\star)\) was given in Eq. (6.12). Finally, in Eq. (6.10), we consider the case \( \chi = \Omega \partial \rho_0 / \partial \zeta \), for which we obtain

\[
\left\| \frac{\partial \rho_0}{\partial \zeta} (\delta^3 F^{-1} T_{GY}^* F - \delta^3) \right\|_\star - \frac{P_{\perp s}}{m_s \Omega_s} \nabla_\perp \delta^3 \times \tilde{b} - \frac{P_{\perp s}^2}{4N_0 m_s^2 \Omega_s^2} \nabla_\perp \nabla_\perp \delta^3 \times \tilde{b} \tag{6.14}
\]

\[
= 2 \frac{P_{\perp s} \delta B}{m_s \Omega_s} \nabla_\perp \delta^3 \times \tilde{b} - \frac{c}{B} \left( \nabla_\perp \delta \phi \times \tilde{b} + \frac{P_{\perp s}}{2m_s \Omega_s^2} \nabla_\perp \nabla_\perp \delta \phi \times \tilde{b} \right) \delta^3 \\
- \frac{cP_{\perp s}}{2m_s \Omega_s^2 B} \left[ 3 (\nabla_\perp^2 \delta^3 \nabla_\perp \delta \phi \times \tilde{b} + \nabla_\perp \delta \phi \nabla_\perp \nabla_\perp \delta^3 \phi) \right] \\
- \frac{cP_{\perp s}}{m_s \Omega_s^2 B} \left[ (\nabla_\perp \delta^3 \nabla_\perp \delta \phi \times \tilde{b} + \nabla_\perp \delta \phi \nabla_\perp \nabla_\perp \delta^3 \phi) \right] \times [(112) \tilde{1} - (211) \tilde{1} + (122) \tilde{2} - (221) \tilde{2}],
\]

where the truncated moments of Eq. (6.9) were used, and \((112) = \tilde{1} \tilde{1} \tilde{2}, \text{ etc.}\)

When we substitute Eq. (6.12), (6.13), or (6.14) into Eq. (6.10), we shall also make use of the following identities involving the delta function \( \delta^3 (X - r) \):

\[
\int d^3X \ f(X) \nabla_\perp \delta^3 (X - r) \times \tilde{b} = - \nabla_\perp \times [f(r) \tilde{b}],
\]

\[
\int d^3X \ f(X) \nabla_\perp^2 \delta^3 (X - r) = \nabla_\perp^2 f(r),
\]

\[
\int d^3X \ g(X) \nabla_\perp f(X) \cdot \nabla_\perp \delta^3 (X - r) = - \nabla_\perp \cdot [g(r) \nabla_\perp f(r)],
\]

where \( f \) and \( g \) are arbitrary functions of position.

6.1.1.5. Linearized Gyrokinetic Maxwell Equations

The gyrocenter residue, Eq. (6.11), appears in the gyrokinetic Maxwell's equations, Eqs. (6.2) and (6.3), and gives them an important role in the derivation of the gyrofluid equations. This observation was made by Lee [1983], who noted the appearance of the ion polarization drift in the gyrokinetic Poisson's equation. The work of Bernstein and Catto [1985], who considered the derivation of their magneto-gyro-dynamics equations based on moments of the gyrokinetic Vlasov equation, seems to have missed this important contribution from the gyrokinetic Maxwell's equations.
The gyrokinetic Maxwell's equations, expressed in terms of gyrocenter phase-space moments, are given as

\[ \nabla_\perp^2 \delta \phi = -4\pi \sum \varepsilon_i \int d^2X \| \delta \phi_i \| F^{-1}T_{GY}^2 F \| \delta \phi, \]  

(6.15)

\[ \nabla_\perp^2 \delta A_\parallel = -\frac{4\pi}{B} \sum m_i \Omega_i \int d^2X \| \delta \phi_i \| F^{-1}T_{GY}^2 F \| \delta \phi, \]  

(6.16)

\[ \vec{b} \times \nabla \delta B_\parallel = \nabla \times \vec{B} - \frac{4\pi}{B} \sum m_i \Omega_i \int d^2X \| \frac{\partial \rho_0}{\partial \zeta} \| F^{-1}T_{GY}^2 F \| \delta \phi, \]  

(6.17)

where equilibrium quasineutrality was assumed in Eq. (6.15), equilibrium return parallel currents were cancelled in Eq. (6.16), and the equilibrium pressure balance equation is given by the unperturbed version of the right-hand side of Eq. (6.17).

Using the expression given by Eq. (6.12), the linearized gyrokinetic Poisson's equation, Eq. (6.15), leads to the following gyrokinetic quasineutrality condition (which holds if \( k_D \rho_e \gg 1 \)) and which up to order \( O(k_D^2 \rho_e^2) \) is given as

\[ N_{6i}^{gv} - N_{6e}^{gv} = -\frac{cN_0}{B \Omega_i} \nabla_\perp^2 \delta \phi - \frac{\nabla_\perp^2 P_{\perp i}}{2m_i \Omega_i^2} - \frac{3\overline{P}_i}{2m_i \Omega_i^2 B} \nabla_\perp^2 \delta B_\parallel, \]  

(6.18)

where all quantities on the right-hand side are expressed as gyrocenter quantities and \( \overline{P}_i \) denotes the equilibrium ion pressure. This gyrokinetic charge-separation relation, Eq. (6.18), will be used later to produce the so-called vorticity equation. (We note that the vorticity equation, a first-moment equation in regular fluid dynamics, is obtained through the zeroth-moment equation.) The first term on the right-hand side of Eq. (6.18) is the aforementioned ion polarization drift, while the second term provides the diamagnetic convective derivative (which generalizes the polarization drift velocity, as is noted in Appendix B). The third term on the right-hand side of Eq. (6.18) represents magnetic compression and will play an important role in the demonstration of energy conservation, given in subsection 6.1.3. [Eq. (6.18) is the gyrokinetic equivalent of the charge quasineutrality in real space: \( N_{6i}^{p} - N_{6e}^{p} = 0 \).]

Next, using Eq. (6.13), the linearized gyrokinetic parallel Ampere's law, Eq. (6.16), gives, up to order \( O(k_D^2 \rho_e^2) \), the expression

\[ \nabla_\perp^2 \delta A_\parallel = -\frac{4\pi}{c} \delta J_\parallel - \frac{2\pi \overline{P}_i}{B \Omega_i} \nabla_\perp^2 \delta v_i, \]  

(6.19)

where the parallel perturbed current density is defined as \( \delta J_\parallel = \sum \varepsilon_i (N \delta v)_s^{gv} \), and the second term on the right-hand side of Eq. (6.19) can be viewed as the parallel
ion gyroviscosity. This equation will be used later to eliminate the parallel perturbed current density $\delta J_\parallel$ in favor of the shear-Alfvén component $\delta A_\parallel$ and the perturbed parallel ion velocity $\delta v_i$.

Finally, using Eq. (6.14), the linearized gyrokinetic perpendicular Ampere’s law, Eq. (6.17), gives, up to order $k_\perp O(k_\perp^3 \rho_i^2)$, the so-called pressure balance equation

$$\tilde{b} \times \nabla_\perp \delta B_\parallel = \nabla \times B - \frac{4\pi}{B} \tilde{b} \times \left( \nabla_\perp \rho_{giv} \right) + \frac{\tilde{P}_i}{2N_0 m_i \Omega_i^2} \nabla_\perp^2 \rho_{gi} \left( \frac{3c \tilde{P}_i}{2B \Omega_i} \right) \tilde{b} \times \nabla_\perp \nabla_\perp^2 \delta \phi. \quad (6.20)$$

If we assume that the equilibrium magnetic field satisfies the equilibrium pressure balance equation,

$$\nabla \times B = \frac{4\pi}{B} \tilde{b} \times \nabla_\perp \tilde{P},$$

and we get rid of the differential operator $(\tilde{b} \times \nabla_\perp)$, then Eq. (6.20) becomes

$$(1 + \tilde{\beta}) \delta B_\parallel + \frac{4\pi}{B} \left( \frac{\rho_{giv}}{\rho_i} - \tilde{P} \right) = -\frac{2\pi \tilde{P}_i}{B} \left( \frac{3c}{B \Omega_i} \frac{\nabla_\perp^2 \delta \phi}{\nabla_\perp \rho_{gi}} + \frac{1}{N_0 m_i \Omega_i^2} \frac{\nabla_\perp^2 \rho_{gi}}{\nabla_\perp \rho_{gi}} \right), \quad (6.21)$$

where $\tilde{\beta} = 8\pi \tilde{P}/B^2$ is the equilibrium beta value. We note that by using the (linearized) relationship between $\rho_{giv}$ and $\rho_{gi}$, obtained by setting $\chi = \mu B$ in Eq. (6.10),

$$P_{\rho_i} = P_{\rho_i} + 2\tilde{P}_s \left( \frac{\delta B_\parallel}{B} + \frac{c}{B \Omega_s} \frac{\nabla_\perp^2 \delta \phi}{\nabla_\perp \rho_{gi}} + \frac{\nabla_\perp^2 \rho_{giv}}{2N_0 m_i \Omega_i^2} \right),$$

we exactly recover the pressure balance equation given by Eq. (5.57). The perturbed pressure balance equation, Eq. (6.21), will be used later to eliminate the compressional magnetic field perturbation $\delta B_\parallel$ in favor of the perpendicular perturbed electron and ion pressures and the perturbed electrostatic potential $\delta \phi$.

### 6.1.2 Nonlinear Gyrofluid Equations

We are now ready to derive a closed set of gyrofluid equations from the moment equations of the gyrokinetic Maxwell-Vlasov system. These gyrofluid equations will represent a seven-field model involving the following fields (and their corresponding evolution equation): the perturbed electrostatic potential $\delta \phi$ (vorticity equation); the perturbed parallel vector potential (shear-Alfvén component) $\delta A_\parallel$ (generalized Ohm's
law); the perturbed parallel gyrofluid velocity $\delta v$ (parallel acceleration law); the perturbed ion and electron perpendicular pressures $(P_{\perp i}, P_{\perp e})$ (perpendicular pressure equations); and the perturbed ion and electron parallel pressures $(P_{||i}, P_{||e})$ (parallel pressure equations). The parallel (compressional-Alfvén) component of the perturbed magnetic field $\delta B_{||}$ evolves in such a way that the perturbed pressure balance [Eq. (6.21)] is satisfied.

### 6.1.2.1. Gyrofluid Vorticity Equation

Setting $\chi = 1$ in the general moment equation, Eq. (6.5), we obtain the gyrokinetic equation of continuity

$$\frac{\partial}{\partial t} ||h|| + \nabla \cdot ||h \hat{X}|| = 0.$$ 

More explicitly, $||h||_s$ and $||h \hat{X}||_s$ are given as $||h||_s = N_{0s}$, and

$$||h \hat{X}||_s = N_{0s} \delta v_s \hat{b}_s + \frac{c N_{0s}}{B} \hat{b} \times \nabla \delta \phi + \delta v_s \left( \frac{P_{\perp s}}{2 m_s \Omega_s B} \right) \nabla^2 \delta \mathbf{B} \perp + \left( \frac{P_{\perp s}}{m_s \Omega_s B} \right) \hat{b} \times \nabla \delta \mathbf{B}_{||}$$

$$+ \frac{\hat{b}}{m_s \Omega_s} \times \left( P_{\perp s} \nabla \ln B + P_{||s} \hat{b} \cdot \nabla \hat{b} \right) \left( \frac{P_{\perp s}}{2 m_s \Omega_s^2} \right) \hat{b} \times \nabla \left( \frac{\nabla^2 \delta \phi}{B} \right),$$

where $\hat{b}^* = \hat{b} + \delta \mathbf{B} \perp / B$. With these expressions, and using the notation $\nabla \parallel = \hat{b}^* \cdot \nabla$, the equation of continuity becomes

$$\frac{\partial}{\partial t} N_{0s} + B \nabla \parallel \left( \frac{N_{0s} \delta v_s}{B} \right) + \nabla \parallel \delta \phi \cdot \nabla \perp \times \left( \frac{c N_{0s} \hat{b}}{B} \right) + \nabla \parallel \frac{\nabla^2 \delta \phi}{B} \cdot \nabla \perp \left( \frac{c P_{\perp s} \hat{b}}{2 m_s \Omega_s^2} \right)$$

$$+ \left( \frac{P_{\perp s}}{2 m_s \Omega_s^2 B} \right) \nabla \parallel \nabla \perp \delta v_s + \frac{c}{e_s B} \left( \nabla \perp P_{\perp s} \hat{b} \times \nabla \perp \ln B + \nabla \perp P_{||s} \hat{b} \times \hat{b} \cdot \nabla \perp \hat{b} \right)$$

$$+ \nabla \parallel \delta \mathbf{B}_{||} \cdot \nabla \perp \left( \frac{P_{\perp s} \hat{b}}{m_s \Omega_s B} \right) = 0,$$

(6.22)

where we have used the identity $\nabla \cdot (f \hat{b} \times \nabla g) = \nabla g \cdot \nabla \times (f \hat{b})$. If we assume that the electrons are behaving adiabatically ($N_{0e} = e N_0 \delta \phi / T_e$), then Eq. (6.18) can be substituted into Eq. (6.22) to give an electromagnetic (high-$\beta$) Hasegawa–Mima equation.

