ELECTROMAGNETIC THEORY
(PY 302)

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1 Electrostatics I

1.1 Electric Field due to a Discrete Distribution of Charges

The electric field \( \mathbf{E}(\mathbf{r}) \) at the field point \( \mathbf{r} \) due to a discrete distribution of charged particles \( \{(q_i, \mathbf{r}_i); \ i = 1, \ldots, N\} \) is expressed as

\[
\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \mathbf{E}_i(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3},
\]

where \( \mathbf{r} - \mathbf{r}_i \) denotes the displacement vector from the source point \( \mathbf{r}_i \) to the field point \( \mathbf{r} \); here, \( \epsilon_0 = 8.854187 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2} \) denotes the permittivity of free space (now taken to be a finite constant related to the speed of light in vacuum). This expression exhibits the characteristic features of Coulomb’s Law: The magnitude of the electric field \( |\mathbf{E}_i| \) produced by a single charge \( q_i \) is inversely proportional to the square of the distance \( |\mathbf{r} - \mathbf{r}_i| \) between the source point \( \mathbf{r}_i \) and the field point \( \mathbf{r} \).

1.2 Electric Field due to a Continuous Distribution of Charges

To treat the more general case of a continuous charge distribution, we first consider the infinitesimal electric field \( d\mathbf{E}(\mathbf{r}; \mathbf{r}') \) evaluated at the field point \( \mathbf{r} \) and produced by an infinitesimal charge element \( dq(\mathbf{r}') \) located at the source point \( \mathbf{r}' \). Using Coulomb’s Law, we
postulate

\[ d\mathbf{E}(\mathbf{r}; \mathbf{r}') = \frac{1}{4\pi \epsilon_0} \frac{dq(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \]

Next, using the Superposition Principle, the electric field \( \mathbf{E}(\mathbf{r}) \) at point \( \mathbf{r} \) due to a continuous distribution of charged particles inside a volume \( V \) is, thus, expressed as

\[
\mathbf{E}(\mathbf{r}) = \int_V d\mathbf{E}(\mathbf{r}; \mathbf{r}') = \frac{1}{4\pi \epsilon_0} \int_V \frac{dq(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{4\pi \epsilon_0} \int_V \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\tau',
\]

(2)

where \( dq(\mathbf{r}') = \rho(\mathbf{r}') \, d\tau' \) is the infinitesimal charge element expressed in terms of the volume charge density \( \rho(\mathbf{r}') \) at the source point \( \mathbf{r}' \).

Note that by substituting the discrete charge distribution

\[ \rho(\mathbf{r}') = \sum_i q_i \delta^3(\mathbf{r}' - \mathbf{r}_i), \]

into Eq. (2) we recover the discrete expression (1) when we use the properties of the three-dimensional delta function \( \delta^3(\mathbf{r}' - \mathbf{r}_i) \) (see Appendix B):

\[
\int_V \mathbf{F}(\mathbf{r}') \left[ \sum_i q_i \delta^3(\mathbf{r}' - \mathbf{r}_i) \right] \, d\tau' = \sum_i q_i \mathbf{F}(\mathbf{r}_i),
\]

where \( \mathbf{F}(\mathbf{r}') \) is an arbitrary vector-valued function of \( \mathbf{r}' \).

Furthermore, when charge is continuously distributed on a surface \( \mathbf{r}' = \mathbf{r}_S(\alpha, \beta) \), where a point on the surface \( S \) is labeled by two parameters \( \alpha \) and \( \beta \), or on a line \( \mathbf{r}' = \mathbf{r}_L(\alpha) \),
where a point on the line $L$ is labeled by the single parameter $\alpha$, the electric field at field point $\mathbf{r}$ is expressed as

$$
E(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int_S \sigma(\alpha, \beta) \left( \frac{\mathbf{r} - \mathbf{r}_S(\alpha, \beta)}{|\mathbf{r} - \mathbf{r}_S(\alpha, \beta)|^3} \right) J_S \, d\alpha \, d\beta, \quad (3)
$$

$$
E(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int_L \lambda(\alpha) \left( \frac{\mathbf{r} - \mathbf{r}_L(\alpha)}{|\mathbf{r} - \mathbf{r}_L(\alpha)|^3} \right) J_L \, d\alpha, \quad (4)
$$

respectively, where $\sigma(\alpha, \beta)$ and $\lambda(\alpha)$ are the surface and line charge densities while $J_S$ and $J_L$ represent appropriate Jacobian terms.

As an example, we consider a uniformly charged loop of radius $a$ and charge $q$ and, hence, with line charge density given as $\lambda = q/(2\pi a)$.

The parametrization of the charge distribution can be expressed as

$$
r' = a \left( \cos \theta' \hat{x} + \sin \theta' \hat{y} \right) = \mathbf{r}_L(\theta'),
$$

where we have assumed that the loop is lying on the $(x, y)$-plane with its center placed at the origin and, thus, $J_L \, d\alpha = a \, d\theta'$. Next, cylindrical coordinates $(r, \theta, z)$ are ideally suited for this problem and, therefore, we find

$$
\mathbf{r} - \mathbf{r}_L(\theta') = \left( r \cos \theta - a \cos \theta' \right) \hat{x} + \left( r \sin \theta - a \sin \theta' \right) \hat{y} + z \hat{z},
$$

$$
|\mathbf{r} - \mathbf{r}_L(\theta')| = \sqrt{r^2 + a^2 + z^2 - 2ar \cos(\theta - \theta')}.
$$
so that the electric field at field point \( \mathbf{r} \) is expressed as

\[
\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{d\theta'}{2\pi} \left\{ \frac{(r \cos \theta - a \cos \theta') \hat{x} + (r \sin \theta - a \sin \theta') \hat{y} + z \hat{z}}{[r^2 + a^2 + z^2 - 2ar \cos(\theta - \theta')]^{3/2}} \right\}.
\]

This integral can be readily calculated for the special case where the field point is on the \( z \)-axis (where \( \mathbf{r} = z\hat{z} \) and \( r = 0 \)) so that

\[
\mathbf{E}(z) = \frac{q}{4\pi\varepsilon_0} \frac{z\hat{z}}{(a^2 + z^2)^{3/2}}.
\]

### 1.3 Gauss’s Law

The integral form of Gauss’s Law was studied previously in your introductory physics course. It states that the electric flux through a closed surface \( \partial V \) is proportional to the electric charge \( q_V \) enclosed by the volume \( V \). More specifically, the integral form of Gauss’s Law is expressed as

\[
\oint_{\partial V} \mathbf{E} \cdot d\mathbf{a} = \frac{q_V}{\varepsilon_0}.
\]

To show that Coulomb’s Law is consistent with Gauss’s Law, we consider the electric field due to a single charge \( q \) located at the origin:

\[
\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2},
\]

and take \( V \) to be a sphere of radius \( r \) centered at the origin, so that \( q_V = q \) and \( d\mathbf{a} = r^2 \sin \theta \, d\theta \, d\phi \, \hat{r} \), and, thus, Gauss’s Law yields

\[
\oint_{\partial V} \mathbf{E} \cdot d\mathbf{a} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \frac{q}{4\pi\varepsilon_0} \frac{\hat{r} \cdot r^2 \sin \theta \, \hat{r}}{r^2} = \frac{q}{4\pi\varepsilon_0} \left( \int_0^\pi \sin \theta \, d\theta \right) \cdot \left( \int_0^{2\pi} d\phi \right) = \frac{q}{\varepsilon_0}.
\]

We now derive Gauss’s Law in differential form. From the definition of \( q_V \) in terms of the volume charge density \( \rho(\mathbf{r}) \):

\[
q_V = \int_V \rho(\mathbf{r}) \, d\tau,
\]

and using the Divergence Theorem of Vector Calculus

\[
\oint_{\partial V} \mathbf{E} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} \, d\tau,
\]

we easily find the differential form of Gauss’s Law as

\[
\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}).
\]
Another way to derive the differential form of Gauss’s Law is to take the divergence of Eq. (2)
\[ \nabla \cdot \mathbf{E} = \int_V \frac{\rho(r')}{4\pi \epsilon_0} \nabla \cdot \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\tau', \]
and use the definition of the delta function \( \delta^3(r - r') \):
\[ \delta^3(r - r') = \frac{1}{4\pi} \nabla \cdot \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right), \]
to obtain
\[ \nabla \cdot \mathbf{E} = \int_V \frac{\rho(r')}{4\pi \epsilon_0} \left( 4\pi \delta^3(r - r') \right) d\tau' = \frac{1}{\epsilon_0} \rho(r). \]

Note that since
\[ \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \left| \mathbf{r} - \mathbf{r}' \right|^{-1}, \]
the curl of the electric field associated with static charge distributions is
\[ \nabla \times \mathbf{E} = \int_V \frac{\rho(r')}{4\pi \epsilon_0} \nabla \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\tau' = 0, \]
which follows from the vector identity \( \nabla \times \nabla f = 0 \). Hence, the line integral of the electric field
\[ \int_{\gamma} \mathbf{E} \cdot d\mathbf{l} \]
is independent of the path \( \gamma \) taken and depends only on the initial and final points. Moreover, from Stokes’s Theorem, we find
\[ \oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{a} = 0. \]

## 1.4 Electric Potential

It was pointed out in an earlier lecture that
\[ \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right), \]
and that the electric field due to a continuous charge distribution can be expressed as a gradient vector field
\[ \mathbf{E}(\mathbf{r}) = \int_V \frac{\rho(r')}{4\pi \epsilon_0} \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\tau' = -\nabla \left( \int_V \frac{\rho(r')}{4\pi \epsilon_0 |\mathbf{r} - \mathbf{r}'|} d\tau' \right). \]
This last expression is consistent with the fact that the electric field due to a static charge distribution is curl-free ($\nabla \times \mathbf{E} = 0$) and that the line integral
\[ \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} \] is independent of the path joining the initial point $a$ and the final point $b$.

By introducing the electric potential $\Phi(\mathbf{r})$, defined as
\[ \Phi(\mathbf{r}) = \int_{V} \frac{\rho(\mathbf{r'}) \, d\tau'}{4\pi\varepsilon_{0} |\mathbf{r} - \mathbf{r'}|}, \] the electric field due to a static charge distribution becomes
\[ \mathbf{E}(\mathbf{r}) = -\nabla \Phi(\mathbf{r}), \] and the line integral (5) becomes
\[ \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} = -\Phi(\mathbf{b}) + \Phi(\mathbf{a}). \]
In particular, we may use this line-integral expression to construct the electric potential $\Phi(\mathbf{r})$ at a field point $\mathbf{r}$ if we know the potential $\Phi(\mathbf{a})$ at some reference point $\mathbf{a}$ and the line integral of the electric field along an arbitrary path that joins the reference point to the field point
\[ \Phi(\mathbf{r}) = \Phi(\mathbf{a}) - \int_{a}^{r} \mathbf{E} \cdot d\mathbf{l}. \]

Additional expressions for the electric potential due to a line charge distribution and surface charge distribution, respectively, are
\[ \Phi(\mathbf{r}) = \int_{L} \frac{\lambda(\alpha) J_{L} d\alpha}{|\mathbf{r} - \mathbf{r}_{L}(\alpha)|} \quad \text{and} \quad \Phi(\mathbf{r}) = \int_{S} \frac{\sigma(\alpha, \beta) J_{S} d\alpha d\beta}{|\mathbf{r} - \mathbf{r}_{S}(\alpha, \beta)|}. \]  

### 1.5 Example

As an example of the second expression in Eq. (8), we consider the electric potential inside and outside a charged spherical shell of radius $R$ with uniform surface density $\sigma$.

