Laplace’s Equation

A special case of Poisson’s equation $\nabla^2 \Phi(r) = -\epsilon_0^{-1} \rho(r)$ involves the case where the charge density $\rho(r)$ at the field point $r$ vanishes, which yields Laplace’s equation

$$\nabla^2 \Phi(r) = \nabla \cdot \nabla \Phi(r) = \sum_i \frac{1}{J} \frac{\partial}{\partial u^i} \left( J \nabla \Phi \cdot \nabla u^i \right) = \sum_{i,j} \frac{1}{J} \frac{\partial}{\partial u^i} \left( J g^{ij} \frac{\partial \Phi}{\partial u^j} \right),$$  

where the inverse metric tensor components are $g^{ij} = \nabla u^i \cdot \nabla u^j$ and $J$ is the Jacobian for the transformation $r \rightarrow u$ from Cartesian to curvilinear coordinates (e.g., cylindrical or spherical coordinates).

- Laplace’s equation in one dimension

The investigation of solutions of Laplace’s equation begins with the one-dimensional case: $\Phi''(x) = 0$, whose solution is

$$\Phi(x) = \Phi_0 - E x,$$

where $\Phi_0$ is the potential at $x = 0$ and $E$ is the magnitude of the constant electric field. Although this solution is trivial, it nonetheless exhibits two general properties that can be generalized to higher dimensions.

Property I $\Phi(x) = \frac{1}{2} [\Phi(x + a) + \Phi(x - a)]$, for any $a$.

Property II $\Phi(x)$ has no minimum or maximum in the range $(x - a, x + a)$; minima and maxima can occur only at the end points $x \pm a$.

- Laplace’s equation in two dimensions

We begin with Laplace’s equation (1) in Cartesian coordinates

$$\frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2} = 0,$$

whose general solution can be expressed as an expansion in powers of $x^n y^m$:

$$\Phi(x,y) = \Phi_0 - E_{x0} x - E_{y0} y + \alpha xy + \frac{\beta}{2} \left( x^2 - y^2 \right) + \cdots,$$  

(2)
where $\Phi_0 = \Phi(0,0)$ and $E_0 = (E_{x0}, E_{y0})$ are the electric potential and the electric field at the origin, respectively, and $(\alpha, \beta, \cdots)$ are constants.

In the Figure below, Eq. (2) is plotted for the case $(\Phi_0 = 0, E_{x0} = 1 = E_{y0}, \alpha = 1, \beta = 5)$, which clearly shows that the potential $\Phi(x,y)$ indeed satisfies Property II, i.e., Eq. (2) has no maximum or minimum in the square $(x,y) \in (-2, 2) \times (-2, 2)$, while minima and maxima occur only on the square’s perimeter. Note the existence of a saddle point$^1$ at the origin, which combines a local minimum in $x$ and a local maximum in $y$.

The potential (2) also satisfies Property I since, for example, the potential at the origin $(0, 0)$ can be found by integrating Eq. (2) on the perimeter of the square $(-a, a) \times (-a, a)$, for any $a$:

$$
\Phi(0, 0) = \frac{1}{8a} \left\{ \int_{-a}^{a} [\Phi(x,-a) + \Phi(x,a)] \, dx + \int_{-a}^{a} [\Phi(a,y) + \Phi(-a,y)] \, dx \right\} = \Phi_0.
$$

Before moving on to the three-dimensional case, we write Laplace’s equation (1) in polar coordinates ($x = r \cos \theta, \ y = r \sin \theta$):

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi(r, \theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi(r, \theta)}{\partial^2 \theta} = 0,
$$

$^1$A saddle point is an extremum which exhibits a minimum in one direction and a maximum in another.
and point out that the azimuthally symmetric solution $\Phi(r)$:

$$(r \Phi')' = 0 \rightarrow \Phi'(r) = -E_a \frac{a}{r} \rightarrow \Phi(r) = \Phi_a - a E_a \ln(r/a),$$

where $\Phi_a = \Phi(a)$ and $E_a$ are constants.

Moreover, according to Property I, the potential $\Phi(r)$ at the field point $r$ can be evaluated in terms of the integral

$$\Phi(r) = \frac{1}{2\pi} \oint \Phi(r + a \hat{r}') d\theta'$$

along the boundary of a disk of arbitrary radius $a$ centered at the field point $r$, where $\hat{r}' = \cos \theta' \hat{x} + \sin \theta' \hat{y}$.