The gyrofluid vorticity equation is obtained by subtracting the electron (drift-kinetic: $P_{\perp s}/m_s \Omega_s^2 = 0$) continuity equation from the ion (gyrokinetic) continuity equation, and using Eqs. (6.18) and (6.19),

$$\frac{c^2 N_{0m_i}}{B^2} \left[ \frac{\partial}{\partial t} + \left( \frac{c}{B} \hat{b} \times \nabla \delta \phi + \frac{\hat{b}}{N_{0m_i} \Omega_i} \times \nabla P_{\perp i} \right) \cdot \nabla \right] \nabla^2 \delta \phi + \frac{c}{4 \pi} \nabla \parallel \nabla^2 \parallel \delta A_{||}$$
\[ + \frac{\dot{\theta}}{B} \left[ (\nabla_\perp P_\perp \times \nabla \ln B) + (\nabla_\perp P_\parallel \times \hat{b} \cdot \nabla \hat{b}) \right] - \frac{\dot{\nabla}_\perp \nabla \nabla_\perp \phi}{B^2 \Omega_i} \nabla \nabla_\perp P_\perp \nabla = \frac{c^2 \dot{P}_i}{2 \Omega_i} \nabla_\perp \nabla_\perp \delta \phi \cdot \nabla \times \left( \frac{\hat{b}}{B^2} \right) - \frac{c}{2B \Omega_i} \nabla_\perp \left( \frac{dP_\perp}{dt} \right) + c \nabla_\perp \delta B_\parallel \cdot \nabla \times \left( \frac{P_\perp \hat{b}}{B^2} \right) - \frac{3c \dot{P}_i}{2B^2 \Omega_i} \frac{d}{dt} \nabla_\perp \delta B_\parallel, \quad (6.23) \]

where the ion parallel velocity is equal to the gyrofluid parallel velocity, \( \delta v_\parallel = \delta v \) (when we ignore electron inertia), and we have used the identity

\[ \frac{d\nabla^2 P_\perp}{dt} = \nabla^2 \left( \frac{dP_\perp}{dt} \right) + \frac{\dot{\theta}}{B} \nabla \nabla_\perp P_\perp \nabla \nabla_\perp \delta \phi - \frac{2c^2}{B} \hat{b} \cdot \nabla \nabla_\perp \delta \phi \cdot \nabla \nabla_\perp P_\perp \nabla, \]

with \( d/dt = \partial/\partial t + \dot{\theta}B \times \nabla \delta \phi \cdot \nabla_\perp \). Finally, we note that if the moment equation, Eq. (6.5), is evaluated in a drifting phase space (Bernstein and Catto [1985]) then we have

\[ h = 1 + \rho_\parallel \hat{b} \cdot \nabla \times \hat{b} + \frac{\hat{b}}{\Omega_i} \cdot \nabla \times \nabla, \]

where \( \nabla \) is the total drift velocity. It is simple to see that when \( \chi = 1 \) we immediately obtain the vorticity equation.

6.1.2.2. Gyrofluid Parallel Equation of Motion

Setting \( \chi = \rho_\parallel \) in the general moment equation, Eq. (6.5), we obtain the gyrofluid parallel equation of motion

\[ \frac{\partial}{\partial t} ||h\rho|| + \nabla \cdot ||h \dot{\nabla} \rho|| = ||h \dot{\rho}||. \]

More explicitly, keeping only the relevant moments, we have \( ||h\rho|| = N_0 \delta v_s / \Omega_s \), and

\[ ||h\rho||_s = \frac{P_{ls}}{m_s \Omega_s} \hat{b}^* + \frac{\delta v_s}{m_s \Omega_s^2} \hat{b} \times (P_{ls} \nabla \ln B + 3P_\parallel \hat{b} \cdot \nabla \hat{b}) \]

\[ + \frac{cN_0}{B \Omega_s} \delta v_s \hat{b} \times \nabla_\parallel \delta \phi + \frac{P_{ls} \delta v_s}{m_s \Omega_s B} \left( \hat{b} \times \nabla_\parallel \delta B_\parallel + \frac{c}{2 \Omega_s} \hat{b} \times \nabla_\perp \delta \phi \right), \]

\[ ||h \dot{\rho}||_s = -\frac{N_0}{B} \left( \frac{\delta A_\parallel}{\partial t} + c \nabla_\parallel \delta \phi \right) - \frac{cP_{ls}}{2m_s \Omega_s^2 B} \nabla_\parallel \nabla_\perp^2 \delta \phi \frac{d}{dt} \left( P_\parallel + P_{ls} \right) \nabla \ln B \]

\[ - \frac{P_{ls}}{m_s \Omega_s B} \nabla_\parallel \delta B_\parallel - \frac{P_{ls}}{2m_s \Omega_s^2 B} \frac{d}{dt} \left( \nabla_\perp^2 \delta A_\parallel \right). \]

The first use of these expressions is concerned with the derivation of the dissipationless generalized Ohm's law, obtained from the electron gyrofluid parallel equation.
of motion. The electron gyrofluid parallel velocity is given in terms of the perturbed parallel current density \( \delta J_\parallel \) and the parallel mass flow velocity \( \delta \nu \) as follows

\[
\delta \nu_e = \delta \nu - \frac{m_i}{M N_0 e} \delta J_\parallel = (1 + \frac{\rho_i^2}{2} \nabla_\perp^2) \delta \nu + \frac{c}{4 \pi N_0 e} \nabla_\perp^2 \delta A_\parallel,
\]

where \( \rho_i^2 = T_i/m_i \Omega_i^2 \) and we have ignored terms of order \( \mathcal{O}(m_e/m_i) \). Using the electron (drift-kinetic) parallel equation of motion with the expression for \( \delta \nu_e \), we obtain the dissipationless generalized Ohm’s law

\[
\frac{d}{dt} \left( 1 - \frac{c^2}{\omega_{pe}^2} \right) \nabla_\parallel \delta A_\parallel + \vec{e} \cdot \nabla \delta \phi = \frac{e}{N_0 e} \left( \nabla_\parallel P_\parallel + \frac{P_{\perp e}}{B} \nabla_\parallel \delta B_\parallel \right) + \frac{m_i c}{e} \frac{d}{dt} \left( 1 + \frac{\rho_i^2}{2} \nabla_\perp^2 \right) \delta \nu - \frac{c}{N_0 e} (P_\parallel - P_{\perp e}) \nabla_\parallel \ln B. \tag{6.24}
\]

Note that the electron-inertia terms provide a useful limit for the generalized Ohm’s law, in the absence of dissipation.

The second use of these expressions is concerned with the derivation of the gyrofluid parallel acceleration law, involving the gyrofluid parallel velocity \( \delta \nu \). The equation for the parallel acceleration is obtained by simply multiplying the electron and ion parallel equations by their respective masses, and adding them to give

\[
m_i N_0 \left\{ \frac{d}{dt} + \frac{\bar{P}_i}{N_0 m_i \Omega_i} \left[ \delta \vec{\times} (\nabla \ln B + 3 \vec{e} \cdot \nabla \delta \vec{b}) \right] + \frac{\bar{b}}{B} \times \left( \nabla_\perp \delta B_\parallel + \frac{c}{2 \Omega_i} \nabla_\perp \nabla_\perp^2 \delta \phi \right) \right\} \delta \nu = -\nabla_\parallel P_\parallel + (P_\parallel - P_{\perp}) \nabla_\parallel \ln B - \frac{P_{\perp}}{B} \nabla_\parallel \delta B_\parallel - \frac{\bar{P}_i}{2 \Omega_i B} \left( c \nabla_\parallel \nabla_\perp^2 \delta \phi + \frac{d \nabla_\perp^2 \delta A_\parallel}{dt} \right), \tag{6.25}
\]

where terms of order \( \mathcal{O}(m_e/m_i) \) have been neglected. Note that the issue concerning the gyroviscous cancellations, discussed in 5.2.3.2. [see Eq. (5.53)], is completely bypassed by the use of the gyrocenter (guiding-center) residue, Eq. (6.11). This is because all FLR effects, discussed in Appendix B, are automatically introduced by the guiding-center transformation \( T_{GC} \).

6.1.2.3. Gyrofluid Pressure Equations

Setting \( \chi = \mu \) in the general moment equation, Eq. (6.5), we obtain the gyrofluid perpendicular pressure equation. Since \( \mu \) is a gyrocenter constant of motion, this
equation takes the simple form
\[ \frac{\partial}{\partial t} ||h\mu|| + \nabla \cdot ||h\dot{X}\mu|| = 0, \]
where \( ||h\dot{\mu}|| = 0 \) is a gyrokinetic identity. More explicitly, we have
\[ ||h\mu||_* = \frac{P_{1\perp}}{m_s B}, \text{ and} \]
\[ ||h\dot{X}\mu||_* = \frac{P_{1\perp}}{m_s B} \left( \delta u_\perp \delta^* + \frac{eB}{\Omega_s} \nabla \cdot \nabla \delta \phi \right). \]

Using these expressions, the gyrofluid perpendicular pressure equation becomes
\[ \frac{D_{s}P_{1\perp}}{D_t} = - \frac{eP_{1\perp}}{B} \nabla \cdot \left( \frac{\nabla \times B}{B} + 3 \delta \dot{\nabla} \ln B \right) + 2P_{1\perp} \delta v_{\perp} \nabla \nabla \ln B - P_{1\perp} \nabla \nabla \delta v_{\perp}, \]
\[ (6.26) \]
where \( D_s/D_t = d/dt + \delta v_{\perp} \nabla \nabla \).

Finally, setting \( \chi = \tilde{\rho}_{\parallel}^2 \), where \( \tilde{\rho}_{\parallel} = \rho_{\parallel} - \delta v_{\parallel}/\Omega_s \), in the general moment equation, Eq. (6.5), we obtain the gyrofluid parallel pressure equation
\[ \frac{\partial}{\partial t} ||h\tilde{\rho}_{\parallel}^2|| + \nabla \cdot ||h\dot{X}\tilde{\rho}_{\parallel}^2|| = 2 \frac{d}{dt} ||h\tilde{\rho}_{\parallel}|| \tilde{\rho}_{\parallel}, \]
where
\[ \frac{\partial}{\partial t} \tilde{\rho}_{\parallel} = \tilde{\rho}_{\parallel} - \frac{1}{\Omega_s} \left( \frac{\partial}{\partial t} \delta v_{\parallel} + \dot{X} \cdot \nabla \delta v_{\parallel} - \delta v_{\parallel} \dot{X} \cdot \nabla \ln B \right). \]

More explicitly, we have
\[ ||h\tilde{\rho}_{\parallel}^2||_* = \frac{P_{1\parallel}}{m_s \Omega_s^2}, \]
\[ ||h\dot{X}\tilde{\rho}_{\parallel}^2||_* = \frac{P_{1\parallel}}{m_s \Omega_s^2} \left( \delta u_{\parallel} \delta^* + \frac{eB}{\Omega_s} \times \nabla \delta \phi \right), \]
\[ 2 ||h\tilde{\rho}_{\parallel} \frac{d}{dt} \tilde{\rho}_{\parallel}||_* = - \frac{2eP_{1\parallel}}{m_s \Omega_s^2 B^2} \nabla \times B \cdot \nabla \delta \phi - \frac{2P_{1\parallel}}{m_s \Omega_s^2} \nabla \nabla \delta v_{\parallel} - \frac{2P_{1\parallel}}{m_s \Omega_s^2} \nabla \nabla \ln B \delta v_{\parallel}. \]

Using these expressions, the gyrofluid parallel pressure equation becomes
\[ \frac{D_s P_{1\parallel}}{D_t} = - \frac{eP_{1\parallel}}{B} \nabla \cdot \left( 3 \frac{\nabla \times B}{B} + 4 \hat{B} \times \nabla \ln B \right) + P_{1\parallel} \delta v_{\parallel} \nabla \nabla \ln B - 3P_{1\parallel} \nabla \nabla \delta v_{\parallel}. \]
\[ (6.27) \]

If we now assume that the pressure remains isotropic during its evolution, by adding the perpendicular pressure with half the parallel pressure we obtain the well-known equation of state
\[ \frac{3}{2} \frac{D_s P_s}{D_t} + \frac{5}{2} P_s \nabla \cdot \left( \delta u_{\perp} \delta^* + \frac{eB}{B} \times \nabla \delta \phi \right) = 0. \]
\[ (6.28) \]
The second term on the left-hand side of Eq. (6.28) is the familiar term representing plasma compressibility.

6.1.3 Energy Conservation Property

6.1.3.1 Consistent Gyrofluid Equations

The gyrofluid equations for the vorticity, Eq. (6.23), and the parallel acceleration, Eq. (6.25), both have time derivatives appearing on their right-hand sides. Before we can proceed with a proof of energy conservation, we wish to eliminate these terms by using the gyrofluid perpendicular ion pressure, Eq. (6.26), and the gyrofluid Ohm's law, Eq. (6.24), respectively.