Making use of the spherical symmetry inherent to the charge distribution, we choose the field point to be on the $z$-axis: $\mathbf{r} = z\hat{z}$, while the surface parametrization of the spherical shell is expressed as
\[ \mathbf{r}_{S}(\theta', \varphi') = R \left[ \sin \theta' (\cos \varphi' \hat{x} + \sin \varphi' \hat{y}) + \cos \theta' \hat{z} \right], \] so that
\[ |\mathbf{r} - \mathbf{r}_{S}(\theta', \varphi')| = \sqrt{z^2 + R^2 - 2zR \cos \theta'}. \]
and $\mathcal{J}_S \, d\alpha \, d\beta = R^2 \sin \theta' \, d\theta' \, d\varphi'$. The electric potential is now calculated as

$$\Phi(z) = \frac{\sigma R^2}{4\pi \epsilon_0} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \frac{\sin \theta'}{\sqrt{z^2 + R^2 - 2zR \cos \theta'}}$$

$$= \frac{\sigma R^2}{2\epsilon_0} \int_0^{\pi} \sin \theta' \, d\theta' \frac{\sin \theta'}{\sqrt{z^2 + R^2 - 2zR \cos \theta'}}$$

$$= \frac{\sigma R}{2z \epsilon_0} \left( \sqrt{z^2 + R^2 + 2zR} - \sqrt{z^2 + R^2 - 2zR} \right).$$

At this point, we need to distinguish between the case where the field point is inside the spherical shell (i.e., $z < R$) and the case where the field point is outside the spherical shell (i.e., $z > R$). For the inside case, we find $\sqrt{z^2 + R^2 \pm 2zR} = |R \pm z|$, and thus the inside potential is

$$\Phi_{in}(z) = \frac{\sigma R}{2z \epsilon_0} \left( (R + z) - (R - z) \right) = \frac{\sigma R}{\epsilon_0}.$$

For the outside case, however, we find $\sqrt{z^2 + R^2 \pm 2zR} = (z \pm R) > 0$, and thus the outside potential is

$$\Phi_{out}(z) = \frac{\sigma R}{2z \epsilon_0} \left( (z + R) - (z - R) \right) = \frac{\sigma R^2}{\epsilon_0 z}.$$

By restoring the spherical symmetry of the charge distribution ($z \to r$), we write the electric potential as

$$\Phi(r) = \frac{\sigma}{\epsilon_0} \begin{cases} 
R & \text{if } r < R \\
R^2 / r & \text{if } r > R 
\end{cases}.$$
From this expression and the definition (7) of the electric field in terms of the electric potential, we find

\[ E(r) = -\nabla \Phi(r) = \frac{q\hat{r}}{4\pi\epsilon_0} \begin{cases} 0 & \text{if } r < R \\ r^{-2} & \text{if } r > R \end{cases} \]

where we have used the definition \( \sigma = q/(4\pi R^2) \) for the uniform surface charge density in terms of the total charge \( q \) carried by the spherical shell. Hence, the electric field outside the charged spherical shell is identical to the field obtained by concentrating the charge \( q \) at the center of the sphere. In fact, we could easily have recovered this result by making use of the integral form of Gauss’s Law.

### 1.6 Poisson’s Equation

By substituting the definition (7) into the differential form of Gauss’s Law \( \nabla \cdot \mathbf{E} = \rho/\epsilon_0 \) we obtain Poisson’s Equation:

\[ \nabla^2 \Phi(r) = -\frac{\rho(r)}{\epsilon_0} \]

which expresses the Laplacian of the electric potential in terms of the charge density \( \rho(r) \).

We can use the definition of the delta function

\[ \delta^3(r - r') = -\nabla^2 \left( \frac{1}{4\pi |r - r'|} \right) \]
Figure 6: Electrostatic boundary conditions

to verify that Eq. (6) is a solution of Poisson’s Equation.

For the special case where $\rho(r) = 0$ at the field point $r$, Poisson’s Equation (9) becomes

$$\nabla^2 \Phi(r) = 0,$$

which is known as Laplace’s Equation.

1.7 Electrostatic Boundary Conditions

Boundary conditions in electrostatics are discussed in terms of a charged surface with surface density $\sigma(\alpha, \beta)$ at a point $r_S(\alpha, \beta)$ and a unit vector $\hat{n}(\alpha, \beta)$ perpendicular to the charged surface.

By defining $E_+$ and $E_-$ as the electric fields just above the surface and just below the surface at point $r_S(\alpha, \beta)$, respectively, applications of Gauss’s Law and Stokes’ Theorem yield the following boundary conditions

$$\begin{align*}
\hat{n} \cdot (E_+ - E_-) &= \sigma / \epsilon_0 \\
\hat{n} \times (E_+ - E_-) &= 0
\end{align*} \rightarrow E_+ - E_- = \frac{\sigma}{\epsilon_0} \hat{n}.
$$

Hence, the components of the electric field perpendicular to the surface, defined as $E_\perp = \hat{n} \cdot E$, exhibit a jump at the surface by an amount $\sigma / \epsilon_0$, while the components of the electric field parallel to the surface, defined as $E_\parallel = (\hat{n} \times E) \times \hat{n}$, is continuous across the
surface. In the case of a conducting surface (at \( z = 0 \)), for example, we find \( \hat{n} = \hat{z} \) and \( E_+ = -E_- = (\sigma/2\epsilon_0) \hat{z} \).

Next, starting with the expression
\[
\Phi(\mathbf{r}_S + \epsilon \hat{n}) - \Phi(\mathbf{r}_S - \epsilon \hat{n}) = -\int_{-\epsilon}^{\epsilon} \mathbf{E}(\mathbf{r}_S + s \hat{n}) \cdot \hat{n} ds
\]
and, taking the limit \( \epsilon \to 0 \), we find
\[
\Phi_+ - \Phi_- = \lim_{\epsilon \to 0} [\epsilon \hat{n} \cdot (E_+ - E_-)] = 0,
\]
i.e., the electric potential is continuous at the surface while the perpendicular components of its gradient, defined as \( \partial \Phi/\partial n = \hat{n} \cdot \nabla \Phi \), are discontinuous
\[
\left( \frac{\partial \Phi}{\partial n} \right)_+ - \left( \frac{\partial \Phi}{\partial n} \right)_- = -\frac{\sigma}{\epsilon_0}.
\]
Note that the electric force per unit area \( f_S \) experienced by the charged surface is expressed as
\[
f_S = \frac{\sigma}{2} (E_+ + E_-) = \sigma E_- + \frac{\sigma^2}{2\epsilon_0} \hat{n},
\]
which includes the force on the surface produced by the electric field below the surface (first term) and the surface tension generated by the surface charge density (second term).

1.8 Electrostatic Energy

The electric energy of a static discrete charge distribution \( \{(q_1, r_1), (q_2, r_2), \ldots, (q_N, r_N)\} \) is defined as the work required to construct the distribution by bringing each charge \( q_i \) successively from infinity to its final position \( r_i \):
\[
W = \frac{1}{2} \sum_{i=1}^{N} q_i \left( \sum_{j \neq i} \frac{q_j}{4\pi \epsilon_0 |r_i - r_j|} \right) = \frac{1}{2} \sum_{i} q_i \Phi(r_i),
\]
where the factor \( \frac{1}{2} \) takes into account that the contributions from the \( (j < i) \) terms and the \( (j > i) \) terms are equal.

For a continuous charge distribution, we generalize this expression to
\[
W = \frac{1}{2} \int_V \rho(\mathbf{r}) \Phi(\mathbf{r}) d\tau = \frac{1}{2} \int_S \sigma(\alpha, \beta) \Phi(\mathbf{r}_S) J d\alpha d\beta,
\]
where the first expression applies to a volume distribution, while the second expression applies to a surface distribution. Note that, making use of Gauss’s Law \( \rho = \epsilon_0 \nabla \cdot \mathbf{E} \), we also find
\[
W = \frac{\epsilon_0}{2} \int_V \mathbf{E} \cdot \nabla \Phi d\tau = \frac{\epsilon_0}{2} \int_V \left( -\mathbf{E} \cdot \nabla \Phi + \nabla \cdot (\mathbf{E} \Phi) \right) d\tau
\]
\[
= \frac{\epsilon_0}{2} \int_V |\mathbf{E}|^2 d\tau + \frac{\epsilon_0}{2} \oint_{\partial V} \Phi \mathbf{E} \cdot d\mathbf{a},
\]
which becomes, for a localized charge density and in the limit $V \to |R^3|$, 

$$W = \frac{\varepsilon_0}{2} \int_{|R^3|} |E|^2 \, d\tau.$$

We now compare the energy of a charged hollow sphere of radius $R$ with the energy of a charged solid sphere of radius $R$, where each sphere carries the same total charge $q$ and the electric potential at the surface of each sphere is $\Phi(R) = q/(4\pi \varepsilon_0 R)$. First, the charge density in the case of the hollow sphere is

$$\rho(r) = \frac{q}{4\pi R^2} \delta(r - R),$$

and the electric potential is

$$\Phi(r) = \frac{q}{4\pi \varepsilon_0} \begin{cases} R^{-1} & (r < R) \\ r^{-1} & (r > R) \end{cases},$$

so that the energy of a charged hollow sphere is

$$W_{\text{hollow}} = \frac{1}{2} \int \frac{q}{4\pi R^2} \delta(r - R) \cdot \frac{q}{4\pi \varepsilon_0 R} \cdot 4\pi r^2 \, dr = \frac{q^2}{8\pi \varepsilon_0 R}.$$

Next, the charge density in the case of the solid sphere (for $r < R$) is

$$\rho(r) = \frac{3q}{4\pi R^3},$$

and the electric potential (for $r < R$) is

$$\Phi(r) = \frac{q}{8\pi \varepsilon_0 R} \left(3 - \frac{r^2}{R^2}\right),$$

so that the energy of a charged hollow sphere is

$$W_{\text{solid}} = \frac{1}{2} \int_0^R \frac{3q}{4\pi R^3} \cdot \frac{q}{8\pi \varepsilon_0 R} \left(3 - \frac{r^2}{R^2}\right) \cdot 4\pi r^2 \, dr = \frac{3q^2}{20\pi \varepsilon_0 R}.$$

Hence, we find that $W_{\text{solid}} > W_{\text{hollow}}$, so that energy is lowered by concentrating the charge $q$ on the surface of the sphere.

