Dirac Delta Function

• Paradox

The Divergence Theorem of Vector Calculus

\[ \int_V d\tau \nabla \cdot \mathbf{A} = \oint_{\partial V} d\mathbf{a} \cdot \mathbf{A} \]  

(1)

presents us with an interesting paradox when we consider the vector field

\[ \mathbf{A} = \frac{\mathbf{r}}{r^3} \]  

(2)

On the one hand, using identities presented in the September 2 lecture notes, we readily find that

\[ \nabla \cdot \mathbf{A} = r^{-3} \nabla \cdot \mathbf{r} + \mathbf{r} \cdot \nabla r^{-3} = \frac{3}{r^3} - 3 \frac{\mathbf{r} \cdot \mathbf{r}}{r^5} = 0, \]  

(3)

and, hence,

\[ \int_V d\tau \nabla \cdot \mathbf{A} = 0. \]

On the other hand, choosing \( V \) to be a sphere of radius \( r \) and denoting its surface as \( \partial S \), we also have

\[ \oint_{\partial V} d\mathbf{a} \cdot \mathbf{A} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \ r^2 \sin \theta \ \hat{\mathbf{r}} \cdot \frac{\mathbf{r}}{r^3} = 2\pi \int_0^\pi d\theta \ \sin \theta = 4\pi, \]

where we used the following expression for the surface element \( d\mathbf{a} \) for a sphere

\[ d\mathbf{a} = \left( \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right) d\theta \wedge d\varphi = r^2 \sin \theta \ d\theta \ d\varphi \ \hat{\mathbf{r}}. \]

We immediately notice the paradox that, according to the Divergence Theorem, we find the nonsensical result \( 0 = 4\pi \).

The paradox is resolved by noting that \( \nabla \cdot (r^{-3} \mathbf{r}) = 0 \) is valid only when \( \mathbf{r} \neq 0 \). To reconcile the two sides of the Divergence Theorem (1), we, therefore, introduce a singular function known as the delta function \( \delta^3(\mathbf{r}) \), defined by the identity\(^1\)

\[ \delta^3(\mathbf{r}) = \nabla \cdot \left( \frac{\mathbf{r}}{4\pi r^3} \right), \]  

(4)

\(^1\)The delta function was first introduced in Physics by P.A.M. Dirac and was, at first, vehemently rejected by mathematicians; delta functions are now part of a branch of Mathematics known as distribution theory.
with the property that \( \delta^3(\mathbf{r}) \) is zero when \( \mathbf{r} \neq 0 \) and is \textit{infinite} when \( \mathbf{r} = 0 \). Additional properties of the delta function include

\[
\int_V d\tau \, f(\mathbf{r}) \, \delta^3(\mathbf{r} - \mathbf{q}) = \begin{cases} 
  f(\mathbf{q}) & \text{if } \mathbf{q} \text{ is located inside } V \\
  0 & \text{if } \mathbf{q} \text{ is located outside } V
\end{cases}
\]

Using the delta function \( \delta^3(\mathbf{r}) \), we now write Eq. (3) as

\[
\nabla \cdot \mathbf{A} = 4\pi \delta^3(\mathbf{r}),
\]
and, thus,

\[
\int_V d\tau \, \nabla \cdot \mathbf{A} = 4\pi \int_V d\tau \, \delta^3(\mathbf{r}) = 4\pi,
\]
which now satisfies the Divergence Theorem (1).

Note that the vector field (2) can also be written in terms of the gradient operator as \( \mathbf{A} = -\nabla r^{-1} \) so that Eq. (5) becomes

\[
\nabla^2 r^{-1} = -4\pi \delta^3(\mathbf{r}).
\]

• Properties of the Delta Function

○ Properties in One Dimension: Let \( f(x) \) be an arbitrary function of \( x \)

\[
\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}
\]

\[
\int_{-\infty}^{\infty} dx \, f(x) \, \delta'(x - a) = -\int_{-\infty}^{\infty} dx \, f'(x) \, \delta(x - a) = -f'(a)
\]

\[
\delta(x - a) = \frac{d}{dx} \Theta(x - a), \quad \text{where } \Theta(x - a) = \begin{cases} 
  1 & (x > a) \\
  0 & (x < a)
\end{cases}
\]

○ Properties in Three Dimensions

\[
\delta^3(\mathbf{r} - \mathbf{q}) = \delta(x - q_x) \, \delta(y - q_y) \, \delta(z - q_z)
\]

\[
\delta^3(\mathbf{r}) = \mathcal{J}^{-1} \delta(u^1) \, \delta(u^2) \, \delta(u^3)
\]

\[
\delta^3(\mathbf{r} - \mathbf{q}) = \frac{1}{r^2} \delta(r - q) \, \delta(\cos \theta - \cos \theta_q) \, \delta(\varphi - \varphi_q)
\]
• Examples
  - Charge distribution on the $z = 0$ plane: $\rho(r) = \sigma(x, y) \delta(z)$.
  - Charge distribution on the surface of a sphere of radius $a$: $\rho(r) = \sigma(\theta, \varphi) \delta(r - a)$.
  - Charge distribution on a circle of radius $a$ on the $z = 0$ plane
    
    $\rho(r) = \begin{cases} 
    \lambda(\theta) \delta(z) \delta(r - a) & \text{in cylindrical geometry}(r, \theta, z) \\
    a^{-1} \lambda(\varphi) \delta(\theta - \frac{\pi}{2}) \delta(r - a) & \text{in spherical geometry}(r, \theta, \varphi)
    \end{cases}$