Firstly, we consider the term \( \nabla_{\perp}^2 dP_{\perp i}/dt \) on the right-hand side of Eq. (6.23). Using the ion pressure equation, Eq. (6.26), it is simple to obtain

\[
\nabla_{\perp}^2 \frac{dP_{\perp i}}{dt} = -\frac{\tilde{b}}{B} \times \nabla_{\perp} \nabla_{\perp}^2 \delta v - \frac{\tilde{b}}{B} \nabla_{\perp} \nabla_{\perp}^2 \delta \phi \nabla \times \left( \frac{\tilde{b}}{B^2} \right).
\]

Next, we combine the term \( \nabla_{\perp}^2 \nabla_{\perp} \delta v \) with the term \( \nabla_{||} \nabla_{\perp} \delta v \), appearing on the right-hand side of Eq. (6.23), to obtain

\[
\nabla_{\perp}^2 \nabla_{\perp} \delta v - \nabla_{||} \nabla_{\perp} \delta v = -\frac{\tilde{b}}{B} \times \nabla_{\perp} \nabla_{\perp} \delta A_{||} - \nabla_{\perp} \delta v - 2\frac{\tilde{b}}{B} \times \nabla_{\perp} \nabla_{\perp} \delta A_{||} : \nabla_{\perp} \nabla_{\perp} \delta v.
\]

Using these expressions, the gyrofluid vorticity equation, Eq. (6.23), becomes

\[
\frac{c^2 N_0 m_i}{B^2} \frac{d}{dt} \nabla_{\perp}^2 \delta \phi + \frac{c^2}{B^2 \Omega_i} \nabla_{\perp} \left( \tilde{b} \times \nabla_{\perp} P_{\perp i} : \nabla_{\perp} \nabla_{\perp} \delta \phi \right) + \frac{c}{4\pi} \nabla_{\perp} \nabla_{\perp}^2 \delta A_{||} = \frac{2 \tilde{b}}{B} \cdot \nabla_{\perp} \nabla_{\perp} \delta v + \frac{\nabla_{\perp} \left( \nabla_{\perp} P_{||} \times \nabla \ln B \right)}{\Omega_i} + \frac{\nabla_{\perp} \left( \nabla_{\perp} \tilde{b} \cdot \nabla \tilde{b} \right)}{B^2 \Omega_i}.
\]

Secondly, we consider the term

\[
c \nabla_{\perp} \nabla_{\perp}^2 \delta \phi + \frac{d}{dt} \nabla_{\perp}^2 \delta A_{||},
\]

appearing on the right-hand side of Eq. (6.25). It is simple to show that this expression can be transformed into the expression

\[
c \nabla_{\perp} \nabla_{\perp}^2 \delta \phi + \frac{d}{dt} \nabla_{\perp}^2 \delta A_{||} = \nabla_{\perp} \left( \frac{\delta \delta A_{||}}{\partial t} + c \nabla_{\perp} \delta \phi \right) + \frac{2 \tilde{b}}{B} \times \nabla_{\perp} \nabla_{\perp} \delta A_{||} : \nabla_{\perp} \nabla_{\perp} \delta \phi.
\]
Next, we use a simplified version of the gyrofluid Ohm's law, Eq. (6.24), given as
\[
\frac{\partial \delta A_{||}}{\partial t} + c\nabla_{||} \delta \phi = \frac{c}{eN_0} \left( \nabla_{||} P_{||e} + \frac{P_{\perp e}}{B} \nabla_{||} \delta B_{||} \right),
\]
where the term \((\nabla_{||} \ln B)\) and electron inertia were neglected. When we substitute these two expressions into the parallel acceleration law, Eq. (6.25), we easily obtain
\[
m_iN_0 \left\{ \frac{d}{dt} + \frac{P_{\perp}}{N_0m_i\Omega_i} \left[ \delta \times (\nabla \ln B + 3\delta \cdot \nabla \delta b) + \frac{\delta b}{B} \times \left( \nabla_{\perp} \delta B_{||} \right) \right] \cdot \nabla_{\perp} \right\} \delta v = - \left( \nabla_{||} P_{||} + \frac{P_{\perp}}{B} \nabla_{||} \delta B_{||} \right) \quad (6.30)
\]
\[- \frac{P_{\perp}}{2N_0m_i\Omega_i^2} \nabla_{\perp}^2 \left( \nabla_{||} P_{||e} + \frac{P_{\perp e}}{B} \nabla_{||} \delta B_{||} \right) \cdot \frac{cP_{\perp}}{B^2\Omega_i} \left( \delta b \times \nabla_{\perp} \nabla_{\perp} \delta A_{||} : \nabla_{\perp} \nabla_{\perp} \delta \phi \right).
\]

The consistent gyrofluid (seven-field) model is, therefore, composed of the vorticity equation, Eq. (6.29), the generalized Ohm's law, Eq. (6.24), the parallel acceleration law, Eq. (6.30), and the pressure equations, Eqs. (6.26) and (6.27).

Finally, instead of substituting the pressure balance equation, Eq. (6.21), into the expressions for the gyrofluid vorticity equation, Eq. (6.29), Ohm's law, Eq. (6.24), and parallel acceleration, Eq. (6.30), it is simpler to use an equation that describes the time evolution of the compressional term \(\delta B_{||}\). Such an equation is derived from the pressure balance equation, Eq. (6.21), and the equations for the perpendicular ion and electron pressures, Eq. (6.26), and is given by the expression
\[
\frac{1}{4\pi} \frac{\partial \delta B_{||}}{\partial t} = - \frac{3cP_{\perp}}{2B^2\Omega_i} \frac{\partial \nabla_{\perp}^2 \delta \phi}{\partial t} + \frac{1}{B} \nabla_{||} \left( P_{\perp} \delta v + \frac{c\nabla_{\perp}^2 \delta A_{||}}{4\pi N_0e} P_{\perp e} + \frac{P_{\perp e}}{2N_0m_i\Omega_i^2} \nabla_{\perp}^2 \delta v \right)
\]
\[
+ \frac{c\delta b}{B^2} \times \nabla_{\perp} \delta \phi \cdot (\nabla_{\perp} P_{\perp} + 3\nabla \ln B), \quad (6.31)
\]
where \((1 + \beta)\) was assumed to be unity, and we have neglected the term \(\rho_i^2 \nabla_{\perp}^2 P_{\perp i}\) compared to \(P_{\perp i}\).

6.1.3.2. Isotropic Gyrofluid Model

A very useful model, which we shall use later in section 6.2., is given by the five-field (isotropic) model which describes the nonlinear evolution of the fluctuating fields \((\delta \phi, \delta A_{||}, \delta v, P_{i}, P_{e})\), with a complimentary equation for the compressional magnetic field component \((\delta B_{||})\). This model is obtained by imposing the isotropic-pressure condition \((P_{||e} = P_{\perp e} = P_{e})\) on the gyrofluid equations, Eqs. (6.24) and (6.29)–(6.31), and by using the isotropic-pressure equation of state, Eq. (6.28).
The reduced equations of the isotropic gyrofluid model are: the isotropic gyrofluid vorticity equation

\[
\frac{c^2}{4\pi v_A^2} \frac{d}{dt} \nabla_\perp^2 \delta \phi + \frac{c^2}{B^2 \Omega_i} \nabla_\perp \cdot (\hat{b} \times \nabla_\perp P_s \nabla_\perp \delta \phi) + \frac{c}{4\pi} \nabla_\parallel \nabla_\perp^2 \delta A_\parallel
\]

(6.32)

\[+ \frac{3c\hat{P}_s}{2B^2 \Omega_i} \frac{d}{dt} \nabla_\parallel^2 \delta B_\parallel - \frac{2c}{B} \nabla_\perp \cdot (P_s \hat{b} \times \nabla \ln B) = \frac{c^2 B \hat{P}_s}{\Omega_i} \nabla_\perp \nabla_\perp^2 \delta \phi \cdot \nabla \times \left( \frac{\hat{b}}{B^2} \right) \]

\[+ c \nabla_\perp \delta B_\parallel \cdot \nabla \times \left( \frac{P_s \hat{b}}{B^2} \right) - \frac{c\hat{P}_s}{B^2 \Omega_i} \nabla_\perp \cdot (\nabla_\perp \delta \phi \hat{b} \times \nabla_\perp \nabla_\perp \delta A_\parallel);\]

the isotropic gyrofluid Ohm's law

\[\frac{\partial \delta A_\parallel}{\partial t} + c \nabla_\parallel \delta \phi = \frac{c}{eN_0} \left( \nabla_\parallel P_e + \frac{P_s}{B} \nabla_\parallel \delta B_\parallel \right); \]

(6.33)

the isotropic gyrofluid parallel acceleration law

\[m_i N_0 \left\{ \frac{d}{dt} + \frac{\hat{P}_s}{N_0 m_i \Omega_i} \left[ 4 \hat{b} \times \nabla \ln B + \frac{\hat{b}}{B} \times \left( \nabla_\perp \delta B_\parallel + \frac{c}{2\Omega_i} \nabla_\perp \nabla_\perp^2 \delta \phi \right) \right] \cdot \nabla_\perp \right\} \delta \nu \]

\[= - \left( \nabla_\parallel P + \frac{P_s}{B} \nabla_\parallel \delta B_\parallel \right) - \frac{\hat{P}_s}{2N_0 m_i \Omega_i^2} \nabla_\perp^2 \left( \nabla_\parallel P_e + \frac{P_s}{B} \nabla_\parallel \delta B_\parallel \right) \]

(6.34)

\[- \frac{c\hat{P}_s}{B^2 \Omega_i} \left( \hat{b} \times \nabla_\perp \nabla_\perp \delta A_\parallel : \nabla_\perp \nabla_\perp \delta \phi \right);\]

the isotropic gyrofluid equation of state (for fluid species s)

\[\left( \frac{d}{dt} + \delta \nu_s \nabla_\parallel \right) P_s + \frac{5}{3} P_s \nabla \cdot \left( \delta \nu_s \hat{b}^* + \frac{c\hat{b}}{B} \times \nabla_\perp \delta \phi \right) = 0; \]

(6.35)

and the gyrofluid equation for the compressional magnetic field

\[\frac{1}{4\pi} \frac{\partial \delta B_\parallel}{\partial t} = - \frac{3c\hat{P}_s}{2B^2 \Omega_i} \frac{\partial \nabla_\perp^2 \delta \phi}{\partial t} + \frac{1}{B} \nabla \left[ P_s \delta \nu + \frac{c\nabla_\parallel^2 \delta A_\parallel}{4\pi N_0 e} + \frac{\hat{P}_s}{2N_0 m_i \Omega_i^2} \nabla_\perp^2 \delta \nu \right] \]

\[+ \frac{c\hat{b}}{B^2} \times \nabla_\perp \delta \phi \cdot (\nabla_\perp P + 3 \nabla \ln B). \]

(6.36)

Finally, several reduced FLR-MHD models (see section 5.2.) use the additional assumption that the pressure is isothermal, and thus ignore ion and electron temperature-gradient effects. In this case, the pressure evolution (for species s) is governed by the isothermal gyrofluid continuity equation

\[\frac{\partial N_{os}}{\partial t} + \nabla \cdot \left[ N_{os} \delta \nu_s \hat{b}^* + \frac{cN_{os} \hat{b}}{B} \times \nabla_\perp \delta \phi + \frac{P_s}{m_s \Omega_s} \left( 2\hat{b} \times \nabla \ln B + \frac{\hat{b}}{B} \times \nabla_\perp \delta B_\parallel \right) \right] \]

\[= - \nabla_\perp \cdot \left\{ \frac{P_s}{2m_s \Omega_s^2} \left[ \hat{b} \times \nabla_\perp \left( \frac{c\nabla_\parallel^2 \delta \phi}{B} \right) - \frac{\delta \nu_s \hat{b}}{B^2} \times \nabla_\perp \nabla_\perp^2 \delta A_\parallel \right] \right\}. \]

(6.37)
6.1.3.3. Energy Invariant for the Isotropic Gyrofluid Model

We now show that the isotropic gyrofluid equations, Eqs. (6.32)–(6.36), conserve a quantity which we call the isotropic gyrofluid (IGF) energy invariant.

The normal procedure used in the derivation of a (reduced) fluid energy invariant involves an explicit construction of the invariant, through the use of the fluid equations. For example, the procedure was carried out in 5.2.3.5. and 5.2.3.6., but as in the case of the HKM model, the procedure sometimes requires a certain amount of guesswork and may also fail to provide an energy invariant [see Eq. (5.85)].

We would like to apply such a procedure to the isotropic gyrofluid equations, given above, and obtain an expression for the gyrofluid energy invariant. We will also show that the gyrofluid energy invariant can be derived from the gyrokinetic energy invariant, Eq. (4.21).