### 1.9 Conductors and Induced Charges

Conductors possess the following electrostatic properties
I. The electric field inside a conductor vanishes: $E_{in} = 0$
II. The charge density inside a conductor is zero: $\rho_{in} = \epsilon_0 \nabla \cdot E_{in} = 0$
III. A conductor’s charge distribution is localized on its surface.
IV. The surface of a conductor is an equipotential surface: $\hat{n} \times \nabla \Phi_S = 0$
V. The electric field at a solid conductor’s surface is $E_S = (\sigma/\epsilon_0) \hat{n}$

To ensure that an external electric field $E_{ext}$ is cancelled inside a conductor, induced charges are created at the surface of the conductor (represented by the induced charge density $\rho_{ind}$) so that the induced electric field $E_{ind}$ (such that $\nabla \cdot E_{ind} = \epsilon_0 \rho_{ind}$) satisfies the condition $E_{net} = E_{ext} + E_{ind} = 0$ everywhere inside the conductor. We note, here, that the induced surface-charge density and induced charge are defined in terms of the electric potential as

$$\sigma_{ind} = -\epsilon_0 \frac{\partial \Phi}{\partial n} \quad \text{and} \quad q_{ind} = -\epsilon_0 \int_S \frac{\partial \Phi}{\partial n} \, da.$$

### 1.10 Capacitors

A capacitor is an electrical device used to store electric charge and electric energy.

First, we consider a parallel-plate capacitor composed of two parallel charged plates with equal area $A$, separated by a distance $d$, and one plate carries the charge $q$ while the other carries the opposite charge $-q$. 

![Parallel-plate capacitor](image)
The electric potential between the two plates is

\[ \Phi(z) = \frac{\sigma}{\epsilon_0} z, \]

where \( \sigma = q/A \) denotes the surface charge density, so that the potential of the lower plate in the Figure above is \( \Phi(0) = 0 \) and the potential of the upper plate is \( \Phi(d) = \sigma d/\epsilon_0 \). Thus, the stored charge \( q \) is proportional to the potential difference \( V = \Phi(d) - \Phi(0) \), with the constant of proportionality called the capacitance

\[ C = \frac{q}{V} = \frac{q}{(qd/\epsilon_0 A)} = \epsilon_0 \frac{A}{d}. \]

The unit of capacitance is the Farad (abbreviated F and named after Michael Faraday) defined as \( F = C \cdot V^{-1} = C^2 \cdot (N \cdot m)^{-1} \) and, thus, the permittivity of free space can also be given as \( \epsilon_0 = 8.85 \text{ pF} \cdot \text{m}^{-1} \). The energy stored in the parallel-plate capacitor is expressed as

\[ W = \frac{1}{2} \int_{-\infty}^{\infty} \left[ \sigma \delta(z - d) - \sigma \delta(z) \right] \Phi(z) A dz = \sigma A \cdot \frac{\sigma d}{2\epsilon_0} = \frac{q^2}{2C}. \]

Next, we consider a spherical-plate capacitor composed of two concentric charged spherical shells: the inner shell has radius \( a \) and carries positive charge \( q \) while the outer shell has radius \( b > a \) and carries negative charge \(-q\).

According to Gauss’s Law, the electric field between the two shells is

\[ \mathbf{E}(r) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^2}. \]
so that the electric potential is
\[ \Phi(r) = \Phi(a) - \int_a^r \frac{q}{4\pi \epsilon_0} \frac{dr}{r^2} = \Phi(a) + \frac{q}{4\pi \epsilon_0} \left( \frac{1}{r} - \frac{1}{a} \right). \]

Note that, since \( r > a \) between the two shells, we find that \( \Phi(r) < \Phi(a) \). The potential difference \( V = \Phi(a) - \Phi(b) \) is
\[ V = \frac{q}{4\pi \epsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right) = \frac{q}{C}, \]
and, thus, the capacitance of a spherical-plate capacitor is
\[ C = 4\pi \epsilon_0 \left( \frac{1}{a} - \frac{1}{b} \right)^{-1} = \epsilon_0 \cdot \frac{4\pi ab}{b-a}. \]

The stored energy, on the other hand, is
\[ W = \frac{1}{2} \int_0^\infty \left[ \sigma_a \delta(r-a) + \sigma_b \delta(r-b) \right] \Phi(r) \, 4\pi r^2 \, dr = \frac{1}{2} q V = \frac{q^2}{2C}, \]
where \( \sigma_a = q/(4\pi a^2) \) and \( \sigma_b = -q/(4\pi b^2) \) are the surface charge densities for the inner and outer shells, respectively.

### 2 Electrostatics II – Special Techniques

#### 2.1 Laplace’s Equation

A special case of Poisson’s equation \( \nabla^2 \Phi(r) = -\epsilon_0^{-1} \rho(r) \) involves the case where the charge density \( \rho(r) \) at the field point \( r \) vanishes, which yields Laplace’s equation
\[ \nabla^2 \Phi(r) = \nabla \cdot \nabla \Phi(r) = \sum_i \frac{1}{J} \frac{\partial}{\partial u^i} \left( J \, \nabla \Phi \cdot \nabla u^i \right) = \sum_{i,j} \frac{1}{J} \frac{\partial}{\partial u^i} \left( J \, g^{ij} \frac{\partial \Phi}{\partial u^j} \right), \]
where the inverse metric tensor components are \( g^{ij} = \nabla u^i \cdot \nabla u^j \) and \( J \) is the Jacobian for the transformation \( r \to u \) from Cartesian to curvilinear coordinates (e.g., cylindrical or spherical coordinates).

##### 2.1.1 Laplace’s equation in one dimension

The investigation of solutions of Laplace’s equation begins with the one-dimensional case: \( \Phi''(x) = 0 \), whose solution is
\[ \Phi(x) = \Phi_0 - E x, \]
where \( \Phi_0 \) is the potential at \( x = 0 \) and \( E \) is the magnitude of the constant electric field. Although this solution is trivial, it nonetheless exhibits two general properties that can be generalized to higher dimensions.
Property I \( \Phi(x) = \frac{1}{2} [\Phi(x + a) + \Phi(x - a)] \), for any \( a \).

Property II \( \Phi(x) \) has no minimum or maximum in the range \((x - a, x + a)\); minima and maxima can occur only at the end points \( x \pm a \).

2.1.2 Laplace’s equation in two dimensions

We begin with Laplace’s equation (11) in Cartesian coordinates

\[
\frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2} = 0,
\]

whose general solution can be expressed as an expansion in powers of \( x^n y^m \):

\[
\Phi(x,y) = \Phi_0 - E_x x - E_y y + \alpha xy + \frac{\beta}{2} (x^2 - y^2) + \frac{\gamma}{6} (x^3 - 3 xy^2 + y^3 - 3 y x^2) + \cdots,
\]

(12)

where \( \Phi_0 = \Phi(0,0) \) and \( E_0 = (E_x, E_y) \) are the electric potential and the electric field at the origin, respectively, and \( (\alpha, \beta, \gamma, \cdots) \) are constants.

In the Figure below, Eq. (12) is plotted for the case \( (\Phi_0 = 0, E_x = 1 = E_y, \alpha = 1, \beta = 5, \gamma = 6) \), which clearly shows that the potential \( \Phi(x,y) \) indeed satisfies Property II, i.e., Eq. (12) has no maximum or minimum in the square \((x, y) \in (-2, 2) \times (-2, 2)\), while minima and maxima occur only on the square’s perimeter. Note the existence of a saddle point\(^1\) at the origin, which combines a local minimum in \( x \) and a local maximum in \( y \).

The potential (12) also satisfies Property I since, for example, the potential at the origin \((0,0)\) can be found by integrating Eq. (12) on the perimeter of the square \((-a, a) \times (-a, a)\), for any \( a \):

\[
\Phi(0,0) = \frac{1}{8a} \left\{ \int_{-a}^{a} [\Phi(x,-a) + \Phi(x,a)] dx + \int_{-a}^{a} [\Phi(a,y) + \Phi(-a,y)] dx \right\} = \Phi_0.
\]

Before moving on to the three-dimensional case, we write Laplace’s equation (11) in polar coordinates \((x = r \cos \theta, \ y = r \sin \theta)\):

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi(r,\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi(r,\theta)}{\partial \theta^2} = 0,
\]

and point out that the azimuthally symmetric solution \( \Phi(r) \):

\[
(r \Phi')' = 0 \rightarrow \Phi'(r) = -E_a \frac{a}{r} \rightarrow \Phi(r) = \Phi_a - a E_a \ln(r/a),
\]

where \( \Phi_a = \Phi(a) \) and \( E_a \) are constants.

\(^1\)A saddle point is an extremum which exhibits a minimum in one direction and a maximum in another.
Moreover, according to Property I, the potential $\Phi(\mathbf{r})$ at the field point $\mathbf{r}$ can be evaluated in terms of the integral

$$\Phi(\mathbf{r}) = \frac{1}{2\pi} \oint \Phi(\mathbf{r} + a \hat{r}') \, d\theta'$$

along the boundary of a disk of arbitrary radius $a$ centered at the field point $\mathbf{r}$, where $\hat{r}' = \cos \theta' \hat{x} + \sin \theta' \hat{y}$.

### 2.1.3 Laplace’s equation in three dimensions

Laplace’s equation (11) in three dimensions can be written in Cartesian, cylindrical, and spherical coordinates as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0,$$

respectively. Although we shall not discuss general solutions of Laplace’s equation (11) in three dimensions, we note that special symmetric solutions $\Phi(r)$ in cylindrical and spherical
geometries are
\[ \Phi(r) = \begin{cases} 
\Phi_a - a E_a \ln(r/a) & \text{(cylindrical geometry)} \\
\Phi_a + E_a a^2/r & \text{(spherical geometry)} 
\end{cases} \]

Moreover, according to Property I, the \( \Phi(r) \) at the field point \( r \) can be evaluated in terms of the integral
\[ \Phi(r) = \frac{1}{4\pi} \int_0^\pi \sin \theta' \int_0^{2\pi} d\varphi' \Phi(r + a \hat{r}') \]
along the boundary of a sphere of arbitrary radius \( a \) centered at the field point \( r \), where \( \hat{r}' = \sin \theta' [\cos \varphi' \hat{x} + \sin \varphi' \hat{y}] + \cos \theta' \hat{z} \).