○ Laplace’s equation in three dimensions

Laplace’s equation (1) in three dimensions can be written in Cartesian, cylindrical, and spherical coordinates as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0,$$

respectively. Although we shall not discuss general solutions of Laplace’s equation (1) in three dimensions, we note that special symmetric solutions $\Phi(r)$ in cylindrical and spherical geometries are

$$\Phi(r) = \begin{cases} 
\Phi_a - a E_a \ln(r/a) & \text{(cylindrical geometry)} \\
\Phi_a + E_a a^2/r & \text{(spherical geometry)}
\end{cases}$$

Moreover, according to Property I, the $\Phi(r)$ at the field point $r$ can be evaluated in terms of the integral

$$\Phi(r) = \frac{1}{4\pi} \int_0^\pi \sin \theta' \int_0^{2\pi} d\varphi' \Phi(r + a \hat{r}')$$

along the boundary of a sphere of arbitrary radius $a$ centered at the field point $r$, where $\hat{r}' = \sin \theta' [\cos \varphi' \hat{x} + \sin \varphi' \hat{y}] + \cos \theta' \hat{z}$.

As an example, consider the case of the potential $\Phi(r')$ associated with a charge $q$ located at $r = z \hat{z}$:

$$\Phi(r') = \frac{q}{4\pi \varepsilon_0} (r'^2 + z^2 - 2 r' z \cos \theta')^{-\frac{1}{2}}.$$
The electric potential $\Phi_0$ at the origin can be calculated as the integral over the boundary of a sphere of radius $a < z$ centered at the origin:

$$\Phi_0 = \frac{1}{4\pi} \int_0^{\pi} \sin \theta' \, d\theta' \int_0^{2\pi} \, d\varphi' \frac{q}{4\pi\epsilon_0} (a^2 + z^2 - 2az \cos \theta')^{-\frac{1}{2}} = \frac{q}{4\pi\epsilon_0 z}.$$ 

Lastly, we prove Earnshaw’s Theorem: A charged particle cannot be held in stable equilibrium by electrostatic forces alone. The proof begins by considering a charge $q$ placed inside a volume $V$, in which the electric potential $\Phi(r)$ satisfies Laplace’s equation (1). A stable equilibrium is associated with a minimum value $W(r) = \frac{1}{2} q \Phi(r)$ for the energy of charge $q$ placed at point $r$ inside $V$. On the one hand, an equilibrium point $r_0$ inside volume $V$ is associated with $\nabla W(r_0) = 0$, which implies that the electric field at point $r_0$ must vanish, i.e., $\nabla \Phi(r_0) = 0$. On the other hand, the equilibrium point $r_0$ is stable if $\nabla^2 W(r_0) > 0$. However, since $\nabla^2 \Phi(r_0) = 0$ for any point inside volume $V$, we find that a stable equilibrium point cannot be found in any region $V$ where $\Phi(r)$ satisfies Laplace’s equation $\nabla^2 \Phi(r) = 0$.

- **Boundary Conditions and Uniqueness Theorems**

  Solutions $\Phi(r)$ of Laplace’s equation $\nabla^2 \Phi(r) = 0$ inside a region $V$ are uniquely defined only if (a) the potential $\Phi_S$ is specified on the boundary $\partial V$ or (b) the electric field normal ($-\partial \Phi_S/\partial n$) to the boundary $\partial V$ is specified. In the first case, called the Dirichlet problem, the boundary $\partial V$ may be divided into domains $\partial V = \bigcup_i \partial V_i$, where each domain $\partial V_i$ is assigned the potential $\Phi_i$ (see Figure below). In the second case, called the Neumann problem, the boundary $\partial V$ is again divided into domains $\partial V = \bigcup_i \partial V_i$, where each domain $\partial V_i$ is now assigned the surface charge density $\sigma_i$ (see Figure below).
We now show that a solution $\Phi(r)$ of Laplace’s equation $\nabla^2 \Phi(r) = 0$ inside region $V$ subject to the Dirichlet or Neumann boundary conditions on $\partial V$ is unique. It is worth pointing out that the same uniqueness theorem can be applied to solutions of Poisson’s equation $\rho(r) = -\varepsilon_0 \nabla^2 \Phi(r)$ inside region $V$ subject to Dirichlet or Neumann boundary conditions on $\partial V$.