First, we proceed with the construction of the isotropic gyrofluid energy invariant by: (a) multiply the vorticity equation with \((-\delta\phi\), the parallel acceleration law with \((\delta v\), and Ohm's law with \((-\nabla_\perp^2 \delta A_\parallel /4\pi\); and (b) adding the resulting equations along with \((3/2\times\) the ion and electron pressure equations. If we integrate over the plasma volume and eliminate terms which appear as exact divergence terms (assuming that surface contributions vanish), we obtain

\[
\int d^3r \frac{\partial}{\partial t} \left[ \frac{c^2}{8\pi v_A^2} \left| \nabla_\perp \delta \phi \right|^2 + \frac{1}{8\pi} \left| \nabla_\perp \delta A_\parallel \right|^2 + \frac{m_i N_0}{2} \delta v^2 + \frac{3}{2} (P_e + P_i) \right] = \int d^3r \left\{ \left[ \left( \frac{3cP_i}{2B^2 \Omega_i} \right) \nabla_\perp^2 \delta \phi \right] \frac{\partial \delta B_\parallel}{\partial t} + \left[ \frac{c\delta \phi}{B^2} b \times (\nabla_\perp P + 3\nabla \ln B) \right] \cdot \nabla_\perp \delta B_\parallel \right. \\
- \left[ \frac{\delta v P}{B} + \frac{P_e}{B} \left( \frac{c\nabla_\perp^2 \delta A_\parallel}{4\pi e N_0} + \frac{\bar{P}_i}{2N_0 m_i \Omega_i^2} \nabla_\perp^2 \delta v \right) \right] \nabla_\parallel \delta B_\parallel \right\}. 
\]

By integrating by parts the right-hand side of Eq. (6.38), and by using Eq. (6.36), the right-hand side of Eq. (6.38) becomes

\[
\int d^3r \left[ \frac{1}{8\pi} \frac{\partial}{\partial t} \delta B_\parallel^2 + \frac{3cP_i}{2B^2 \Omega_i} \frac{\partial}{\partial t} \left( \delta B_\parallel \nabla_\perp^2 \delta \phi \right) \right]. 
\]

If we combine this expression to the left-hand side of Eq. (6.38), we obtain the isotropic gyrofluid energy invariant

\[
E_{IGF} = \int d^3r \left[ \frac{1}{2} N_0 m_i \delta v^2 + \frac{c^2}{8\pi v_A^2} \left| \nabla_\perp \delta \phi \right|^2 + \frac{1}{8\pi} \left| \nabla_\perp^2 \delta A_\parallel \right|^2 + \frac{3}{2} (P_e + P_i) \right].
\]
\[
-\frac{\delta B^2}{8\pi} - \delta B\| \left( \frac{3c\bar{P}_i}{2B^2\Omega_i} \nabla_\perp^2 \delta \phi \right) \right].
\] (6.40)

The first term in this expression represents the energy due to (ion) parallel motion, the second term represents the energy due to perpendicular \(E \times B\) motion, and the third term represents the energy due to the perpendicular magnetic field. The next two terms represent the usual internal energy associated with ion and electron pressure. Finally, the last two terms can be transformed according to the pressure balance equation, Eq. (6.21), and obtain

\[
-\frac{\delta B^2}{8\pi} - \frac{3c\bar{P}_i}{2B^2\Omega_i} \nabla_\perp^2 \delta \phi \delta B\| = \frac{\delta B^2}{8\pi} + (P_\perp - \bar{P}) \frac{\delta B\|}{B},
\] (6.41)

where the first term on the right-hand side is the familiar parallel-magnetic-field energy, and the last term on the right-hand side is the energy associated with the perturbed compressional magnetic component. With this expression, the isotropic gyrofluid energy invariant, Eq. (6.38), becomes

\[
\mathcal{E}_{IGF} = \int d^3r \left[ \frac{1}{2} N_0 m_i \delta v^2 + \frac{c^2 |\nabla_\perp \delta \phi|^2}{8\pi v_A^2} + \frac{3}{2} P + \frac{1}{8\pi} (\delta B^2 + |\nabla_\perp \delta A|^2) + (P - \bar{P}) \frac{\delta B\|}{B} \right].
\] (6.42)

We would like to conclude this section by showing that the gyrofluid energy can also be derived from the gyrokinetic energy invariant, given by Eq. (4.21),

\[
\mathcal{E}_{GY} = \sum_s m_s \int d^3X \| (T^*_{GY})^{-1} H_0 \| + \frac{1}{8\pi} \int d^3r \left[ |\nabla_\perp \delta \phi|^2 + |B + \delta B|^2 \right],
\] (6.43)

where \(h = B^*_{\|}/B\) was taken to be unity, and the gyrocenter kinetic energy is

\[
(T^*_{GY})^{-1} H_0 = H_0 - G_1(H_0) + \frac{1}{2} G_2^2(H_0) - G_2(H_0) + \cdots.
\] (6.44)

In the symplectic representation, which is the representation used in this chapter, we have

\[
-\left< G_1(H_0) \right> = -\frac{\Omega^2}{B} \left< \delta A_0 \cdot \frac{\partial \rho_0}{\partial \zeta} \right> \approx \mu \delta B\|,
\]

\[
\frac{1}{2} \left< G_1^2(H_0) \right> \approx \frac{ec}{2mB} \frac{\partial}{\partial \mu} \left( |\delta \phi_0|^2 \right) \approx \frac{c^2}{2B^2} |\nabla_\perp \delta \phi|^2,
\]
and \( G_2(H_0) \) gives higher-order terms that are neglected. Next, we evaluate the moments contained in the first integral of Eq. (6.43), where

\[
\begin{align*}
    m_s||H_0||_s &= P_{\perp s} + \frac{1}{2} P_{\parallel s} + \frac{1}{2} N_0 m_s \delta v^2_s, \\
    m_s||G_1(H_0)||_s &= -P_{\perp s} \delta B_{\parallel} / B, \\
    (m_s/2)||G_2^2(H_0)||_s &\approx \frac{c^2 N_0 m_s}{B^2} |\nabla_\perp \delta \phi|^2.
\end{align*}
\]

Finally, if we let the electron mass vanish and assume pressure isotropy, we obtain

\[
\mathcal{E}_{GY} = \int d^3 r \left[ \frac{1}{2} N_0 m_s \delta v^2 + \frac{1}{8 \pi} \left( 1 + \frac{c^2}{v^2_A} \right) |\nabla_\perp \delta \phi|^2 + \frac{3}{2} P + P \frac{\delta B_{\parallel}}{B} + \frac{1}{8 \pi} \left( B^2 + 2 B \delta B_{\parallel} + \delta B^2_{\parallel} + |\nabla_\perp \delta A_{\parallel}|^2 \right) \right].
\]

We note that the term \( B^2 \) is constant in time and can be omitted. We recover the expression given by Eq. (6.42) if we add the following two terms,

\[
- \left( \frac{P}{B} + \frac{B}{4 \pi} \right) \delta B_{\parallel},
\]

on the right-hand side of Eq. (6.46). This addition does not modify the invariance of \( \mathcal{E}_{GY} \) since the evolution equation for \( \delta B_{\parallel} \) is given as an exact divergence and vanishes upon integration. Furthermore, by using the condition \( k_{De} \rho_s \gg 1 \), the term \( 1 + c^2/v^2_A \) is replaced by \( c^2/v^2_A \) in Eq. (6.46) and Eq. (6.42) follows.
6.2 Comparison with Nonlinear Reduced Fluid Models

We present in this section a detailed comparison between the gyrofluid equations of the (third-generation) isotropic five-field model, derived in section 6.1., and two sets of reduced FLR-MHD (second-generation) equations given by the four-field of Hasegawa and Wakatani [1983], and the four-field model of Hazeltine, Kotschenreuther, and Morrison [1985].

We proceed with our comparative analysis as follows. First, each second-generation reduced FLR-MHD model is shown to be contained in our (third-generation) isotropic gyrofluid model. Next, we discuss the importance of the additional terms found in the gyrofluid model, compared to each FLR-MHD model.

2.0.1. Four-field Model of Hasegawa and Wakatani

The four-field model of Hasegawa and Wakatani [1983] was given in 5.2.1., and its dissipationless version contains the following reduced FLR-MHD equations:

\[
\frac{c^2}{4\pi v_A^2} \left( \frac{d}{dt} + \hat{\mathbf{z}} \times \nabla_\perp P_i \cdot \nabla_\perp \right) \nabla_\perp^2 \delta \phi + \frac{c}{4\pi} \nabla_\parallel \nabla_\perp^2 \delta A_\parallel \tag{6.47}
\]

\[
= \frac{2c\hat{\mathbf{z}}}{B} \times \nabla \ln B \cdot \nabla_\perp P_i;
\]

\[
\frac{\partial \delta A_\parallel}{\partial t} + c \nabla_\parallel \delta \phi = \frac{c}{e N_0} \nabla_\parallel \dot{P}_e; \tag{6.48}
\]

\[
\frac{dP_e}{dt} = - \frac{cT_e}{e4\pi} \nabla_\parallel (\nabla_\perp^2 \delta A_\parallel); \tag{6.49}
\]

\[
\frac{dP_i}{dt} = 0. \tag{6.50}
\]

The Hasegawa–Wakatani model neglects the compressional part of the magnetic field perturbation \(\delta B_\parallel\) as well as ion parallel motion \(\delta v\). In addition, it assumes that the ions represent an incompressible fluid, i.e., \(dP_i/dt = 0\), which also implies that the perturbed \(E \times B\) velocity is given in its lowest-order incompressible form. With these assumptions, the isotropic gyrofluid vorticity equation, Eq. (6.32), becomes

\[
\frac{m_i N_0 c^2}{B^2} \frac{d}{dt} \nabla_\perp^2 \delta \phi + \frac{c^2}{B^2 \Omega_i} \nabla_\perp \cdot (\hat{\mathbf{z}} \times \nabla_\perp P_i \cdot \nabla_\perp \delta \phi) + \frac{c}{4\pi} \nabla_\parallel \nabla_\perp^2 \delta A_\parallel = \frac{2c}{B} \nabla_\perp \dot{P} \cdot \hat{\mathbf{z}} \times \nabla \ln B. \tag{6.51}
\]
The remaining difference between Eqs. (6.47) and (6.51) is the energy-conserving term

\[ \frac{c^2}{B^2 \Omega_i} \nabla \times \nabla \perp p \cdot \nabla \perp \delta \phi, \]

which was discussed in 5.2.3.6. in relation to the energy-conservation property. Because this term does not appear in Eq. (6.47), the Hasegawa–Wakatani model does not conserve energy.

The gyrofluid and Hasegawa–Wakatani’s generalized Ohm laws are represented by the same equation, Eq. (6.48), if we make use, in Eq. (6.33), of the assumptions indicated above. The generalized Ohm’s law contains the Hall term which allows the consideration of the so-called electron Boltzmann’s response. Furthermore, if we neglect ion-pressure-gradient terms and consider a uniform equilibrium magnetic field, the Hasegawa–Wakatani model reduces to the three-field model of Hazeltine [1983], given in 5.2.2.

Finally, the consideration of a nonuniform magnetic field coupled with the pressure-gradient term, appearing on the right-hand side of the vorticity equation, Eq. (6.47), provides the necessary driving force for drift-ballooning modes in nonuniform magnetized plasmas.

2.0.2. Four-field HKM Model

The four-field model of Hazeltine, Kotschenreuther, and Morrison [1985] was given in 5.2.3 in its dissipationless version, and contains the following reduced FLR-MHD equations:

\[ \frac{\partial \nabla^2 \phi}{\partial t} + \left[ \phi + \delta \tau p, \nabla^2 \phi \right] + \nabla \parallel \nabla^2 \psi + (1 + \tau) \left[ h, p \right] = \delta \tau \left[ \nabla \perp \phi; \nabla \perp p \right], \tag{6.52} \]

\[ \frac{\partial \psi}{\partial t} + \nabla \parallel \psi = \delta \nabla \parallel p; \tag{6.53} \]

\[ \frac{\partial v}{\partial t} + \left[ \phi, v \right] + \frac{1}{2} (1 + \tau) \nabla \parallel p = \delta \tau \beta_e \left\{ \frac{1}{2} (1 + \tau) [p, v] + 4[h, v] \right\}; \tag{6.54} \]

\[ \frac{\partial p}{\partial t} + \left[ \phi, p \right] = \beta_e \left\{ 2[h, (\phi - \delta p)] - \nabla \parallel (v + 2\delta \nabla^2 \psi) \right\}, \tag{6.55} \]

where Hazeltine’s normalization [ Eqs. (5.28)–(5.31) and (5.42)–(5.44) ] was used with \( \tau = T_i / T_e \). The HKM model assumes that the ion and electron temperatures are uniform, and considers equilibrium-magnetic-field nonuniformity through the lowest-order (normalized) term \( \nabla \ln B = -\epsilon \nabla \perp h / a \).
In order to compare the reduced FLR-MHD equations of the HKM model with our gyrofluid isotropic model, we must first obtain the normalized form of the gyrofluid equations. The normalized gyrofluid isotropic equations are given as follows: the normalized gyrofluid isotropic vorticity equation is

\[
\frac{\partial}{\partial t} \nabla_\perp^2 \phi + \left[ \phi + \delta \tau p, \nabla_\perp^2 \phi \right] + \nabla_\parallel \nabla_\perp^2 \psi + (1 + \tau)[h, p] - \delta \tau \left[ \nabla_\perp \phi, \nabla_\perp p \right] = \frac{3}{2} \delta \tau \beta_e \frac{d \nabla_\parallel^2 b}{dt} + \frac{1}{2} (1 + \tau) \left[(p + 3 \beta_e h), b\right] + \delta \tau \beta_e \left\{ 3[\nabla_\perp^2 \phi, h] + \nabla_\perp \cdot [\nabla_\perp \psi, v] \right\};
\]