As an example, consider the case of the potential \( \Phi(r') \) associated with a charge \( q \) located at \( r = z \hat{z} \):
\[ \Phi(r') = \frac{q}{4\pi \epsilon_0} (r'^2 + z^2 - 2r'z \cos \theta')^{-\frac{1}{2}}. \]

The electric potential \( \Phi_0 \) at the origin can be calculated as the integral over the boundary of a sphere of radius \( a < z \) centered at the origin:
\[ \Phi_0 = \frac{1}{4\pi} \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} d\varphi' \frac{q}{4\pi \epsilon_0} (a^2 + z^2 - 2az \cos \theta')^{-\frac{1}{2}} = \frac{q}{4\pi \epsilon_0 z}. \]

Lastly, we prove Earnshaw’s Theorem: A charged particle cannot be held in stable equilibrium by electrostatic forces alone. The proof begins by considering a charge \( q \) placed inside a volume \( V \), in which the electric potential \( \Phi(r) \) satisfies Laplace’s equation (11). A stable equilibrium is associated with a minimum value \( W(r) = \frac{1}{2} q \Phi(r) \) for the energy of charge \( q \) placed at point \( r \) inside \( V \). On the one hand, an equilibrium point \( r_0 \) inside volume \( V \) is associated with \( \nabla W(r_0) = 0 \), which implies that the electric field at point \( r_0 \) must vanish, i.e., \( \nabla \Phi(r_0) = 0 \). On the other hand, the equilibrium point \( r_0 \) is stable if \( \nabla^2 W(r_0) > 0 \). However, since \( \nabla^2 \Phi(r_0) = 0 \) for any point inside volume \( V \), we find that a stable equilibrium point cannot be found in any region \( V \) where \( \Phi(r) \) satisfies Laplace’s equation \( \nabla^2 \Phi(r) = 0 \).

### 2.2 Boundary Conditions and Uniqueness Theorems

Solutions \( \Phi(r) \) of Laplace’s equation \( \nabla^2 \Phi(r) = 0 \) inside a region \( V \) are uniquely defined only if (a) the potential \( \Phi_S \) is specified on the boundary \( \partial V \) or (b) the electric field normal \( (-\partial \Phi_S/\partial n) \) to the boundary \( \partial V \) is specified. In the first case, called the Dirichlet problem, the boundary \( \partial V \) may be divided into domains \( \partial V = \cup_i \partial V_i \), where each domain \( \partial V_i \) is assigned the potential \( \Phi_i \) (see Figure below). In the second case, called the Neumann problem, the boundary \( \partial V \) is again divided into domains \( \partial V = \cup_i \partial V_i \), where each domain \( \partial V_i \) is now assigned the surface charge density \( \sigma_i \) (see Figure below).
We now show that a solution $\Phi(r)$ of Laplace’s equation $\nabla^2 \Phi(r) = 0$ inside region $V$ subject to the Dirichlet or Neumann boundary conditions on $\partial V$ is unique. It is worth pointing out that the same uniqueness theorem can be applied to solutions of Poisson’s equation $\rho(r) = -\varepsilon_0 \nabla^2 \Phi(r)$ inside region $V$ subject to Dirichlet or Neumann boundary conditions on $\partial V$.

The proof of the Uniqueness Theorem for solutions of Poisson’s equation (and Laplace’s equation as a special case) proceeds by assuming that, instead, two solutions $\Phi_a(r)$ and $\Phi_b(r)$ exist and both satisfy the same boundary conditions. Next, we construct the difference potential $\Psi(r) = \Phi_a(r) - \Phi_b(r)$ and easily verify that $\Psi(r)$ is a solution of Laplace’s equation

$$\nabla^2 \Psi(r) = \nabla^2 \Phi_a(r) - \nabla^2 \Phi_b(r) = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0,$$

but with boundary conditions $\Psi_S = 0$ (Dirichlet) or $\partial \Psi_S / \partial n = 0$ (Neumann) on the boundary $\partial V$. Lastly, we use the identity

$$\int_V \left( \Psi \nabla^2 \Psi + |\nabla \Psi|^2 \right) d\tau = \oint_{\partial V} \Psi_S \frac{\partial \Psi_S}{\partial n} da = 0,$$

where the surface integral vanishes because either $\Psi_S = 0$ or $\partial \Psi_S / \partial n = 0$ on the boundary. Since $\nabla^2 \Psi = 0$ inside region $V$, our identity becomes

$$\int_V |\nabla \Psi|^2 d\tau = 0,$$

which implies that $\nabla \Psi = 0$ everywhere inside $V$. 

---

**Figure 10: Dirichlet and Neumann boundary conditions**

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The proof of the Uniqueness Theorem for solutions of Poisson’s equation (and Laplace’s equation as a special case) proceeds by assuming that, instead, two solutions $\Phi_a(r)$ and $\Phi_b(r)$ exist and both satisfy the same boundary conditions. Next, we construct the difference potential $\Psi(r) = \Phi_a(r) - \Phi_b(r)$ and easily verify that $\Psi(r)$ is a solution of Laplace’s equation

$$\nabla^2 \Psi(r) = \nabla^2 \Phi_a(r) - \nabla^2 \Phi_b(r) = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0,$$

but with boundary conditions $\Psi_S = 0$ (Dirichlet) or $\partial \Psi_S / \partial n = 0$ (Neumann) on the boundary $\partial V$. Lastly, we use the identity

$$\int_V \left( \Psi \nabla^2 \Psi + |\nabla \Psi|^2 \right) d\tau = \oint_{\partial V} \Psi_S \frac{\partial \Psi_S}{\partial n} da = 0,$$

where the surface integral vanishes because either $\Psi_S = 0$ or $\partial \Psi_S / \partial n = 0$ on the boundary. Since $\nabla^2 \Psi = 0$ inside region $V$, our identity becomes

$$\int_V |\nabla \Psi|^2 d\tau = 0,$$

which implies that $\nabla \Psi = 0$ everywhere inside $V$. 

---

**Figure 10: Dirichlet and Neumann boundary conditions**
For the Dirichlet problem (First Uniqueness Theorem), since $\Psi(r)$ is a constant inside $V$ and vanishes on $\partial V$, we find that $\Psi(r) = 0$ everywhere inside $V$ (since $\Psi$ cannot have a minimum or maximum inside $V$). Hence, the two solutions $\Phi_a(r)$ and $\Phi_b(r)$ are, in fact, identical $\Phi_a(r) = \Phi_b(r)$ everywhere inside $V$. A unique solution of Poisson’s equation (and Laplace’s equation) inside a region $V$ was, therefore, determined solely by specifying the potential on the boundary $\partial V$ (Dirichlet problem).

A similar argument can be used for the Uniqueness Theorem applied to the Neumann problem (Second Uniqueness Theorem). The condition $\nabla \Psi = 0$ inside $V$ implies that the electric fields $E_a$ and $E_b$ are identical inside $V$ as well as on the boundary $\partial V$ since $\partial \Psi_S / \partial n = 0$.

## 2.3 Method of Images

Applications of the Uniqueness Theorems lead us to an important method used in solving Laplace’s equation subject to Dirichlet or Neumann boundary conditions.

### 2.3.1 Infinite Grounded Plane

As a first example, let us consider the physical problem where a charge $q$ is placed at a distance $h$ above an infinite conductor (at $z = 0$) held at potential $\Phi_S(x, y) = 0$. This is a Dirichlet problem and the First Uniqueness Theorem tells us that the solution $\Phi(x, y, z)$ in the region $z > 0$ is unique subject to the boundary condition $\Phi_S(x, y) = 0$. It is clear that induced charges must be created on the infinite conductor at $z = 0$; for instance, without induced charges, the potential due to $q$ at the origin is $\Phi_q(0, 0, 0) = q/(4\pi \epsilon h) \neq 0$, which does not satisfy the boundary condition. The physical problem is difficult to solve because the induced surface charge density $\sigma_S(x, y)$ is unknown.

We note that the boundary condition $\Phi_S(x, y) = 0$ would be easily satisfied if we placed a second charge $-q$ of equal magnitude but opposite sign below the infinite plane at a distance $h$ (see Figure below).

The solution for the electric potential for this image problem is

$$\Phi(x, y, z) = \frac{q}{4\pi \epsilon_0} \left( \frac{1}{\sqrt{x^2 + y^2 + (z - h)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + h)^2}} \right),$$

which satisfies Laplace’s equation everywhere above the plane except at $(x, y, z) = (0, 0, h)$ and satisfies the boundary condition $\Phi_S(x, y) = \Phi(x, y, z = 0) = 0$ everywhere on the infinite plane. By the First Uniqueness Theorem, we have, therefore, found the unique solution $\Phi(x, y, z)$ to the physical problem.
Note that the induced surface charge density \( \sigma_S(x, y) \) can also be calculated from

\[
\sigma_S(x, y) = -\varepsilon_0 \frac{\partial \Phi_S}{\partial n} = -\varepsilon_0 \frac{\partial \Phi}{\partial z} \bigg|_{z=0} = -\frac{qh}{2\pi (x^2 + y^2 + h^2)^{3/2}}.
\]

As expected that the sign of the induced charge is opposite to \( q \) and the total induced charge \( q_{\text{ind}} \) is calculated to be

\[
q_{\text{ind}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \sigma_S(x, y) = \int_{0}^{\infty} dr \int_{0}^{2\pi} d\theta \left( \frac{-qh}{2\pi (r^2 + h^2)^{3/2}} \right) = -q.
\]

Moreover, since the induced surface charge density is opposite to \( q \), the charge \( q \) will be attracted to the infinite conductor with a force

\[
F = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2h)^2} \hat{z},
\]

calculated as if the conductor was absent and only the two charges \( q \) and \(-q\) (separated by a distance \( 2h \)) exist.

### 2.3.2 Grounded Sphere

As a second example, let us consider the problem of finding the outside potential \( \Phi(r, \theta) \) associated with a charge \( q \) placed at a distance \( a \) (on the \( z \)-axis) from the center of a
grounded sphere of radius $R$, i.e., $\Phi(R, \theta) = 0$. To determine the outside potential ($r > R$), an image charge $q'$ is placed inside the sphere at a distance $b$ on the $z$-axis.

The outside potential is, therefore, expressed as

$$\Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2br \cos \theta}} \right).$$

We now use the boundary conditions $\Phi(R, 0) = 0 = \Phi(R, \pi)$ to obtain the relations

$$q' = -\varepsilon q \quad \text{and} \quad b = \varepsilon R,$$

where $\varepsilon = R/a < 1$. Hence, the outside potential for this problem is

$$\Phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} - \frac{\varepsilon}{\sqrt{\varepsilon^2 + \varepsilon^4 a^2 - 2 \varepsilon^2 ar \cos \theta}} \right), \quad (14)$$

which satisfies the boundary condition

$$\Phi(R, \theta) = \frac{q}{4\pi\epsilon_0 a} \left( \frac{1}{\sqrt{1 + \varepsilon^2 - 2 \varepsilon \cos \theta}} - \frac{\varepsilon}{\sqrt{\varepsilon^2 + \varepsilon^4 - 2 \varepsilon^3 \cos \theta}} \right) = 0.$$

We calculate the induced surface charge density to be

$$\sigma_s(\theta) = -\epsilon_0 \frac{\partial \Phi}{\partial r}(R, \theta) = -\frac{q}{4\pi a R} \frac{(1 - \varepsilon^2)}{(1 + \varepsilon^2 - 2 \varepsilon \cos \theta)^{3/2}}.$$
so that the induced charge \( q_{\text{ind}} \) is
\[
q_{\text{ind}} = 4\pi R^2 \int_0^\pi \sigma S(\theta) \sin \theta \, d\theta = -\varepsilon q = q'.
\]

We, therefore, see from these two examples that the image charge(s) must be placed outside of the region where the potential is to be evaluated and that the induced charge on the grounded surface is equal to the total image charge.

