• Electric Potential

It was pointed out in an earlier lecture that

\[ \frac{r - r'}{|r - r'|^3} = -\nabla\left(\frac{1}{|r - r'|}\right), \]

and that the electric field due to a continuous charge distribution can be expressed as a gradient vector field

\[ \mathbf{E}(r) = \int_V \frac{\rho(r')}{4\pi\epsilon_0} \left(\frac{r - r'}{|r - r'|^3}\right) d\tau' = -\nabla\left(\int_V \frac{\rho(r')}{4\pi\epsilon_0} \frac{d\tau'}{|r - r'|}\right). \]

This last expression is consistent with the fact that the electric field due to a static charge distribution is curl-free (\(\nabla \times \mathbf{E} = 0\)) and that the line integral

\[ \int_a^b \mathbf{E} \cdot d\mathbf{l} \]

is independent of the path joining the initial point \(a\) and the final point \(b\).

By introducing the electric potential \(\Phi(r)\), defined as

\[ \Phi(r) = \int_V \frac{\rho(r')}{4\pi\epsilon_0} \frac{d\tau'}{|r - r'|}, \]

the electric field due to a static charge distribution becomes

\[ \mathbf{E}(r) = -\nabla\Phi(r), \]

and the line integral (1) becomes

\[ \int_a^b \mathbf{E} \cdot d\mathbf{l} = -\Phi(b) + \Phi(a). \]

In particular, we may use this line-integral expression to construct the electric potential \(\Phi(r)\) at a field point \(r\) if we know the potential \(\Phi(a)\) at some reference point \(a\) and the line integral of the electric field along an arbitrary path that joins the reference point to the field point

\[ \Phi(r) = \Phi(a) - \int_a^r \mathbf{E} \cdot d\mathbf{l}. \]
Additional expressions for the electric potential due to a line charge distribution and surface charge distribution, respectively, are

\[
\Phi(\mathbf{r}) = \int_{L} \frac{\lambda(\alpha) \mathcal{J}_L \, d\alpha}{|\mathbf{r} - \mathbf{r}_L(\alpha)|} \quad \text{and} \quad \Phi(\mathbf{r}) = \int_{S} \frac{\sigma(\alpha, \beta) \mathcal{J}_S \, d\alpha d\beta}{|\mathbf{r} - \mathbf{r}_S(\alpha, \beta)|},
\]  

(4)

**Example**

As an example of the second expression in Eq. (4), we consider the electric potential inside and outside a charged spherical shell of radius \( R \) with uniform surface density \( \sigma \).

Making use of the spherical symmetry inherent to the charge distribution, we choose the field point to be on the \( z \)-axis: \( \mathbf{r} = z \mathbf{\hat{z}} \), while the surface parametrization of the spherical shell is expressed as

\[
\mathbf{r}_S(\theta', \varphi') = R \left[ \sin \theta' \left( \cos \varphi' \mathbf{\hat{x}} + \sin \varphi' \mathbf{\hat{y}} \right) + \cos \theta' \mathbf{\hat{z}} \right],
\]

so that

\[
|\mathbf{r} - \mathbf{r}_S(\theta', \varphi')| = \sqrt{z^2 + R^2 - 2zR \cos \theta'},
\]

and \( \mathcal{J}_S \, d\alpha d\beta = R^2 \sin \theta' \, d\theta' d\varphi' \). The electric potential is now calculated as

\[
\Phi(z) = \frac{\sigma R^2}{4\pi \epsilon_0} \int_{0}^{\pi} d\theta' \int_{0}^{2\pi} d\varphi' \frac{\sin \theta'}{\sqrt{z^2 + R^2 - 2zR \cos \theta'}}
\]

\[
= \frac{\sigma R^2}{2\epsilon_0} \int_{0}^{\pi} \frac{\sin \theta' \, d\theta'}{\sqrt{z^2 + R^2 - 2zR \cos \theta'}}
\]

\[
= \frac{\sigma R}{2z \epsilon_0} \left( \sqrt{z^2 + R^2 + 2zR} - \sqrt{z^2 + R^2 - 2zR} \right).
\]
At this point, we need to distinguish between the case where the field point is inside the spherical shell (i.e., \( z < R \)) and the case where the field point is outside the spherical shell (i.e., \( z > R \)). For the inside case, we find \( \sqrt{z^2 + R^2} \pm 2zR = (R \pm z) > 0 \), and thus the inside potential is

\[
\Phi_{\text{in}}(z) = \frac{\sigma R}{2z \epsilon_0} \left( (R + z) - (R - z) \right) = \frac{\sigma}{\epsilon_0} R.
\]

For the outside case, however, we find \( \sqrt{z^2 + R^2} \pm 2zR = (z \pm R) > 0 \), and thus the outside potential is

\[
\Phi_{\text{out}}(z) = \frac{\sigma R}{2z \epsilon_0} \left( (z + R) - (z - R) \right) = \frac{\sigma R^2}{\epsilon_0 z}.
\]

By restoring the spherical symmetry of the charge distribution (\( z \to r \)), we write the electric potential as

\[
\Phi(r) = \frac{\sigma}{\epsilon_0} \begin{cases} 
R & \text{if } r < R \\
R^2/r & \text{if } r > R
\end{cases}
\]

From this expression and the definition (3) of the electric field in terms of the electric potential, we find

\[
E(r) = -\nabla \Phi(r) = \frac{q \hat{r}}{4\pi \epsilon_0} \begin{cases} 
0 & \text{if } r < R \\
0 & \text{if } r > R
\end{cases}
\]

where we have used the definition \( \sigma = q/(4\pi R^2) \) for the uniform surface charge density in terms of the total charge \( q \) carried by the spherical shell. Hence, the electric field outside the charged spherical shell is identical to the field obtained by concentrating the charge \( q \)
at the center of the sphere. In fact, we could easily have recovered this result by making use of the integral form of Gauss’s Law.

- Poisson’s Equation

By substituting the definition (3) into the differential form of Gauss’s Law \( \nabla \cdot \mathbf{E} = \rho/\epsilon_0 \) we obtain Poisson’s Equation:

\[
\nabla^2 \Phi(r) = -\frac{\rho(r)}{\epsilon_0},
\]

which expresses the Laplacian of the electric potential in terms of the charge density \( \rho(r) \).

We can use the definition of the delta function

\[
\delta^3(r - r') = -\nabla^2 \left( \frac{1}{4\pi |r - r'|} \right)
\]

to verify that Eq. (2) is a solution of Poisson’s Equation.

For the special case where \( \rho(r) = 0 \) at the field point \( r \), Poisson’s Equation (5) becomes

\[
\nabla^2 \Phi(r) = 0,
\]

which is known as Laplace’s Equation.