(6.56)

the normalized gyrofluid isotropic Ohm's law is

\[
\frac{\partial \psi}{\partial t} + \nabla_\parallel \phi - \delta \nabla_\parallel p = \delta \beta_e \nabla_\parallel b;
\]

(6.57)

the normalized gyrofluid isotropic parallel acceleration law is

\[
\frac{\partial v}{\partial t} + [\phi, v] + \frac{1}{2} (1 + \tau) \nabla_\parallel p - \delta \tau \beta_e \left\{ \frac{1}{2} (1 + \tau) [p, v] + 4 [h, v] \right\} = - \beta_e \left\{ \frac{1}{2} (1 + \tau) \nabla_\parallel b \\
+ \delta \tau \left[b + \frac{1}{2} (1 + \tau) p + \delta \nabla_\perp^2 \phi, v\right] + \frac{\delta \tau}{2} \nabla_\perp^2 \left( \nabla_\parallel p \right) + \delta \tau [\nabla_\perp \psi; \nabla_\perp \phi] \right\};
\]

(6.58)

and the normalized gyrofluid isotropic (isothermal) electron pressure equation is

\[
\frac{\partial p}{\partial t} + [\phi, p] + 2 \beta_e \left[(\phi - \delta p), h\right] + \beta_e \nabla_\parallel (v + 2 \delta \nabla_\perp^2 \psi) = \delta \beta_e \left\{ 3 b + (p + 3 \beta_e h) - \delta \tau \beta_e \nabla_\parallel \nabla_\perp^2 v \right\}.
\]

(6.59)

Next, we give the normalized form of the gyrofluid Maxwell's equations, given by Eqs. (6.18)–(6.19) and (6.21), as follows:

\[
(N_{0i} - N_{0e})/\epsilon N_0 = -2 \delta \left( \nabla_\perp^2 \phi + \frac{\delta \tau}{2} \nabla_\perp^2 p + \frac{3}{2} \delta \tau \beta_e \nabla_\perp^2 b \right),
\]

(6.60)

\[
\delta J_\parallel /ee N_0 v_A = -2 \delta \left( \nabla_\perp^2 \psi + \frac{1}{2} \delta \tau \beta_e \nabla_\perp^2 v \right),
\]

(6.61)

\[
(\beta/\beta_e) b + \frac{1}{2} (1 + \tau) p = -\frac{\delta \tau \beta_e}{2} \left( 3 \nabla_\perp^2 \phi + \delta \tau \nabla_\perp^2 p \right),
\]

(6.62)

where \( \beta = \beta_e [1 + (1 + \tau) \beta_e] \).

We now show that the HKM reduced equations, Eqs. (6.52)–(6.55), are recovered from the gyrofluid isothermal equations, Eqs. (6.56)–(6.59), by applying a few simple (consistent) limits. The HKM vorticity equation and Ohm's law, Eqs. (6.52) and
(6.53), are recovered from Eqs. (6.56) and (6.57), by simply neglecting all terms of order $O(\beta_\epsilon)$. We note that, from Eqs. (6.59) and (6.62), we have $[p, b] = O(\beta_\epsilon) = dp/dt$. The HKM electron pressure equation, Eq. (6.55), is recovered from the gyrofluid isothermal electron pressure equation, Eq. (6.59), if we neglect all terms of $O(\beta_\epsilon^2)$.

The case of the HKM parallel acceleration law, Eq. (6.54), is slightly more subtle to consider. If we retain only the lowest-order term in $\nabla_\parallel p$ and neglect terms of $O(k_\parallel^2 \beta_\parallel^2)$, in the gyrofluid parallel acceleration law, Eq. (6.58), then Eq. (6.54) is recovered.
PART IV: SUMMARY AND DISCUSSION

The reduced gyrokinetic description of low-frequency nonlinear tokamak dynamics was presented in this dissertation in three different versions: the gyrocenter (test-particle) description; the gyrokinetic (self-consistent) description; and the gyrofluid description. These three reduced descriptions are members of a hierarchy based on the gyrokinetic ordering.

The Hamiltonian perturbation method known as the extended Phase-space Lagrangian Lie perturbation method is used throughout the dissertation and represents a powerful and versatile tool in the development of the gyrokinetic formalism.

The reduced equations presented in this dissertation should have wide applicability in the nonlinear analysis of nonuniform magnetized plasmas.
Chapter 7

Summary and Discussion

This dissertation is concerned with the perturbation of an equilibrium tokamak (guiding-center) plasma by electromagnetic fields which satisfy the gyrokinetic ordering (introduced in Section 1.1). A unified gyrokinetic hierarchy of descriptions is identified and contains the gyrocenter description, the gyrokinetic description, and the gyrofluid description. Each member of the hierarchy is derived from the previous description, with the gyrocenter description being derived from the unperturbed guiding-center description (Chapter 2) and the electromagnetic field perturbation. The reduced equations given at each level of the hierarchy have an (adiabatic) energy invariant which can be derived from the guiding-center energy invariant.

An important element in our discussion of single-particle dynamics is the use of the extended phase-space Lagrangian (Section 1.2) in which the Hamiltonian function and the Poisson-bracket structure both appear explicitly in the phase-space Lagrangian. The extended phase-space Lagrangian Lie perturbation method (Section 1.3), based on the theory of Lie transforms, is used extensively throughout the dissertation. This method is especially relevant to the analysis of electromagnetic perturbations since the unperturbed Hamiltonian function and the Poisson-bracket structure are affected simultaneously by such perturbations.

The Hamiltonian gyrocenter theory (Chapter 3) describes the non-self-consistent behavior of test-particles under the influence of electromagnetic fields which satisfy the gyrokinetic ordering. Gyrocenter dynamics are expressed in terms of a gyrocenter Hamiltonian function, a gyrocenter Poisson-bracket structure, and a transformation from the guiding-center phase space to the gyrocenter phase space. Two representations (Hamiltonian and symplectic) are identified and refer to the specific form
adopted for the gyrocenter Poisson-bracket structure. In both cases the corresponding magnetic moment is an adiabatic invariant to all orders in perturbation amplitude. The gyrocenter Hamilton's equations are given in both representations (Section 3.3) and contain linear as well as nonlinear (quadratic) terms in perturbation amplitude. The gyrocenter drift equations generalize the guiding-center drift equations and should find many applications in the study of the diffusion of test-particles in turbulent media with dominant low-frequency, small-perpendicular-wavelength fluctuations (Chan [1989]).

The self-consistent gyrokinetic description is given in terms of the nonlinear gyrokinetic Maxwell-Vlasov system. The gyrokinetic Vlasov equation and the gyrokinetic Maxwell's equations are obtained through the application of the guiding-center and gyrocenter transformations (Sections 3.1 and 3.2). The resulting gyrokinetic Maxwell-Vlasov system is shown to have an adiabatic gyrokinetic energy invariant given as the sum of the gyrocenter kinetic energy integral and the electromagnetic field energy integral. Two sets of energy-conserving gyrokinetic equations are given with one set corresponding to the linear approximation (linear gyrokinetics) and the other set corresponding to the quadratic approximation — each set has an exact gyrokinetic energy conservation law (Section 3.3). The nonlinear gyrokinetic Maxwell-Vlasov system, in the electrostatic or finite-β electromagnetic approximation, has been used in gyrokinetic particle simulations of drift turbulence (Lee [1983, 1987] and Lee et al. [1984]) and ion-temperature-gradient physics (Lee and Tang [1988]) in slab geometry.

A third and final reduced description is given by the gyrofluid reduced description which is derived from the moments of the gyrokinetic Maxwell-Vlasov equations (Chapter 6). The reduced fluid equations have a natural energy invariant which is simply the gyrokinetic energy invariant of the Maxwell-Vlasov system expressed in terms of the gyrocenter velocity-space moments. The main advantages of this new generation of reduced fluid models over the previous generations (RMHD and FLR-RMHD) is the simple yet powerful procedure used to derive these reduced (gyrofluid) equations and the physical transparency of the gyrofluid energy invariant (compared to the HKM model, for example). Furthermore, the isotropic gyrofluid model can be used in the electrostatic limit to give a four-field \((δφ, δu, P_{||}, P_e)\) model which can be used to study drift-wave turbulence (including ion-temperature-gradient effects) in toroidal geometry.
It is hoped that the reduced equations of the gyrokinetic hierarchy presented in this dissertation will find wide application, especially in the research effort towards understanding the anomalous transport processes in nonuniform magnetized (tokamak) plasmas.

Finally, we would like to mention the works of Kaufman and Boghosian [1984] and Boghosian [1987], who considered the use of Lie-Poisson structures in their Hamiltonian formulations of gyrokinetics (Kaufman and Boghosian [1984]) and relativistic guiding-center and oscillation-center theories (Boghosian [1987]). Such formulations treat both particles (Vlasov equation) and fields (Maxwell's equations) on the same level — through the use of a covariant action principle — and thus the (Lie) Poisson-bracket structure involves functional derivatives with respect to the dynamical field variables \((f, E, B)\). Kaufman and Boghosian [1984] rederived the gyrokinetic equations of Dubin et al. [1983].
Appendix A

Axisymmetric Guiding-center and Gyrocenter Dynamics

Magnetic coordinates provide a natural representation for magnetic fields, since the magnetic-field topology is intrinsically characterized by such coordinates. In addition, under certain conditions, magnetic coordinates can be viewed as canonical variables in a multi-dimensional Hamiltonian system, representing the magnetic-field-line flow.

The Hamiltonian guiding-center theory of White and Chance (1984) emphasized this point by using canonical (magnetic-coordinate) variables to describe the drift motion of charged particles in tokamak plasmas. This formulation proved to be useful in studying the drift motion of energetic particles and their effects on the stability of auxiliary-heated tokamak discharges.

We present in this Appendix the phase-space Lagrangian formulation of the Hamiltonian theory of the magnetic-field-line flow of Cary and Littlejohn (1983) and Littlejohn (1984c), and the Hamiltonian guiding-center theory of Littlejohn (1985).

Finally, recent numerical work by Chan et al. (1989), based on the application of the Poincaré-map method, looks at the stochasticity-induced transport of magnetospheric high-energy protons due to resonance overlap of high-\(m\) MHD modes. Because these modes are characterized by perpendicular wavelengths of the order of the ion gyroradius, a gyrokinetic (gyrocenter) description was used. In this Appendix, the gyrocenter equations of motion (derived in chapter 3) are given in terms of the magnetic coordinates used to describe an axisymmetric tokamak configuration (Brizard [1989a]).
A.1 Magnetic Coordinate Representation

A.1.1. Magnetic Coordinates

Using straight-field-line magnetic coordinates \((\psi_p, \theta, \xi)\) where field lines on the \((\theta, \xi)\)-plane are straight lines, the covariant representation for the axisymmetric tokamak magnetic field is (White and Boozer [1982], and White and Chance [1984])

\[
B = g(\psi_p) \nabla \xi + I(\psi_p) \nabla \theta + \delta(\psi_p, \theta) \nabla \psi_p,
\]

where \(g\) contains contributions from the vacuum toroidal field and the plasma diamagnetic field, and \(I\) contains the poloidal magnetic field component due to (toroidal) plasma currents. The role of \(\delta\) follows from the condition \(B \cdot \nabla \psi_p = 0\), which implies from Eq. (A.1) that

\[
-\delta(\psi_p, \theta) = \frac{g \left( \nabla \psi_p \cdot \nabla \xi \right)}{\mid \nabla \psi_p \mid^2} + I \frac{\left( \nabla \psi_p \cdot \nabla \theta \right)}{\mid \nabla \psi_p \mid^2}.
\]

Hence, \(\delta\) represents the degree of non-orthogonality in our spatial coordinate system, and for unshifted circular surfaces we have \(\delta = 0\) (White and Chance [1984]).

The contravariant representation for the magnetic field is given as

\[
B = \nabla \psi_p \times \nabla [q(\psi_p) \theta - \xi],
\]

where \(q(\psi_p) = B \cdot \nabla \xi / B \cdot \nabla \theta\) is the safety factor. From this expression, we immediately obtain the magnetic vector potential

\[
A = \psi \nabla \theta - \psi_p \nabla \xi,
\]

where \((2\pi)\psi\) is the toroidal flux, and \(d\psi / d\psi_p = q(\psi_p)\). Finally, the Jacobian for the transformation from physical spatial coordinates to magnetic coordinates \((\psi_p, \theta, \xi)\) is

\[
J_B^{-1} = (\nabla \psi_p \times \nabla \theta) \cdot \nabla \xi = B^2(\psi_p, \theta) / (gq + I).
\]

A.1.2. Hamiltonian Formulation of Magnetic-field-line Flow

We wish to show that the magnetic-field-line flow, given by Eq. (A.2), is a 1-D Hamiltonian system, with simple canonical coordinates, when the magnetic field configuration is axisymmetric. Specifically, we will show that the general magnetic-field-line flow can be represented by a 2-D Hamiltonian system, and that in the case of axisymmetry, we have a 1-D Hamiltonian system.
The divergenceless nature of a general magnetic field is made explicit by the use of the contravariant representation (c.f., Boozer [1986])

\[ B = \nabla \psi \times \nabla \theta + \nabla \phi \times \nabla \chi, \]  \hspace{1cm} (A.5)

where \( \psi, \chi, \theta, \) and \( \phi \) are functions of position \( r \). (In toroidal geometry, \( \theta \) and \( \phi \) are the poloidal and toroidal angles, respectively, whereas \( 2\pi \chi \) and \( 2\pi \psi \) are the poloidal and toroidal flux functions.) The differential equation which defines a general magnetic-field line is given by the well-known expression

\[ \frac{dr^1}{B^1} = \frac{dr^2}{B^2} = \frac{dr^3}{B^3} = ds, \]  \hspace{1cm} (A.6)

where \( s \) is the field-line parameter. This differential equation can also be given in vector form as \((dr/ds) \times B = 0\), which is especially well suited for a description in terms of a differential one-form.