### 2.4 Separation of Variables in Two Dimensions

#### 2.4.1 Two-Dimensional Cartesian Problems

Solutions \( \Phi(x, y) \) of Laplace’s equation for two-dimensional problems can be expressed as the product
\[
\Phi(x, y) = X(x) Y(y) = \left( Ae^{kx} + B e^{-kx} \right) \left( Ce^{iky} + D e^{-iky} \right),
\]
where the constant \( k \) can be either real or imaginary depending on the boundary conditions imposed along the \( x \)-axis and \( y \)-axis, which also determine the values of the four constants \( A, B, C, D \). Note that \( X''(x) = k^2 X(x) \) and \( Y''(y) = -k^2 Y(y) \), so that the ansatz (15) is indeed a general solution of Laplace’s equation:
\[
\nabla^2 \Phi(x, y) = X''(x) Y(y) + X(x) Y''(y) = \left(k^2 - k^2\right) X(x) Y(y) = 0,
\]
for all values of \((A, B, C, D)\).

Boundary conditions not only determine uniquely solutions of Laplace’s equation but they also provide clues as to which set of coordinates is ideally suited for a given problem. In the present case, two-dimensional Cartesian coordinates \((x, y)\) are ideally suited to rectangular boundary conditions at the left boundary \( x = x_- \) and the right boundary \( x = x_+ \) on the \( x \)-axis, the upper boundary \( y = y_\uparrow \) and the lower boundary \( y = y_\downarrow \) on the \( y \)-axis. Hence, the boundary conditions can now be generally expressed as
\[
\begin{align*}
\Phi(x_-, y) &= \Phi_{\text{left}}(y) \\
\Phi(x_+, y) &= \Phi_{\text{right}}(y) \\
\Phi(x, y_\downarrow) &= \Phi_{\text{lower}}(x) \\
\Phi(x, y_\uparrow) &= \Phi_{\text{upper}}(x)
\end{align*}
\]
\[
(16)
\]
in terms of prescribed functions of \( x \) on the upper and lower boundaries and functions of \( y \) on the left and right boundaries.
As our first example, we consider the case of a semi-infinite slot with boundary conditions
\[
\begin{align*}
\Phi(0, y) &= V_0(y) \\
\lim_{x \to \infty} \Phi(x, y) &= 0 \\
\Phi(x, 0) &= 0 \\
\Phi(x, a) &= 0
\end{align*}
\]

The boundary condition \(\lim_{x \to \infty} \Phi(x, y) = 0\) means that the constant \(k\) must be real (and positive) and that the constant \(A\) must be zero, while the boundary conditions in \(y\) imply that
\[
C + D = 0 \quad \text{and} \quad Ce^{ika} + De^{-ika} = 0,
\]
which means that \(D = -C\), so that the second boundary condition becomes
\[
\sin k a = 0 \quad \rightarrow \quad k = \frac{n\pi}{a} = k_n \quad (n = 1, 2, \ldots).
\]

Hence, our general solution (15) for this problem at this point has the form
\[
\Phi_n(x, y) = E_{n} e^{-k_{n}x} \sin (k_{n} y), \quad (17)
\]
for \(n = 1, 2, \ldots\), where we have introduced the single constant coefficient \(E_n\).

If we now try to match our solution (17) to the last boundary condition \(\Phi(0, y) = V_0(y)\), we find
\[
E_{n} \sin (k_{n}y) = V_0(y),
\]
which is impossible unless \(V_0(y)\) is itself expressed as a sinusoidal function. As a way to save the method of Separation of Variables, we need to use Fourier's Trick, which assumes that we can satisfy the boundary condition \(\Phi(0, y) = V_0(y)\) through the infinite Fourier series
\[
V_0(y) = \sum_{n=1}^{\infty} E_{n} \sin \left( \frac{n\pi y}{a} \right), \quad (18)
\]
provided we can find expressions for the Fourier coefficients \((E_1, E_2, \cdots)\).

To obtain these expressions, we first need to introduce the integral identity
\[
\int_{0}^{a} \sin \left( \frac{n\pi y}{a} \right) \sin \left( \frac{m\pi y}{a} \right) dy = \frac{1}{2} \int_{0}^{a} \left\{ \cos \left( \frac{(n-m)\pi y}{a} \right) - \cos \left( \frac{(n+m)\pi y}{a} \right) \right\}
\]
\[
= \frac{a}{2} \left[ \frac{\sin((n-m)\pi)}{(n-m)\pi} - \frac{\sin((n+m)\pi)}{(n+m)\pi} \right]
\]
\[
= \frac{a}{2} \delta_{nm}, \quad (19)
\]
where \(\delta_{nm}\) denotes the Kroenecker delta (which equals 1 if \(n = m\) or 0 if \(n \neq m\)). Next, we multiply Eq. (18) by \(\sin(m\pi y/a)\) and integrate from 0 to \(a\) to obtain, using the identity
(19), the result

\[ \int_0^a V_0(y) \sin \left( m\pi \frac{y}{a} \right) \, dy = \sum_{n=1}^{\infty} a \frac{E_n}{2} \delta_{nm} = \frac{a}{2} E_m, \]

and thus the general Fourier coefficient \( E_n \) is

\[ E_n = \frac{2}{a} \int_0^a V_0(y) \sin \left( n\pi \frac{y}{a} \right) \, dy. \]  \hspace{1cm} (20)

Note that, as \( n \) becomes large, we expect the integral (20) to decrease if \( V_0(y) \) is a smooth function of \( y \).

To provide a concrete example, we consider the simple boundary condition \( V_0(y) = V_0 \) for which the Fourier coefficients (20) are

\[ E_n = \frac{2V_0}{a} \int_0^a \sin \left( n\pi \frac{y}{a} \right) \, dy = \frac{2V_0}{n\pi} \left( 1 - (-1)^n \right) \]

\[ = \frac{4V_0}{n\pi} \quad \text{(for} \ n = 1, 3, 5, \cdots \text{)}, \]

and the solution to the boundary-condition problem is

\[ \Phi(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\cdots} \frac{1}{n} \exp \left( -n\pi \frac{x}{a} \right) \sin \left( n\pi \frac{y}{a} \right), \] \hspace{1cm} (21)

This infinite sum can actually be evaluated explicitly by pursuing the following approach. As a first step, we define the function

\[ Z(x,y) = \exp \left( -\frac{\pi}{a} (x + iy) \right) \quad \text{so that} \quad \Im Z = -e^{-\pi x/a} \sin(\pi y/a), \] \hspace{1cm} (22)

and Eq. (21) becomes

\[ \Phi(x, y) = -\frac{4V_0}{\pi} \Im \left( \sum_{n=1,3,5,\cdots} \frac{Z^n}{n} \right). \] \hspace{1cm} (23)

As a second step, we introduce the Taylor-expansion formulas

\[ \sum_{n=1}^{\infty} \frac{Z^n}{n} \, n = -\ln(1 - Z) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} Z^n \, n = \ln(1 + Z), \]

so that

\[ \sum_{n=1,3,5,\cdots} \frac{Z^n}{n} = \frac{1}{2} \ln \left( \frac{1 + Z}{1 - Z} \right). \] \hspace{1cm} (24)

As a third step, we write

\[ \frac{1 + Z}{1 - Z} = \frac{(1 + Z) \cdot (1 - Z^*)}{|1 - Z|^2} = \left( 1 - \frac{|Z|^2}{|1 - Z|^2} \right) + i \left( \frac{2 \Im Z}{|1 - Z|^2} \right), \]
Figure 13: Approximate solution to Laplace’s equation

and use the identity
\[ \text{Im}[\ln(X + iY)] = \text{Im}\left\{ \frac{1}{2} \ln(X^2 + Y^2) + i \arctan\left( \frac{Y}{X} \right) \right\} = \arctan\left( \frac{Y}{X} \right) \]
for any \( X \) and \( Y \), so that Eq. (24) becomes

\[ \text{Im}\left( \sum_{n=1,3,5,\ldots} \frac{Z^n}{n} \right) = \arctan\left( \frac{2\text{Im}Z}{1 - |Z|^2} \right). \]

By substituting the definition (22) into Eq. (23), we finally obtain

\[ \Phi(x, y) = \frac{2V_0}{\pi} \arctan\left( \frac{\sin(\pi y/a)}{\sinh(\pi x/a)} \right), \]
which obviously satisfies the boundary conditions at \( y = 0 \) and \( y = a \) as well as (separately) \( x = 0 \) and \( x \to \infty \).

The Figure shown below shows the approximate solution (truncated at a finite \( N \))

\[ \Phi(x, y; N) = \frac{4V_0}{\pi} \sum_{n=1}^{N} \frac{\exp[-(2n-1)\pi x/a]}{2n-1} \sin\left( (2n-1)\pi \frac{y}{a} \right), \]
for \( N = 10 \) (left) and \( N = 100 \) (right). Note that, for \( x \neq 0 \), the higher-order terms have very little effect on the overall function \( \Phi(x, y; N) \). For small values of \( x \) (and especially \( x = 0 \)), however, the higher-order terms play a very important role. In the Figure above,
Figure 14: Boundary behavior of approximate solutions
each plot begins at \( x = 0.01 \) to highlight the effect of higher-order terms. Notice, for example, how rapidly the approximate solution tries to approach the boundary solution as \( x \to 0 \) for \( N = 100 \) as compared to the case \( N = 10 \).

The Figure shown below shows the approximate boundary solution (truncated at a finite \( N \))

\[
\Phi_0(y; N) = \frac{4 V_0}{\pi} \sum_{n=1}^{N} \frac{\sin[(2n-1)\pi y/a]}{(2n-1)},
\] (26)

for \( N = 2 \) (upper left), \( N = 10 \) (upper right), \( N = 20 \) (lower left), and \( N = 100 \) (lower right). Note that as \( N \) increases, the approximate boundary solution (26) approaches the boundary condition (shown by a dotted line) except near the boundary points \( y_\downarrow = 0 \) and \( y_\uparrow = a \), where we easily observe the Gibbs phenomenon (i.e., overshooting) associated with the fact that we are trying to use smooth functions \( \{ \sin(k_n y) \} \) in the closed interval \([0, a]\) to fit a function \( V_0(y) \) with discontinuities at the end points of the interval\(^2\). Note that,

\(^2\)Note that even the theoretical value of 18% for the maximum overshoot is observed in the Figures above.
when evaluated at $x = 0$ and $y = a/2$, Eq. (21) yields the identity
\[
\frac{\pi}{4} = \sum_{n=1,3,5,\ldots} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \tag{27}
\]
which converges slowly (i.e., 2 significant-digit accuracy requires more than 25 terms in the sum).