The proof of the Uniqueness Theorem for solutions of Poisson’s equation (and Laplace’s equation as a special case) proceeds by assuming that, instead, two solutions $\Phi_a(r)$ and $\Phi_b(r)$ exist and both satisfy the same boundary conditions. Next, we construct the difference potential $\Psi(r) = \Phi_a(r) - \Phi_b(r)$ and easily verify that $\Psi(r)$ is a solution of Laplace’s equation

$$\nabla^2 \Psi(r) = \nabla^2 \Phi_a(r) - \nabla^2 \Phi_b(r) = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0,$$

but with boundary conditions $\Psi_S = 0$ (Dirichlet) or $\partial \Psi_S / \partial n = 0$ (Neumann) on the boundary $\partial V$. Lastly, we use the identity

$$\int_V \left( \Psi \nabla^2 \Psi + |\nabla \Psi|^2 \right) d\tau = \oint_{\partial V} \Psi_S \frac{\partial \Psi_S}{\partial n} da = 0,$$

where the surface integral vanishes because either $\Psi_S = 0$ or $\partial \Psi_S / \partial n = 0$ on the boundary. Since $\nabla^2 \Psi = 0$ inside region $V$, our identity becomes

$$\int_V |\nabla \Psi|^2 d\tau = 0,$$

which implies that $\nabla \Psi = 0$ everywhere inside $V$.

For the Dirichlet problem (First Uniqueness Theorem), since $\Psi(r)$ is a constant inside $V$ and vanishes on $\partial V$, we find that $\Psi(r) = 0$ everywhere inside $V$ (since $\Psi$ cannot have a minimum or maximum inside $V$). Hence, the two solutions $\Phi_a(r)$ and $\Phi_b(r)$ are, in fact, identical $\Phi_a(r) = \Phi_b(r)$ everywhere inside $V$. A unique solution of Poisson’s equation (and Laplace’s equation) inside a region $V$ was, therefore, determined solely by specifying the potential on the boundary $\partial V$ (Dirichlet problem).

A similar argument can be used for the Uniqueness Theorem applied to the Neumann problem (Second Uniqueness Theorem). The condition $\nabla \Psi = 0$ inside $V$ implies that the electric fields $E_a$ and $E_b$ are identical inside $V$ as well as on the boundary $\partial V$ since $\partial \Psi_S / \partial n = 0$.

**Method of Images**

Applications of the Uniqueness Theorems lead us to an important method used in solving Laplace’s equation subject to Dirichlet or Neumann boundary conditions.

For example, let us consider the physical problem where a charge $q$ is placed at a distance $h$ above an infinite conductor (at $z = 0$) held at potential $\Phi_S(x, y) = 0$. This is a Dirichlet problem and the First Uniqueness Theorem tells us that the solution $\Phi(x, y, z)$ in the region...
$z > 0$ is unique subject to the boundary condition $\Phi_S(x, y) = 0$. It is clear that induced charges must be created on the infinite conductor at $z = 0$; for instance, without induced charges, the potential due to $q$ at the origin is $\Phi_q(0, 0, 0) = q/(4\pi \epsilon h) \neq 0$, which does not satisfy the boundary condition. The physical problem is difficult to solve because the induced surface charge density $\sigma_S(x, y)$ is unknown.

We note that the boundary condition $\Phi_S(x, y) = 0$ would be easily satisfied if we placed a second charge $-q$ of equal magnitude but opposite sign below the infinite plane at a distance $h$ (see Figure below).

![Physical Problem vs Image Problem](image)

The solution for the electric potential for this image problem is

$$
\Phi(x, y, z) = \frac{q}{4\pi \epsilon_0} \left( \frac{1}{\sqrt{x^2 + y^2 + (z-h)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+h)^2}} \right),
$$

which satisfies Laplace’s equation everywhere above the plane except at $(x, y, z) = (0, 0, h)$ and satisfies the boundary condition $\Phi_S(x, y) = \Phi(x, y, z = 0) = 0$ everywhere on the infinite plane. By the First Uniqueness Theorem, we have, therefore, found the unique solution $\Phi(x, y, z)$ to the physical problem.

Note that the induced surface charge density $\sigma_S(x, y)$ can also be calculated from

$$
\sigma_S(x, y) = -\epsilon_0 \frac{\partial \Phi_S}{\partial n} = -\epsilon_0 \frac{\partial \Phi}{\partial z} \bigg|_{z=0} = -\frac{q h}{2\pi (x^2 + y^2 + h^2)^{3/2}}.
$$

As expected that the sign of the induced charge is opposite to $q$ and the total induced charge $q_{\text{ind}}$ is calculated to be

$$
q_{\text{ind}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \sigma_S(x, y) = \int_{0}^{\infty} dr \int_{0}^{2\pi} d\theta \left( \frac{-q h}{2\pi (r^2 + h^2)^{3/2}} \right).
$$
\[ = -\frac{qh}{2} \int_0^{\infty} \frac{ds}{(s + h^2)^2} = -q. \]

Moreover, since the induced surface charge density is opposite to \( q \), the charge \( q \) will be attracted to the infinite conductor with a force
\[
F = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2h)^2} \hat{z},
\]
calculated as if the conductor was absent and only the two charges \( q \) and \( -q \) (separated by a distance \( 2h \)) exist.