Indeed, let us consider the extended magnetic-field-line one-form \( \gamma_B \),

\[ \gamma_B = A \cdot dr - Hds, \]  \hspace{1cm} (A.7)

in extended coordinate space \((r, H)\). The real magnetic-field line, defined in Eq. (A.6), is assumed to correspond to the submanifold \( H^{-1}(0) \), so that we refer to \( H \) as the extended magnetic-field-line Hamiltonian, and \( s \) is the usual extended-time parameter. It is quite simple to show that the requirement that \( i_V \omega_B = i_V \cdot d\gamma_B \) vanish, for any arbitrary (spatial) vector field \( V \), implies that \( dr \times B = 0 \), which defines the magnetic-field-line flow, Eq. (A.6).

From the contravariant representation of a general magnetic field, given in Eq. (A.5), we obtain the (covariant) expression for the vector potential

\[ A = \psi \nabla \theta - \chi \nabla \phi + \nabla \sigma, \]

where \( \sigma(r) \) is a gauge potential. As is well known from the theory of electromagnetics, the choice of this gauge potential does not affect the expression for the magnetic field \( B = \nabla \times A \). In the context of phase-space Lagrangian dynamics, the term \( \nabla \sigma \cdot dr = d\sigma \) in Eq. (A.7) represents a phase-space gauge transformation and does not affect the Hamiltonian dynamics.

If we substitute the expression for \( A \) into the extended magnetic-field one-form, Eq. (A.7), we obtain the canonical one-form

\[ \gamma_B = \psi \, d\theta - \chi \, d\phi - H(\psi, \theta, \phi, \chi) \, ds, \]  \hspace{1cm} (A.8)
in extended magnetic-coordinate space \((\psi, \theta, \phi, \chi)\). It is simple to show that the general magnetic-field-line flow can be represented by the canonical Hamilton's equations

\[
\begin{align*}
\frac{d\theta}{ds} &= \frac{\partial H}{\partial \psi}, & \frac{d\psi}{ds} &= -\frac{\partial H}{\partial \theta}, \\
\frac{d\phi}{ds} &= -\frac{\partial H}{\partial \chi}, & \frac{d\chi}{ds} &= \frac{\partial H}{\partial \phi}.
\end{align*}
\tag{A.9}
\]

When the extended Hamiltonian \(H\) is independent of the (toroidal) angle \(\phi\), we can choose the extended Hamiltonian to be given as (Boozer [1986])

\[
H = \chi_H(\psi, \theta) - \chi,
\tag{A.10}
\]

in direct analogy with the theory of extended Hamiltonian dynamics presented in chapter 1. In this case, the extended-time parameter is simply the toroidal angle \(\phi\), and the axisymmetric magnetic-field-line flow is represented as a 1-D Hamiltonian system:

\[
\frac{d\theta}{d\phi} = \frac{\partial \chi_H}{\partial \psi}, \quad \text{and} \quad \frac{d\psi}{d\phi} = -\frac{\partial \chi_H}{\partial \theta}.
\tag{A.11}
\]

Furthermore, when the Hamiltonian \(\chi_H\) is independent of the angle \(\theta\), the canonical coordinates \(\psi\) and \(\theta\) represent action-angle variables. The coordinate \(\psi\) is the action and is a constant of the motion (i.e., field lines lie on a constant-\(\psi\) surface), and the frequency associated with the Hamiltonian flow, \(d\theta/d\phi = q(\chi)^{-1}\), is known as the rotational transform.

A.1.3. Guiding-center Phase-space Lagrangian

The guiding-center phase-space Lagrangian was derived in chapter 2 [see Eq. (2.57)], and was given by the expression

\[
\gamma = e^{-1} (A + e\rho_B B) \cdot dX + e \mu d\zeta - \left(\frac{1}{2} \rho_B^2 B^2 + \mu B\right) dt,
\tag{A.12}
\]

where units are used for which \(e = m = c = 1\), and Eq. (A.12) is correct up to order \(O(e_B^0)\). Using the covariant representation for the magnetic field, Eq. (A.1), and the expression for the vector potential, Eq. (A.3), we obtain the magnetic-coordinate representation for the guiding-center symplectic structure of Eq. (A.12) (Littlejohn [1985])

\[
\tilde{\gamma} = e^{-1} \psi d\theta - e^{-1} \psi \rho_p d\xi + \rho_B (g d\xi + L d\theta + \delta d\psi_p) + e \mu d\zeta.
\tag{A.13}
\]
The determinant of the Lagrange matrix \((\mathbf{\omega})\), whose coefficients are the components of the symplectic two-form \(\mathbf{\omega} = d\mathbf{\gamma}\), is given as \(\det(\mathbf{\omega}) = D^2\), where
\[
D = (gq + I) + \epsilon\rho_\parallel \left[ g(I' - \partial\delta/\partial\theta) - g'I \right] \equiv J_B BB_\parallel^*,
\] (A.14)
where a derivative with respect to \(\psi_p\) is denoted by a prime. The contravariant representation for \(B^* = B + \epsilon\rho_\parallel \nabla \times B\) is given as
\[
B^* = \nabla \psi_p \times \nabla \theta \left[ q + \epsilon\rho_\parallel \left( I' - \partial\delta/\partial\theta \right) \right] - \nabla \psi_p \times \nabla \xi \left( 1 - \epsilon\rho_\parallel g' \right),
\]
where the contravariant representation was used for \(B\), and the covariant representation was used for \(\nabla \times B\).

Finally, using the inversion algorithm given in Chapter 1 [see Eqs. (1.43) and (1.44)], we obtain the Poisson bracket
\[
\{F, G\} = \frac{1}{\epsilon} \left( \frac{\partial F}{\partial \xi} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \xi} \right) - \frac{1}{D} \left( g \frac{\partial G}{\partial \theta} - I \frac{\partial G}{\partial \xi} \right) \frac{\partial F}{\partial \psi_p}
+ \frac{1}{D} \left[ \frac{\partial G}{\partial \rho_\parallel} + \epsilon \left( g \frac{\partial G}{\partial \psi_p} - \rho_\parallel g' \frac{\partial G}{\partial \psi_p} \right) \right] \frac{\partial F}{\partial \theta}
+ \frac{1}{D} \left[ q \frac{\partial G}{\partial \rho_\parallel} - \epsilon \left( I \frac{\partial G}{\partial \psi_p} - \rho_\parallel I' \frac{\partial G}{\partial \psi_p} \right) \right] \frac{\partial F}{\partial \xi}
- \frac{1}{D} \left[ \frac{\partial G}{\partial \theta} + q \frac{\partial G}{\partial \xi} \right] - \epsilon\rho_\parallel \left( g' \frac{\partial G}{\partial \theta} - I' \frac{\partial G}{\partial \xi} \right) \frac{\partial F}{\partial \rho_\parallel}
+ \frac{\epsilon}{D} \left[ \delta \frac{\partial F}{\partial \theta} - \rho_\parallel \frac{\partial \delta F}{\partial \theta} \right] \frac{\partial F}{\partial \xi} - \left( \delta \frac{\partial F}{\partial \theta} - \rho_\parallel \frac{\partial \delta F}{\partial \theta} \right) \frac{\partial G}{\partial \rho_\parallel},
\] (A.15)
where \(F\) and \(G\) are arbitrary functions. The equations of motion in magnetic-coordinate phase space are given as
\[
\dot{\psi}_p = \{\psi_p, H\} = -\frac{1}{D} \left( g \frac{\partial H}{\partial \theta} - I \frac{\partial H}{\partial \xi} \right),
\]
\[
\dot{\theta} = \{\theta, H\} = \frac{1 - \epsilon\rho_\parallel g'}{D} \frac{\partial H}{\partial \rho_\parallel} + \frac{\epsilon}{D} \left( g \frac{\partial H}{\partial \psi_p} - \delta \frac{\partial H}{\partial \xi} \right),
\] (A.16)
\[
\dot{\xi} = \{\xi, H\} = \frac{1}{D} \left[ q + \epsilon\rho_\parallel \left( I' - \partial\delta/\partial\theta \right) \right] \frac{\partial H}{\partial \rho_\parallel} - \frac{\epsilon}{D} \left( I \frac{\partial H}{\partial \psi_p} - \delta \frac{\partial H}{\partial \theta} \right),
\]
\[
\dot{\rho}_\parallel = \{\rho_\parallel, H\} = -\frac{1 - \epsilon\rho_\parallel g'}{D} \frac{\partial H}{\partial \theta} - \frac{1}{D} \left[ q + \epsilon\rho_\parallel \left( I' - \partial\delta/\partial\theta \right) \right] \frac{\partial H}{\partial \xi},
\]
where the function \(H\) is a Hamiltonian function, yet to be determined.
A.1.4. Canonical Formulation

By inspection of Eq. (A.13), the toroidal component of the generalized momentum is easily obtained as

\[ P_\xi = -\epsilon^{-1} \psi_p + \rho_{||} g(\psi_p). \]  

(A.17)

Using the toroidal momentum \( P_\xi \) as a phase-space coordinate, the 2-D Hamiltonian system for the magnetic-field-line flow can be represented by the phase-space Lagrangian \( \gamma = \tilde{\gamma} - H dt \), where

\[ \tilde{\gamma} = P_\xi d\xi + \epsilon^{-1} (\psi + \epsilon \rho_{||} I) d\theta + \rho_{||} \delta d\psi_p, \]  

(A.18)

where \( \rho_{||}(\psi_p, P_\xi) = (\psi_p + \epsilon P_\xi) g(\psi_p) \). When the magnetic-field-line flow is axisymmetric, Eq. (A.18) is used to describe reduced Hamiltonian dynamics in the reduced phase space \( (\psi_p, \theta) \), as was explained in 1.2.3.5.

The symplectic structure is given by the two-form \( d\tilde{\gamma} \),

\[ d\tilde{\gamma} = \frac{D}{\epsilon g} d\psi_p \wedge d\theta + dP_\xi \wedge \left( \frac{B \cdot dr}{g} \right). \]

Using the inversion algorithm of Littlejohn [1981], introduced in chapter 1, we obtain the Poisson bracket

\[ \{F, G\} = \left( \frac{\partial F}{\partial \xi} \frac{\partial G}{\partial P_\xi} - \frac{\partial F}{\partial P_\xi} \frac{\partial G}{\partial \xi} \right) + \epsilon g \left( \frac{\partial F}{\partial \theta} \frac{\partial G}{\partial \psi_p} - \frac{\partial F}{\partial \psi_p} \frac{\partial G}{\partial \theta} \right) + \frac{\delta}{\partial \xi} \left( I \frac{\partial F}{\partial \psi_p} - \xi \frac{\partial G}{\partial \xi} \right) - \left\{ \frac{\partial G}{\partial \psi_p} - \delta \frac{\partial F}{\partial \theta} \right\} \frac{\partial F}{\partial \xi}. \]  

(A.19)

It is clear from this expression that the coordinates \((P_\xi, \xi, \psi_p, \theta)\) are almost canonical.