### 2.4.2 Expansion in terms of Orthonormal Functions

In general, an arbitrary function $F(s)$ on an interval $a < s < b$ can be expanded as an infinite series
\[
F(s) = \sum_{n=1}^{\infty} f_n \psi_n(s) \tag{28}
\]
involving the basis functions $\{\psi_n(s); n = 1, 2, \cdots\}$, where the coefficient $f_n$ are expressed as
\[
f_n = \int_{a}^{b} F(s) \psi_n(s), \tag{29}
\]
provided the basis functions satisfy the Orthonormality condition
\[
\int_{a}^{b} \psi_n(s) \psi_m(s) \, ds = \delta_{nm}
\]
and the Completeness condition
\[
\sum_{n=1}^{\infty} \psi_n(s) \psi_n(s') = \delta(s' - s).
\]
These two conditions provide necessary and sufficient conditions for the validity of the expansion (28) with coefficients (29).

### 2.4.3 Two-Dimensional Polar Problems

Laplace’s equation is written in polar coordinates $(r, \theta)$ as
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi(r, \theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi(r, \theta)}{\partial \theta^2} = 0,
\]
whose general solution $\Phi(r, \theta)$ can be separated as $\Phi(r, \theta) = R(r) \Theta(\theta)$. Requiring that $\Theta(\theta)$ be a periodic function of the angle $\theta$, we find
\[
\frac{1}{\Theta(\theta)} \frac{d^2 \Theta(\theta)}{d\theta^2} = -n^2 = -\frac{r}{R(r)} \frac{d}{dr} \left( r \frac{dR(r)}{dr} \right),
\]
where \( n = 0, 1, 2, \cdots \); the case \( n = 0 \) shall be treated independently from \( n > 0 \).

First, if \( n = 0 \), then the angular function \( \Theta_0(\theta) \) can be chosen to be \( \Theta_0(\theta) = 1 \) (this is the only allowed periodic solution in this case), while the radial function \( R_0(r) \) satisfies the equation

\[
(r R_0')' = 0 \Rightarrow R_0' = b_0/r \Rightarrow R_0 = a_0 + b_0 \ln r,
\]

and, thus, the general solution for \( n = 0 \) and \( r \neq 0 \) is

\[
\Phi_0(r, \theta) = a_0 + b_0 \ln r. \tag{30}
\]

Next, for \( n > 0 \), the general solution for \( \Theta(\theta) \) is expressed as

\[
\Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta,
\]

while the radial function \( R_n(r) \) satisfies the equation

\[
r (r R_n')' = n^2 R_n(r) \Rightarrow R_n(r) = C_n r^n + D_n r^{-n},
\]

where the constants \((A_n, B_n, C_n, D_n)\) can be re-arranged to form the general solution for \( n \neq 0 \)

\[
\Phi_n(r, \theta) = r^n (a_n \cos n\theta + b_n \sin n\theta) + r^{-n} (c_n \cos n\theta + d_n \sin n\theta), \tag{31}
\]

where the last two terms apply only when \( r \neq 0 \). Boundary conditions in polar coordinates can be expressed in terms of Dirichlet’s boundary condition \( \Phi(R, \theta) = V_0(\theta) \) or Neumann condition \( \sigma_0(\theta) \) on a circle of radius \( R \) and whether we are interested in the inside region \( r < R \) or the outside region \( r > R \).

Let us consider the inside solution for the Dirichlet boundary condition \( \Phi(R, \theta) \). For the inside solution (which includes \( r = 0 \)), we immediately find \( c_n = 0 = d_n \ (n > 1) \) and \( b_0 = 0 \). Hence, the general inside solution \( \Phi_{in}(r, \theta) \) is of the form

\[
\Phi_{in}(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta). \tag{32}
\]

Here, to match the boundary condition \( \Phi(R, \theta) = V_0(\theta) \), we find

\[
a_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(R, \theta) \, d\theta,
\]

while, using the identities

\[
\frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} \cos n\theta & \cos m\theta \\ \sin n\theta & \sin m\theta \end{pmatrix} d\theta = \frac{1}{2} \delta_{nm},
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} \cos n\theta \sin m\theta \, d\theta = 0,
\]
we find

\[ a_n = \frac{1}{\pi R^n} \int_0^{2\pi} \Phi(R, \theta) \cos n\theta \, d\theta, \]

\[ b_n = \frac{1}{\pi R^n} \int_0^{2\pi} \Phi(R, \theta) \sin n\theta \, d\theta. \]

When we combine these results, we find the integral expression

\[ \Phi_{in}(r, \theta) = \int_0^{2\pi} \frac{d\theta'}{2\pi} \Phi(R, \theta') \left[ 1 + 2 \sum_{n=1}^\infty \varepsilon^n \cos(n(\theta - \theta')) \right]. \]

where \( \varepsilon = r/R < 1 \). Next, by using the identity

\[ 1 + 2 \sum_{n=1}^\infty \varepsilon^n \cos(n(\theta - \theta')) = \frac{1 - \varepsilon^2}{1 + \varepsilon^2 - 2\varepsilon \cos(\theta - \theta')}, \]

we obtain the Poisson integral formula

\[ \Phi_{in}(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\Phi(R, \theta') (R^2 - r^2)}{R^2 + r^2 - 2rR \cos(\theta - \theta')} \, d\theta'. \]

The previous identity is constructed as follows: we define \( z = \exp i(\theta - \theta') \) so that

\[ 1 + 2 \sum_{n=1}^\infty \varepsilon^n \cos(n(\theta - \theta')) = 1 + \sum_{n=1}^\infty \varepsilon^n (z^n + z^{-n}) \]

\[ = -1 + \sum_{n=0}^\infty \varepsilon^n (z^n + z^{-n}) \]

\[ = -1 + \frac{1}{1 - \varepsilon z} + \frac{1}{1 - \varepsilon z^{-1}} \]

\[ = \frac{1 - \varepsilon^2}{1 + \varepsilon^2 - \varepsilon (z + z^{-1})}. \]

Note here that, from the completeness condition

\[ 2\pi \delta(\theta - \theta') = \sum_{n=-\infty}^{\infty} e^{in(\theta - \theta')} = \sum_{n=-\infty}^{\infty} \cos(n(\theta - \theta')) = 1 + 2 \sum_{n=1}^\infty \cos(n(\theta - \theta')), \]

we recover the boundary condition \( \lim_{r \to R} \Phi_{in}(r, \theta) = \Phi(R, \theta) \).

For example, let us consider the following boundary condition

\[ V_0(\theta) = \begin{cases} V_0 & \text{for } 0 < \theta < \pi \\ -V_0 & \text{for } \pi < \theta < 2\pi \end{cases} \]
for which \( a_0 = 0 = a_n \) (for all \( n \)) and
\[
b_n = \frac{1}{\pi R^n} \left( V_0 \int_0^\pi \sin n\theta \, d\theta - V_0 \int_0^{2\pi} \sin n\theta \, d\theta \right) = \frac{2V_0}{n\pi R^n} \left( 1 - (-1)^n \right) = \frac{4V_0}{n\pi R^n} \text{ (for } n = 1, 3, 5, \cdots) .
\]
Hence, the inside solution for the Dirichlet problem is
\[
\Phi_{in}(r, \theta) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\cdots} \frac{1}{n} \left( \frac{r}{R} \right)^n \sin n\theta .
\]
By following the same approach as above, we evaluate the infinite sum explicitly and find
\[
\Phi_{in}(r, \theta) = \frac{2V_0}{\pi} \arctan \left( \frac{2rR \sin \theta}{R^2 - r^2} \right) . \tag{33}
\]
With this solution, we can define equipotential lines \( \Phi(x, y) = \Phi \) by the relation
\[
r(\theta; \alpha) = \frac{R}{\alpha} \left( - \sin \theta + \sqrt{\sin^2 \theta + \alpha^2} \right) ,
\]
where \( \alpha = \tan((\pi/2) \Phi/V_0) \). The Figure below shows the equipotential lines in the upper-half of the disk associated with \( \Phi \) equal to 0, 0.295 \( V_0 \), 0.500 \( V_0 \), 0.874 \( V_0 \), and \( V_0 \) or, equivalently, \( \alpha \) equal to 0, \( \frac{1}{2} \), 1, 5, and \( \infty \).

The electric field \( \mathbf{E}_{in} = -\nabla \Phi_{in} \) inside the disk is obtained from Eq. (33) as
\[
\mathbf{E}_{in}(r, \theta) = - \frac{4V_0 R}{\left[ (R^2 - r^2)^2 + (2rR \sin \theta)^2 \right]} \left[ r^2 \sin(2\theta) \mathbf{\hat{x}} + (R^2 - r^2 \cos(2\theta)) \mathbf{\hat{y}} \right] ,
\]
which yields the expected result
\[
\mathbf{E}_S(\theta) = \mathbf{E}_{in}(R, \theta) = - \frac{2V_0}{R \sin \theta} \mathbf{\hat{r}} ,
\]
on the circular boundary at $r = R$ for $0 < \theta < \pi$.

As a second example, we now consider the case of a wedge of angle $\alpha = \pi/\kappa$ (where $\kappa = 1/2, 1, 2, 3, \ldots$) held at a constant potential $V_0$ (see Figure 16). The potential inside the wedge can be expressed as

$$\Phi(r, \theta) = V_0 + \sum_{n=1}^{\infty} b_n r^n \sin n\theta,$$

where the boundary condition $\Phi(r, 0) = V_0$ is easily satisfied. The boundary condition $\Phi(r, \alpha) = V_0$, on the other hand, requires that $n\alpha = m\pi$ or $n = m\kappa$ (for $m \geq 1$), so that the potential inside the wedge (for small values of $r$) is finally expressed as

$$\Phi(r, \theta) = V_0 + b_1 r^\kappa \sin \kappa \theta + \cdots.$$

Hence, the electric field near the corner of the wedge is

$$\mathbf{E} = -\nabla \Phi = -\kappa b_1 r^{\kappa-1} \left( \sin \kappa \theta \hat{r} + \cos \kappa \theta \hat{\theta} \right),$$

so that the magnitude of the electric field scales as $|\mathbf{E}| \simeq \kappa b_1 r^{\kappa-1}$ near the corner. The electric field near the corner ($r \to 0$) is, therefore, either strong when $\kappa < 1 (\alpha > \pi)$, weak when $\kappa > 1 (\alpha < \pi)$, or uniform when $\kappa = 1 (\alpha = \pi)$.