In fact, if we neglect terms of order \( O(\rho^2_{||}) \), it is possible to find simple expressions for canonical coordinates. By inspection of Eq. (A.13), we find the poloidal component of the generalized momentum as

\[ P_\theta = \psi + \rho_{||} I, \]  

(A.20)

where we omit showing \( \epsilon \) for simplicity. The canonical toroidal angle is the original toroidal angle, \( \xi_c = \xi \), whereas the canonical poloidal angle \( \theta_c \) is defined as follows. In order to make the term \((\rho_{||} \delta d\psi_p)\) disappear in Eq. (A.13), we define the canonical poloidal angle as \( \theta_c = \theta - \rho_{||} \delta / q \). Thus, neglecting terms of order \( O(\rho^2_{||}) \), we have

\[ (\psi + \rho_{||} I) d\left( \theta_c - \frac{\rho_{||} \delta}{q} \right) + \rho_{||} \delta d\psi_p = P_\theta d\theta_c - \frac{\rho_{||} \delta}{q} d\psi + d\left( \frac{\rho_{||} \delta \psi}{q} \right) + \rho_{||} \delta d\psi_p. \]
Since phase-space Lagrangian dynamics ignore exact differential one-forms (phase-space gauge transformations), and \( d\psi = q \, d\psi_p \), then the guiding-center phase-space Lagrangian, Eq. (A.12), acquires the canonical form

\[
\gamma_c = P_\theta \, d\theta + P_\xi \, d\xi - H_0 \, dt,
\]

with the canonical variables defined as (White and Chance [1984])

\[
\begin{pmatrix}
P_\theta \\
P_\xi \\
\theta_c \\
\xi_c
\end{pmatrix} =
\begin{pmatrix}
\psi + \rho_\parallel \frac{I}{q} \\
-\psi_p + \rho_\parallel g \\
\theta - \rho_\parallel \frac{\delta q^{-1}}{}
\end{pmatrix}.
\]

(A.21)

## A.2 Equations of Motion

### A.2.1. Guiding-center Drifts

Using Eq. (A.16), the unperturbed guiding-center equations of motion in an axisymmetric tokamak magnetic-field are (White and Chance [1984])

\[
\dot{\psi}_p = -\frac{g}{D}(\mu + \rho_\parallel^2 B) \frac{\partial B}{\partial \psi_p},
\]

(A.22)

\[
\dot{\theta} = (1 - \epsilon \rho_\parallel g') \frac{\rho_\parallel B^2}{D} + \frac{g}{D}(\mu + \rho_\parallel^2 B) \frac{\partial B}{\partial \psi_p},
\]

(A.23)

\[
\dot{\xi} = \left[ q + \epsilon \rho_\parallel \left( I' - \frac{\delta q}{\partial \theta} \right) \right] \frac{\rho_\parallel B^2}{D} - \frac{\epsilon}{D}(\mu + \rho_\parallel^2 B) \left( I \frac{\partial B}{\partial \psi_p} - \frac{\delta B}{\partial \theta} \right),
\]

(A.24)

where the unperturbed guiding-center Hamiltonian is given by the expression

\[
H_0(\psi_p, \theta, \rho_\parallel ; \mu) = \frac{1}{2} \rho_\parallel^2 B^2(\psi_p, \theta) + \mu B(\psi_p, \theta).
\]

(A.25)

As is well-known, Eq. (A.22) describes the radial motion corresponding to untrapped particles (shifted drift surface) and trapped particles (banana orbits). The first term on the right-hand side of Eqs. (A.23) and (A.24) represents the parallel motion of guiding-centers along a magnetic-field line.

The unperturbed parallel acceleration law is given by the expression (White and Chance [1984])

\[
\dot{\rho}_\parallel = \{ \rho_\parallel , H_0 \} = \left( 1 - \frac{\epsilon \rho_\parallel g'}{D} \right)(\mu + \rho_\parallel^2 B) \frac{\partial B}{\partial \theta}.
\]

(A.26)
Finally, we obtain the toroidal momentum equation, either from Eq. (A.17) with Eqs. (A.22) and (A.26) or from Eq. (A.19),

$$\dot{P}_\xi = 0,$$

(A.27)

which holds for axisymmetric tokamak configurations.

A.2.2. Gyrocenter Drifts

Using the phase-space coordinates \((\psi_p, \theta, \xi, P_\xi)\), the expressions for the gyrocenter equations of motion (in the Hamiltonian representation) are given in terms of the Poisson bracket, Eq. (A.19), and the gyrocenter Hamiltonian

$$H = H_0(\psi_p, \theta, P_\xi ; \mu) + \epsilon \Phi^*(\psi_p, \theta, \xi, P_\xi ; \mu),$$

(A.28)

where the perturbed (effective) potential \(\Phi^*\) was derived in chapter 3, and the parallel momentum function \(\rho_\parallel\) is given as

$$\rho_\parallel(\psi_p, P_\xi) = \frac{(\psi_p + \epsilon P_\xi)}{\epsilon g(\psi_p)}.$$

The guiding-center drift equations are given as

$$\dot{\psi}_p = \dot{\psi}_p|_{gc} - \frac{\epsilon \epsilon_\xi}{D} \left( g \frac{\partial \Phi^*}{\partial \theta} - I \frac{\partial \Phi^*}{\partial \xi} \right),$$

(A.29)

$$\dot{\theta} = \dot{\theta}|_{gc} + \frac{\epsilon \epsilon_\xi}{D} \left( g \frac{\partial \Phi^*}{\partial \psi_p} - \delta \frac{\partial \Phi^*}{\partial \xi} \right),$$

(A.30)

$$\dot{\xi} = \dot{\xi}|_{gc} + \epsilon \frac{\partial \Phi^*}{\partial P_\xi} - \frac{\epsilon \epsilon_\xi}{D} \left( I \frac{\partial \Phi^*}{\partial \psi_p} - \delta \frac{\partial \Phi^*}{\partial \theta} \right),$$

(A.31)

where \(\dot{\psi}_p|_{gc}\), etc., are given by Eqs. (A.22)–(A.24). Finally, the toroidal momentum equation is given as

$$\dot{P}_\xi = -\epsilon \frac{\partial}{\partial \xi} (\delta \Phi^*).$$

(A.32)
Appendix B

FLR Formulation of Plasma Physics

FLR physics play an important role in the dynamics of high-temperature tokamak plasmas. In order to investigate the inclusion of FLR effects in the fluid equations, we shall consider solving the FLR-modified Vlasov equation, expressed in (lowest-order) drift phase space. The solution is then used to write down expressions for the particle flux (in the continuity equation), and the collisionless transport coefficients such as the gyroviscous stress tensor (in the equation of motion), and the collisionless heat flux (in the energy equation).

B.1. Vlasov Equation in Drift Phase Space

The presentation given here is based on a paper by Rosenbluth and Simon [1965].

The investigation of how FLR effects are introduced into the Vlasov equation is better conducted in a phase space which drifts with the (lowest-order) drift velocity. Consider the transformation from regular phase space \((r, v)\) to drift phase space \((x, w)\) given as

\[
  r = x \quad \text{and} \quad v = w + u(x_t),
\]

where \(u\) is the lowest-order drift velocity, and the particle velocity \(w\) is given as

\[
  w = w_{\|} \tilde{z} + w_{\perp} \tilde{I},
\]

where \(\tilde{I} = \tilde{\zeta} \times \tilde{z} = e^{i\zeta} \tilde{I} + e^{-i\zeta} \tilde{\omega},\) and \(\tilde{I} = \tilde{z}^* = (\tilde{x} - i\tilde{y})/2.\) With these definitions, the Vlasov equation, in drift phase space, is given by the expression

\[
  \frac{df}{dt} + w \cdot \nabla f + \left[(u + w) \times \Omega + \frac{e}{m}E - \frac{du}{dt} - w \cdot \nabla u\right] \cdot \frac{\partial f}{\partial w} = 0, \quad (B.1)
\]
where
\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad \text{with} \quad \nabla = \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial \mathbf{w}} = \frac{\mathbf{b} \cdot \partial}{\partial w_\parallel} + \frac{\mathbf{i}}{w_\perp} \frac{\partial}{\partial \zeta}. 
\]

The drift velocity \( \mathbf{u}(\mathbf{x}, t) \) is assumed to satisfy the following simplifying requirements: (1) \( \mathbf{b} \cdot \mathbf{u} = 0 \); and (2) \( \nabla \cdot \mathbf{u} = 0 \). To simplify Eq. (B.1), we define the lowest-order drift velocity to be given as
\[
\mathbf{u} = \frac{c}{B} \mathbf{E} \times \mathbf{b},
\]
so that Eq. (B.1) becomes
\[
\frac{df}{dt} + \mathbf{w} \cdot \nabla f + \left( \mathbf{w} \times \Omega - \frac{d\mathbf{u}}{dt} - \mathbf{w} \cdot \nabla \mathbf{u} \right) \cdot \frac{\partial f}{\partial \mathbf{w}} = 0. \tag{B.2}
\]

In addition, the equilibrium magnetic field is assumed to be uniform (with \( \mathbf{b} = \mathbf{z} \)). The drift-phase-space formulation of the Vlasov equation has explicitly introduced three important FLR effects in Eq. (B.2) as follows: (1) the term \( d\mathbf{u}/dt \) will lead to the appearance of the so-called nonlinear polarization drift; (2) the term \( df/dt \) will lead to the generalization of the polarization drift by including the energy-conserving diamagnetic convection term; and (3) the term \( \nabla \mathbf{u} \) will lead to the so-called gyroviscous stress tensor.

If we show explicitly the gyrophase-dependence of the drift Vlasov vector field, we obtain
\[
X_0(f) + e^{i\zeta} X_1(f) + e^{-i\zeta} X_{-1}(f) + e^{2i\zeta} X_2(f) + e^{-2i\zeta} X_{-2}(f) = 0,
\]
where we have neglected parallel gradients \( \partial/\partial z \equiv 0 \), and have defined
\[
X_0 = \frac{d}{dt} - \left( \Omega + \frac{1}{2} \mathbf{b} \cdot \nabla \times \mathbf{u} \right) \frac{\partial}{\partial \zeta},
\]
\[
X_1 = w_\perp \mathbf{i} \cdot \nabla - \mathbf{i} \frac{d\mathbf{u}}{dt} \left( \frac{\partial}{\partial w_\perp} + \frac{i}{w_\perp} \frac{\partial}{\partial \zeta} \right) = X_{-1}, \tag{B.3}
\]
\[
X_2 = -\mathbf{i} \cdot \nabla \mathbf{u} \cdot \mathbf{i} \left( w_\perp \frac{\partial}{\partial w_\perp} + i \frac{\partial}{\partial \zeta} \right) = X_{-2}.
\]

Finally, using the Fourier expansion for \( f \),
\[
f = \sum_{n=-\infty}^{\infty} f_n e^{i n \zeta},
\]
we obtain the Vlasov equation for the $n$th gyrophase-harmonic coefficient:

$$
\bar{X}_0(f_n) + \bar{X}_1(f_{n-1}) + \bar{X}^{-1}(f_{n+1}) + \bar{X}_2(f_{n-2}) + \bar{X}^{-2}(f_{n+2}) = 0, \quad (B.4)
$$

where the notation $\bar{X}_k(f_{n-k})$ is used to indicate that in the $X_k$'s given in Eq. (B.3) the partial derivative $\partial / \partial \zeta$ is replaced by $i(n - k)$.

**B.2. Low-Frequency Ordering**

We define the small parameter $\varepsilon \equiv v_i / \Omega L$ as the ratio of the particle gyroradius to the characteristic scale length $L$. The low-frequency ordering, used in this section, assumes that $u \sim e\nu_T$ and $\omega \sim \varepsilon^2 \Omega$, so that

$$
\left| \frac{1}{\Omega} \frac{d}{dt} \right| \sim O(\varepsilon^2), \quad \text{and} \quad \left| \frac{\nabla u}{\Omega} \right| \sim O(\varepsilon^2).
$$

The same low-frequency ordering, applied to Eq. (B.4), gives

$$
\bar{X}_k(f_{n-k}) = \sum_m \varepsilon^m \bar{X}_{km}(f_{n-k}),
$$

where

$$
\begin{align*}
\bar{X}_{00} &= -in\Omega, \\
\bar{X}_{02} &= \frac{d}{dt} - \left( in/2 \right) \hat{z} \cdot \nabla \times u, \\
\bar{X}_{11} &= \bar{X}_{-11} = w_\perp \hat{1} \cdot \nabla, \\
\bar{X}_{13} &= -\hat{1} \cdot \frac{du}{dt} \left[ \frac{\partial}{\partial w_\perp} - \frac{(n-1)}{w_\perp} \right], \\
\bar{X}_{-13} &= -\hat{2} \cdot \frac{du}{dt} \left[ \frac{\partial}{\partial w_\perp} + \frac{(n+1)}{w_\perp} \right], \\
\bar{X}_{22} &= -\hat{1} \cdot \nabla u \cdot \hat{1} \left[ w_\perp \frac{\partial}{\partial w_\perp} - (n-2) \right], \\
\bar{X}_{-5} &= -\hat{2} \cdot \nabla u \cdot \hat{2} \left[ w_\perp \frac{\partial}{\partial w_\perp} + (n+2) \right].
\end{align*}
$$

Finally, the distribution function is given by the double expansion

$$
f = (f_{00} + \varepsilon f_{01} + \varepsilon^2 f_{02} + \cdots) + \varepsilon \varepsilon^c (f_{11} + \varepsilon f_{12} + \varepsilon^2 f_{13} + \cdots) \quad (B.6)
$$

$$
+ \varepsilon^2 \varepsilon^{2c} (f_{22} + \cdots) + \cdots + (c.c.).
$$
B.3. Solution of Vlasov Equation

The $O(\epsilon^0)$ solution of Eq. (B.4), with Eqs. (B.5) and (B.6), is trivial and corresponds to $f_{00}$ being independent of the gyrophase. At order $O(\epsilon)$, the gyrophase-independent term implies that $f_{01} \equiv 0$, while the gyrophase-dependent terms give

$$f_{11} = -i \frac{w_\perp}{\Omega} \hat{1} \cdot \nabla f_{00} = f_{-11}^*.$$  \hfill (B.7)