### 2.5 Separation of Variables in Three Dimensions

#### 2.5.1 Two-Dimensional Spherical Problems

Laplace’s equation $\nabla^2 \Phi(r) = 0$ is written in spherical coordinates $(r, \theta, \varphi)$ as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0.$$
If we restrict our attention to problems with azimuthal symmetry, i.e., \( \Phi(r) = \Phi(r, \theta) \), so that Laplace’s equation in this case becomes

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 0.
\]  

(34)

Once again we proceed by the Method of Separation of Variables and assume that \( \Phi(r, \theta) = R(r) \Theta(\theta) \) and

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \ell (\ell + 1) = - \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right),
\]

where \( \ell = 0, 1, 2, \cdots \), is an arbitrary non-negative integer. Solutions of the radial equation are

\[
R_\ell(r) = A_\ell r^\ell + B_\ell r^{\ell+1},
\]  

(35)

where \( A_\ell \) and \( B_\ell \) are constants to be determined from boundary conditions. Solutions of the angular equation are expressed in terms of the Legendre polynomials

\[
P_\ell(x) = \frac{1}{2^{\ell} \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell 
\] 

which satisfy the second-order ordinary differential equation

\[
\frac{d}{dx} \left( (1 - x^2) \frac{dF(x)}{dx} \right) + \ell (\ell + 1) F(x) = 0,
\]

with solutions \( F(x) = P_\ell(x) \), which are regular at the end points \( x = \pm 1 \) and

\[
F(x) = Q_\ell(x) = \frac{1}{2} P_\ell(x) \ln \left( \frac{1 + x}{1 - x} \right) - \sum_{k=1}^{\ell} P_{k-1}(x) P_{\ell-k}(x),
\]

\[
\rightarrow \quad Q_0(x) = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \quad Q_1(x) = \frac{x}{2} \ln \left( \frac{1 + x}{1 - x} \right) - 1, \cdots
\]

which are singular at the end points \( x = \pm 1 \). Note that the Legendre polynomials satisfy the following recurrence relations

\[
(\ell + 1) P_{\ell+1}(x) = (2\ell + 1) x P_\ell(x) - \ell P_{\ell-1}(x),
\]

\[
(x^2 - 1) P_\ell'(x) = \ell x P_\ell(x) - \ell P_{\ell-1}(x),
\]
they are generated as coefficients in the Taylor expansion

\[
\frac{1}{\sqrt{1 - 2xz + x^2}} = \sum_{n=0}^{\infty} P_n(x) \, z^n,
\]

and that they obviously possess the parity property: \( P_\ell(-x) = (-1)^\ell P_\ell(x) \).

The only regular solution for the angular function \( \Theta_\ell(\theta) \) is, therefore, expressed as

\[
\Theta_\ell(\theta) = P_\ell(\cos),
\]

Hence, the general solution for an azimuthally-symmetric spherical problem is expressed as

\[
\Phi(r, \theta) = \sum_{\ell=0}^{\infty} \left( A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta),
\]

where the coefficients \( B_\ell \) are zero (for all \( \ell \)) if we are interested in the inside solution \((r < R)\) while the coefficients \( A_\ell \) are zero (for \( \ell \geq 1 \)) if we are interested in the outside solution \((r > R)\).

### 2.5.2 Example

Let us consider the inside solution of a Dirichlet boundary problem in which the potential is specified on the surface of a sphere of radius \( R \) as

\[
V_0(\theta) = \Phi(R, \theta) = \sum_{\ell=0}^{\infty} A_\ell \, R^\ell \, P_\ell(\cos \theta).
\]

Using the Orthogonality condition

\[
\int_{-1}^{1} P_\ell(x) \, P_\ell'(x) \, dx = \int_{0}^{\pi} P_\ell(\cos \theta) \, P_\ell'(\cos \theta) \, \sin \theta \, d\theta = \frac{2 \, \delta_{\ell\ell'}}{2\ell + 1},
\]

we find the coefficients

\[
A_\ell = \frac{2\ell + 1}{2R^\ell} \int_{0}^{\pi} V_0(\cos \theta) \, P_\ell(\cos \theta) \, \sin \theta \, d\theta.
\]

For example, we consider

\[
V_0(\theta) = \begin{cases} 
V_0 & \text{for } 0 \leq \theta < \pi/2 \\
-V_0 & \text{for } \pi/2 < \theta \leq \pi
\end{cases}
\]

so that the coefficients \( A_\ell \) are

\[
A_\ell = (2\ell + 1) \frac{V_0}{2R^\ell} \int_{0}^{1} [P_\ell(x) - P_\ell(-x)] \, dx = (2\ell + 1) \frac{V_0}{R^\ell} \int_{0}^{1} P_\ell(x) \, dx \quad \text{for } \ell = 1, 3, 5, \ldots
\]
Using properties of Legendre polynomials, we find
\[
\int_0^1 P_{2n-1}(x) \, dx = \frac{P_{2n-2}(0) - P_{2n}(0)}{(4n-1)} = -\frac{P_{2n}(0)}{(2n-1)} = \frac{(-1)^{n+1} (2n)!}{2^{2n} (2n-1) (n)!^2},
\]
and, thus,
\[
A_1 = \frac{3 V_0}{2 R}, \quad A_3 = -\frac{7 V_0}{8 R^3}, \quad A_5 = \frac{11 V_0}{16 R^5}, \ldots,
\]
and the inside solution for our Dirichlet problem is
\[
\Phi_{in}(r, \theta) = V_0 \left( \frac{3r}{2 R} P_1(\cos \theta) - \frac{7r^3}{8 R^3} P_3(\cos \theta) + \frac{11r^5}{16 R^5} P_5(\cos \theta) - \cdots \right).
\]

2.6 Electric Multipoles

2.6.1 Multipole Expansion of the Electric Potential

We begin by calculating the electric potential \( \Phi(\mathbf{r}) \) due to a localized charge distribution
\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}') \, d\tau'}{|\mathbf{r} - \mathbf{r}'|},
\]
where \( \rho(\mathbf{r}') \) denotes the charge density and we write \( |\mathbf{r} - \mathbf{r}'| \) as
\[
|\mathbf{r} - \mathbf{r}'| = r \sqrt{1 + \eta^2 - 2 \eta \cos \theta'},
\]
with \( \mathbf{r} = r \hat{z} \) and \( \eta = r'/r < 1 \) since we are interested in calculating the electric potential outside of the localized charge distribution (see Figure 17).

We first introduce the generating function for the Legendre polynomials
\[
\frac{1}{\sqrt{1 + \eta^2 - 2 \eta \cos \theta'}} = \sum_{\ell=0}^\infty P_\ell(\cos \theta') \eta^\ell,
\]
so that Eq. (6) yields the Legendre-polynomial expansion
\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^\infty \frac{1}{r^{\ell+1}} \int r'^\ell P_\ell(\cos \theta') \rho(\mathbf{r}') \, d\tau'. \tag{38}
\]

Next, we introduce the Legendre-polynomial expansion of the charge density (assumed to be azimuthally symmetric)
\[
\rho(r, \theta) = \sum_{k=0}^\infty \rho_k(r) P_k(\cos \theta),
\]
so that Eq. (38) becomes

\[ \Phi(r) = \frac{1}{4\pi\epsilon_0} \sum_{\ell,k=0}^{\infty} \frac{1}{r^{\ell+1}} \left( \int \rho_k(r') \, r'^{2+\ell} \right) 2\pi \int_{0}^{\pi} p_{\ell}(\cos \theta') \, P_{k}(\cos \theta') \, \sin \theta' \, d\theta' \]

\[ = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \left( \int \frac{4\pi}{2\ell + 1} \rho_{\ell}(r') \, dr' \right) = \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{4\pi \epsilon r^{\ell+1}}. \]

Here, the coefficients \( C_0, C_1, \) and \( C_2 \) are, respectively, expressed in terms of moments of the charge distribution as

\[ C_0 = 4\pi \int r'^2 \, \rho_0(r') \, dr' = q, \]

\[ C_1 = \frac{4\pi}{3} \int r'^3 \, \rho_1(r') \, dr' = p_z, \]

\[ C_2 = \frac{4\pi}{5} \int r'^4 \, \rho_2(r') \, dr' = \frac{1}{2} Q_{zz}, \]

where

\[ q = \int \rho(r') \, dr', \quad (39) \]

\[ p = \int r' \, \rho(r') \, dr', \quad (40) \]

\[ Q = \int \left( 3 r' r' - r'^2 I \right) \rho(r') \, dr', \quad (41) \]

are the monopole scalar term (i.e., the total charge), the dipole-moment vector term \( p, \) and the (traceless) quadrupole tensor term \( Q, \) respectively. Thus, the multipole expansion

Figure 17: Electric multipole decomposition of a localized charge distribution
of $\Phi(r, \theta)$ can now be expressed as

$$\Phi(r) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{r} + \frac{\mathbf{p} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \mathbf{Q} \cdot \hat{r}}{2r^3} + \cdots \right),$$

(42)

where we note that, in general, the $\ell$th-order multipole is associated with a $r^{-(\ell+1)}$ dependence for the electric potential.

The electric monopole term is obviously associated with the net charge of the charge distribution. If the charge distribution is globally neutral, then the next multipole moment is the electric dipole moment

$$\mathbf{p} = \int_0^\infty r' d^2 r' \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} d\phi' \rho(r') \left[ r' \sin \theta' \left( \cos \varphi' \hat{x} + \sin \varphi' \hat{y} \right) + r' \cos \theta' \hat{z} \right].$$

Moreover, since a change of origin ($\mathbf{r} \to \mathbf{r} + q \hat{a}$) in the calculation of the dipole moment yields the shift $\mathbf{p} \to \mathbf{p} + q \mathbf{a}$, we conclude that the dipole moment of a neutral charge distribution is independent of where the origin placed. In addition, if the charge distribution is azimuthally symmetric, i.e., $\rho(r', \theta') = \rho(r', \theta')$, then the electric dipole moment is now aligned along the $z$-axis

$$\mathbf{p} = 2 \int_0^\infty r'^2 \sin \theta' \cos \theta' d\phi' \rho(r', \theta') = p \hat{z}.$$

Hence, the charge distribution has a finite electric dipole moment if there exists a separation of charge when projected along the $z$-axis. The quadrupole moment tensor of an azimuthally-symmetric charge distribution can be shown to be represented as

$$\mathbf{Q} = Q_{zz} \left[ \hat{z} \hat{z} - \frac{1}{2} (\hat{x} \hat{x} + \hat{y} \hat{y}) \right],$$

where

$$Q_{zz} = 2 \int r'^2 P_2(\cos \theta') \rho(r', \theta') d\tau'.$$

Since $\mathbf{p} \cdot \hat{r} = p P_1(\cos \theta)$ and $\hat{r} \cdot \mathbf{Q} \cdot \hat{r} = Q_{zz} P_2(\cos \theta)$, the multipole expansion (42) becomes

$$\Phi(r) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^\infty \frac{C_\ell}{r^{\ell+1}} P_\ell(\cos \theta),$$

for an azimuthally-symmetric charge distribution, where the expansion coefficients $C_0 = q$, $C_1 = p_z$, and $C_2 = \frac{1}{2} Q_{zz}$, etc., are now given in terms of multipole moments of the charge distribution.

### 2.6.2 Electric Field of a Dipole

The electric potential associated with a dipole moment $\mathbf{p} = p \hat{z}$ is obtained from the multipole expansion (42) as

$$\Phi(r) = \frac{\mathbf{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} = -\nabla \cdot \left( \frac{\mathbf{p}}{4\pi\epsilon_0 r} \right).$$
See the equipotential surfaces $r(\theta) = \pm \sqrt{(p/4\pi \varepsilon_0 \Phi)} \cos \theta$ for a dipole field in the Figure shown below.