At order $O(\epsilon^2)$, the terms associated with $\exp(\pm i\zeta)$ give $f_{\pm 12} \equiv 0$. The terms associated with $\exp(\pm i2\zeta)$ give

$$f_{22} = -\frac{1}{2} \left( \frac{w_\perp}{\Omega} \hat{1} \cdot \nabla \right)^2 f_{00} + i(\hat{1} \cdot \nabla u \cdot \hat{1}) \frac{w_\perp}{2\Omega} \frac{\partial f_{00}}{\partial w_\perp} = f_{-5}^*,$$  \hfill (B.8)

and the gyrophase-independent terms give

$$\frac{df_{00}}{dt} = 0.$$  \hfill (B.9)

At order $O(\epsilon^3)$, only the terms associated with $\exp(\pm i\zeta)$ will be needed in our analysis. These terms give

$$
\begin{align*}
if_{13} & = -i\Omega f_{-13} \\
& = (d/dt - i/2 \hat{2} \cdot \nabla x u) f_{11} + w_\perp \hat{1} \cdot \nabla f_{02} + w_\perp \hat{2} \cdot \nabla f_{22} \\
& \quad - \hat{1} \cdot \frac{du}{dt} \frac{\partial f_{00}}{\partial w_\perp} - \hat{1} \cdot \nabla u \cdot \hat{1} \left( w_\perp \frac{\partial f_{11}}{\partial w_\perp} + f_{-11} \right).
\end{align*}
$$  \hfill (B.10)

Finally, the only undetermined coefficient in the expressions given by Eqs. (B.7)–(B.10) is $f_{02}$, appearing in Eq. (B.10). This coefficient is determined at order $O(\epsilon^4)$ by the gyrophase-independent term, given as

$$
\begin{align*}
\frac{df_{02}}{dt} & = -w_\perp \hat{1} \cdot \nabla f_{-13} - w_\perp \hat{2} \cdot \nabla f_{13} + \hat{1} \cdot \frac{du}{dt} \left( \frac{\partial f_{11}}{\partial w_\perp} + f_{-11} \right) + \hat{2} \cdot \frac{du}{dt} \left( \frac{\partial f_{11}}{\partial w_\perp} + f_{11} \right) \\
& \quad + \hat{1} \cdot \nabla u \cdot \hat{1} \left( w_\perp \frac{\partial f_{-5}}{\partial w_\perp} + 2f_{-5} \right) + \hat{2} \cdot \nabla u \cdot \hat{2} \left( w_\perp \frac{\partial f_{22}}{\partial w_\perp} + 2f_{22} \right).
\end{align*}
$$  \hfill (B.11)

B.4. Continuity Equation

The first example illustrating the modification introduced by FLR effects on the fluid equations is given by considering the continuity (mass conservation) equation and the FLR-modified particle flux. The mass density is defined as

$$\rho \equiv 2\pi M \int w_\perp dw_\perp \left( f_{00} + \epsilon^2 f_{02} + \cdots \right),$$
where the gyrophase-dependent terms have disappeared. The application of the convective derivative to the mass density $\rho$ gives
\[
\frac{d\rho}{dt} = 2\pi M \epsilon^2 \int w_1 dw_1 \left( \frac{df_{02}}{dt} + O(\epsilon) \right)
= -2\pi M \epsilon^2 \int w_1^2 dw_1 (\mathbf{2} \cdot \nabla f_{13} + \mathbf{1} \cdot \nabla f_{-13}),
\] (B.12)
where we have used Eq. (B.11), with
\[
\int w_1 dw_1 \left( \frac{\partial g}{\partial w_1} + \frac{g}{w_1} \right) = 0 = \int w_1 dw_1 \left( w_1 \frac{\partial h}{\partial w_1} + 2h \right),
\]
which hold provided $(w_1 g)$ and $(w_1^2 h)$ vanish at infinity.

The manipulations involved in Eq. (B.12) are quite tedious, but straightforward, and we obtain
\[
\frac{d\rho}{dt} = \frac{\pi M}{\Omega^2} \int dw_1 \left[ w_1^2 \nabla \cdot \left( \frac{d}{dt} \nabla f_{00} - w_1 \nabla \cdot \left( \frac{\partial f_{00}}{\partial w_1} \frac{du}{dt} \times \Omega \right) \right) \right]
= \frac{1}{\Omega^2} \nabla \cdot \left( \frac{d}{dt} \nabla P_\perp \right) + \frac{1}{\Omega^2} \nabla \cdot \left( \rho \frac{du}{dt} \times \Omega \right),
\] (B.13)
where we have defined the lowest-order expression for the perpendicular pressure as
\[
P_\perp = 2\pi \int w_1 dw_1 \frac{Mw_1^2}{2} f_{00}.
\] (B.14)

Using the lowest-order drift velocity $u = (c/B)E \times \hat{z}$, Eq. (B.13) becomes
\[
\frac{d\rho}{dt} + \frac{1}{\Omega^2} \nabla \cdot \left( \frac{en}{dt} \frac{dE}{dt} - \frac{d\nabla P_\perp}{dt} \right) = 0.
\] (B.15)

Finally, using $dP_\perp/dt = 0$ [which follows from Eqs. (B.9) and (B.14)] and $\nabla \cdot u = 0$, we obtain
\[
\frac{d}{dt} \nabla P_\perp = -\frac{c^2}{B} \times \nabla P_\perp \cdot \nabla E,
\]
and the continuity equation, Eq. (B.15), takes the final form
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{V} \rho) = 0,
\] (B.16)
where the fluid velocity is given by the expression
\[
\mathbf{V} = u + \frac{c}{B\Omega} \left( \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \mathbf{E},
\] (B.17)
and
\[ U = u + \frac{\vec{z} \times \nabla P_\perp}{mn\Omega}. \] (B.18)

The last term on the right-hand side of Eq. (B.18) is the so-called diamagnetic drift velocity, and \((V - u)\) in Eq. (B.17) is the generalized polarization drift velocity (see Horton, Estes, and Biskamp [1980]). Hence, the consideration of FLR effects has allowed the generalized polarization drift to appear. The importance of the polarization drift to the nonlinear theories of drift-wave turbulence is well-known. We note that at the same order, FLR effects introduced the \(E \times B\) and diamagnetic velocities in the convective derivative of the polarization drift.

B.5. Collisionless Transport Coefficients

In order to evaluate the collisionless transport coefficients for the equations of motion, pressure tensor, and energy, we must consider the moments (in drift phase space)

\[ \int f \, d^3w = n, \]
\[ \int f \, w \, d^3w = 0, \]
\[ \int f \, mww \, d^3w = P, \]
\[ \int f \, \frac{m^2}{2} w \, d^3w = q, \]
\[ \int f \, mw^2 \, d^3w = S, \]

where \(n\) is the fluid density, \(P\) is the pressure tensor, \(q\) is the heat flux, and \(S\) can be given as \(S = 2qI/3 + \bar{S}\), where

\[ \bar{S} = \int f \, mw \left(ww - \frac{w^2}{3} I\right) \, d^3w. \]

Taking successively the \((mw)\), \((mww)\), and \((mw^2/2)\) moments of Eq. (B.1), one obtains the following fluid equations

\[ mn \frac{du}{dt} = mn \left( \frac{e}{m} E + u \times \Omega \right) - \nabla \cdot P, \] (B.19)
\[ \frac{dP}{dt} + \nabla \cdot S + (\Omega \times P - P \times \Omega) + \left[ (P \cdot \nabla u) + \text{(transpose)} \right] = 0, \] (B.20)
\[ \frac{3}{2} \frac{d}{dt} (nT) + \nabla \cdot q + P : \nabla u = 0. \] (B.21)
In the equation of motion, Eq. (B.19), it is customary (Braginskii [1965]) to take the form for the pressure tensor

\[ P = p\mathbf{I} + \Pi_g, \]

where \( p \) is the lowest-order scalar pressure and \( \Pi_g \) is known as the gyroviscous stress tensor, which introduces the FLR corrections to the scalar press. On the other hand, in the pressure equation, Eq. (B.20), the work by Chew, Goldberger, and Low [1956] has shown that in the collisionless regime the pressure tensor is given by the expression

\[ P = p_{\parallel}\mathbb{b}\cdot\mathbb{b} + p_{\perp}(I - \mathbb{b}\cdot\mathbb{b}) + \mathbf{\dot{P}} = P_{CGL} + \mathbf{\dot{P}}, \]

where \( P_{CGL} \) satisfies the lowest-order pressure equation

\[ \Omega \times P_{CGL} - P_{CGL} \times \Omega = 0. \]

We note that the two representations for the pressure tensor are equivalent if the lowest-order pressure tensor is isotropic. Finally, assuming pressure isotropy again, the lowest-order equation for the energy, Eq. (B.21), is given by the expression

\[ \frac{3n}{2} \frac{dT}{dt} + \nabla \cdot \mathbf{q} = 0. \]

It is our purpose to derive expressions for \( \Pi_g \) and \( \mathbf{q} \) and to show that in the collisionless regime these non-vanishing expressions are due to the inclusion of FLR effects.

We begin our analysis with the assumption that the parallel component of the drift velocity, \( u_{\parallel} = \mathbb{b} \cdot \mathbf{u} \), no longer vanishes (we still use \( \nabla \cdot \mathbf{u} = 0 \)), and that \( \mathbf{u} \) is given by the solution of the lowest-order equation of motion

\[ \frac{d\mathbf{u}}{dt} = \frac{e}{m} \mathbf{E} + \mathbf{u} \times \Omega - \frac{\nabla P}{mn}. \]

The drift Vlasov equation, Eq. (B.1), therefore becomes

\[ \frac{df}{dt} + \mathbf{w} \cdot \nabla f + \left( \mathbf{w} \times \Omega + \frac{\nabla P}{mn} - \mathbf{w} \cdot \nabla \mathbf{u} \right) \cdot \frac{\partial f}{\partial \mathbf{w}} = 0. \]

We continue the analysis with the following assumptions: (1) we let \( f = f_M(1 + \Phi) \), where \( f_M \) is the Maxwellian distribution function and \( \Phi \) is the correction term which must be solved for; (2) we neglect the term \( df/dt \); and (3) we neglect parallel-gradient terms. The linear equation for \( \Phi \) is, therefore, given by the expression

\[ -\Omega \frac{\partial \Phi}{\partial \zeta} = w_{\perp} \mathbb{\hat{1}} \cdot \nabla (\ln f_M - \ln P) + \frac{m}{T} \left[ w_{\parallel} (\mathbb{\hat{1}} \cdot \mathbf{b}) + \frac{w_{\perp}^2}{2} (\mathbb{\hat{1}} \cdot \mathbb{\hat{1}} - \zeta \zeta) \right] : \mathbf{\nabla u}. \quad (B.22) \]
Finally, using the well-known relation for Maxwellian distributions

\[ \nabla \ln f_M = \nabla \ln n + \nabla \ln T \left( \frac{mw^2}{2T} - \frac{3}{2} \right) = \nabla \ln P + \nabla \ln T \left( \frac{mw^2}{2T} - \frac{5}{2} \right), \]

Eq. (B.39) is easily solved, and \( \Phi \) is given by the expression

\[ \Phi = -\frac{w_1}{\Omega} \zeta \cdot \nabla \ln T \left( \frac{mw^2}{2T} - \frac{5}{2} \right) - \frac{mw_1}{\Omega} \zeta \cdot \nabla u_1 - \frac{mw^2}{4\Omega T} (\nabla \zeta + \zeta \cdot \nabla) : \nabla u. \quad \text{(B.23)} \]

One can easily verify that

\[ \int f_M \Phi \, d^3w = 0 = \int f_M \Phi \, d^3w. \]

Higher moments of \( f_M \Phi \) include the gyroviscous stress tensor

\[ \Pi_g = \int f_M \Phi \, m \left[ w w - (w^2/3) \mathbf{I} \right] \, d^3w = \int f_M \Phi \, mww \, d^3w, \]

and the collisionless heat flux

\[ q = \int f_M \Phi \frac{mw^2}{2} w \, d^3w. \]

Using the expression for \( \Phi \), given by Eq. (B.23), the gyroviscous stress tensor becomes

\[ \Pi_g = \frac{nT}{\Omega} \left[ \vec{b} \times \nabla u || + \text{(transpose)} \right] + \frac{nT}{4\Omega} \left[ (\vec{b} \times \nabla u \perp + \nabla \perp \vec{b} \times u \perp) + \text{(transpose)} \right], \quad \text{(B.24)} \]

and the heat flux is given by the expression

\[ q = \frac{5}{2} \frac{nT}{m\Omega} \vec{b} \times \nabla T. \quad \text{(B.25)} \]

This then completes our study of FLR effects on fluid equations. The next section will describe how these FLR-fluid equations have been used, by various authors, to derive nonlinear reduced fluid equations which include FLR effects. We note that Eqs. (B.24) and (B.25) represent the collisionless versions of the corresponding Braginskii transport coefficients (Braginskii [1965]).
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Love and knowledge, so far as they were possible, led upward toward the heavens. But always pity brought me back to earth. Echoes of cries of pain reverberate in my heart. Children in famine, victims tortured by oppressors, helpless old people a hated burden to their sons, and the whole world of loneliness, poverty, and pain make a mockery of what human life should be. I long to alleviate the evil, but I cannot, and I too suffer.

What I have lived for, by Bertrand Russell