We now find the electric field of a dipole to be expressed as

$$E = \nabla \left[ \nabla \cdot \left( \frac{p}{4\pi \varepsilon_0 r} \right) \right] = \frac{p}{4\pi \varepsilon_0} \cdot \nabla \left( \nabla r^{-1} \right).$$

Noting that $\nabla r^{-1} = -r/r^3$ and, using $\nabla r = \mathbf{I}$, we find

$$E = \frac{p}{4\pi \varepsilon_0} \cdot \left( -\frac{\mathbf{I}}{r^3} + 3 \frac{\hat{r}}{r^3} \right) = \frac{3(p \cdot \hat{r}) \hat{r} - p}{4\pi \varepsilon_0 r^3}$$

$$= \frac{p}{4\pi \varepsilon_0 r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right). \quad (43)$$

Note that this expression, which is valid for $r \neq 0$, is incomplete and needs to be supplemented by the term $-\frac{p}{3\varepsilon_0} \delta^3(\mathbf{r})$, so that the complete expression for the electric field of a dipole is

$$E(\mathbf{r}) = \frac{p}{4\pi \varepsilon_0 r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) - \frac{p}{3\varepsilon_0} \delta^3(\mathbf{r}). \quad (44)$$

We now prove the need for this extra term in Eq. (44) as follows.

### 2.6.3 Volume-integrated Electric field

Let us consider the volume integral of the electric field

$$\int_V E(\mathbf{r}) \, d\tau = \int_V d\tau \int_{V'} d\tau' \frac{\rho(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|^3} = -\int_V \nabla \left( \int_{V'} \frac{\rho(\mathbf{r}') \, d\tau'}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|^3} \right) d\tau,$$
where we shall consider the volume \( V \) to be a sphere of radius \( R \) either totally enclosed in \( V' \) or totally outside of \( V' \). Integration by parts using the identity \( \int_V \nabla f \, d\tau = \oint_{\partial V} f \hat{n} \, da \) yields the expression

\[
\int_V \mathbf{E}(\mathbf{r}) \, d\tau = -\oint_{\partial V} \mathbf{r} \cdot \hat{n} \, da \left( \int_{V'} \frac{\rho(\mathbf{r}') \, d\tau'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \right) = -\int_{V'} \frac{\rho(\mathbf{r}') \, d\tau'}{4\pi\epsilon_0} \left( \oint_{\partial V} \frac{\hat{r} \, da}{|\mathbf{r} - \mathbf{r}'|} \right),
\]

where \( \mathbf{r} = R\hat{r} \) and \( da = R^2 \sin \theta \, d\theta \, d\varphi \). Next, we use the Legendre-polynomial identity

\[
\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<}{r_>^{\ell+1}} P_\ell(\cos \gamma),
\]

where \((r_<, r_>) = (r', R)\) when \( V \) is outside of \( V' \) or \((r_<, r_>) = (R, r')\) when \( V \) is inside \( V' \), and

\[
\cos \gamma = \hat{r} \cdot \hat{r}' = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi').
\]

Since the unit vector \( \hat{r} \) involves either \( \cos \theta \) or \( \sin \theta \), it turns out that only the \( \ell = 1 \) terms contributes to the surface integral and, thus, we find

\[
\oint_{\partial V} \frac{\hat{r} \, da}{|\mathbf{r} - \mathbf{r}'|} = \frac{r_< R^2}{r_>^2} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} \cos \gamma \hat{r} \, d\varphi = \frac{4\pi}{3} \frac{r_< R^2}{r_>^2} \int_0^{2\pi} \cos \gamma \, d\varphi,
\]

so that we finally obtain

\[
\int_V \mathbf{E}(\mathbf{r}) \, d\tau = -\oint_{\partial V} \mathbf{r} \cdot \hat{n} \, da \left( \int_{V'} \frac{\rho(\mathbf{r}') \, d\tau'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \right) = -\int_{V'} \frac{\rho(\mathbf{r}') \, d\tau'}{4\pi\epsilon_0} \left( \oint_{\partial V} \frac{\hat{r} \, da}{|\mathbf{r} - \mathbf{r}'|} \right), \tag{45}
\]

We now consider the cases \((r_<, r_>) = (r', R)\) and \((r_<, r_>) = (R, r')\) separately. In the first case, Eq. (45) yields

\[
\int_V \mathbf{E}(\mathbf{r}) \, d\tau = \frac{4\pi}{3} \int_{V'} \frac{\rho(\mathbf{r}') \, d\tau'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}, \tag{46}
\]

where \( \mathbf{p} \) denotes the dipole-moment of the charge distribution, while in the second case, Eq. (45) yields

\[
\int_V \mathbf{E}(\mathbf{r}) \, d\tau = -\frac{4\pi R^3}{3} \int_{V'} \frac{\rho(\mathbf{r}') \, d\tau'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} = -\frac{4\pi R^3}{3} \mathbf{E}_0, \tag{47}
\]

where \( \mathbf{E}_0 \) denotes the electric field at the origin.

As an application of Eq. (46), we prove that the expression (43) for the electric field of a dipole is incomplete. We easily check that

\[
\frac{\mathbf{p}}{3\epsilon_0} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\varphi \left[ 2 \cos \theta \, \hat{r} + \sin \theta \, \hat{\theta} \right] = \frac{\mathbf{p} \hat{z}}{3\epsilon_0} \int_0^\pi \sin \theta \, P_2(\cos \theta) \, d\theta = 0,
\]

so that without the last term in Eq. (44), the identity (46) would not be satisfied.
3 Polarization

3.1 Atomic and Molecular Polarizations

This chapter deals with electric properties of matter, loosely divided into conductors and insulators (or dielectrics). In contrast to conductors, electrons in dielectrics are attached to specific atoms or molecules. Although atoms and molecules are electrically neutral, they nonetheless exhibit intrinsic and extrinsic polarizability properties defined, respectively, in the absence or presence of an external electric field.

3.1.1 Atomic Polarizability

When a neutral atom is placed in an external electric field \( E \), the electron cloud responds by moving in the opposite direction to the electric field \( E \) (while the heavier positively-charged nucleus almost remains in place), thereby creating a dipole moment

\[
p = \alpha E
\]

in the same direction as the electric field \( E \), where the constant of proportionality is called the atomic polarizability \( \alpha \).

Note that atomic polarizability \( \alpha \) must have units \([\alpha] = [\epsilon_0] \cdot m^3\) and is a strong function of the atomic number (see Figure below).

From the Figure above, we see that alkali metals have the largest polarizabilities while the smallest values correspond to noble gases. This is simply due to their electronic configurations: alkali metals have a single electron outside of closed electronic shells while noble gases have closed electronic shells. Hence, alkali atoms are easily polarized by an external electric field while, as can be seen from the Figure above, noble-gas atoms are weakly polarizable. From the Figure above, we can check that the polarizability of noble gases scales.
Figure 20: Atomic polarizability as a function of atomic number

linearily with atomic number $Z$.

The displacement shown in the first Figure above is, of course, grossly exaggerated. In fact, assuming that the electron cloud (of net charge $-q$) maintains its spherical shape (adequate for noble gases) and that the nucleus (of net charge $q$) has shifted to the right (in the direction of the electric field $E\hat{x}$) by a distance $z$, the electric field due to the electron cloud as experienced by the nucleus is

$$E_e = -\frac{qd}{4\pi\epsilon_0 a^3} \hat{x},$$

where $a$ denotes the atomic radius. Because atomic polarizability is a property of matter in equilibrium, the displacement $d$ adjusts itself so that the electron-cloud field matches the external field: $E_e = E$. Hence, the electric dipole moment $p = qd$ is now expressed as $p = 4\pi\epsilon_0 a^3 E$, which yields the following expression for the atomic polarizability

$$\frac{\alpha}{4\pi\epsilon_0} = a^3.$$

A best-fit analysis of the atomic radii of noble gases yields $a^3(Z) = (2.9 + 0.14Z) \times 10^{-30}$ m$^3$, which overestimates real atomic polarizability values for noble gases by a factor of 10 or more, although it reproduces the linear $\alpha - Z$ relationship.

In general, for a spherically-symmetric charge distribution $\rho(r)$, the magnitude of the electric field at a distance $z$ from the center of the distribution is

$$E(z) = \frac{1}{\epsilon_0 z^2} \int_0^z \rho(r) r^2 \, dr,$$
and the linear relation \( p = qz = \alpha E(z) \) holds for sufficiently small values of \( z \). For example, using the electronic charge distribution for a hydrogen atom

\[
\rho(r) = \frac{q}{\pi a_0^3} \exp \left( -\frac{2r}{a_0} \right),
\]

where \( a_0 = 0.5 \times 10^{-10} \text{ m} \) denotes the Bohr radius, we find after integrating by parts

\[
E(z) = \frac{q}{4\pi \varepsilon_0 z^2} \left[ 1 - e^{-2z/a_0} \left( 1 + 2 \frac{z}{a_0} + 2 \frac{z^2}{a_0^2} \right) \right],
\]

which becomes (in the limit \( z \ll a_0 \))

\[
E(z) \to \frac{4}{3} \frac{q}{4\pi \varepsilon_0} \frac{z}{a_0^3} \Rightarrow \frac{\alpha}{4\pi \varepsilon_0} = \frac{3}{4} a_0^3.
\]

Substituting the value for the Bohr radius, we find \( \frac{\alpha}{(4\pi \varepsilon_0)} = 0.09 \times 10^{-30} \text{ m}^3 \), compared to the experimental value 0.667 \( \times 10^{-30} \text{ m} \).

### 3.1.2 Molecular Polarizability

The situation gets more complicated when one considers molecular polarizabilities. For completely asymmetrical molecules, for example, we write the dipole moment

\[
p = \alpha \cdot E \quad \text{or} \quad \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},
\]

in terms of the \( 3 \times 3 \) polarizability tensor \( \alpha \), which can be diagonalized by choosing appropriate principal axes \((\hat{1}, \hat{2}, \hat{3})\), such that

\[
\alpha = \alpha_1 \hat{1} \hat{1} + \alpha_2 \hat{2} \hat{2} + \alpha_3 \hat{3} \hat{3},
\]

where the principal polarizabilities \((\alpha_1, \alpha_2, \alpha_3)\) are generally different from each other. Hence, in the general asymmetric case, the dipole moment \( p \) may be written as

\[
p = \alpha_1 E_1 \hat{1} + \alpha_2 E_2 \hat{2} + \alpha_3 E_3 \hat{3},
\]

and we note that \( p \) may not be parallel to the electric field \( E \) itself since

\[
p \times E = (\alpha_2 - \alpha_3) E_2 E_3 \hat{1} + (\alpha_3 - \alpha_1) E_3 E_1 \hat{2} + (\alpha_1 - \alpha_2) E_1 E_2 \hat{3}.
\]

Many polar molecules have finite electric dipole moments \( p_0 \) even in the absence of an external electric field (e.g., the magnitude of the electric dipole for the water molecule is \( p_0 = 6.1 \times 10^{-30} \text{ C} \cdot \text{m} \) or \( d_0 = p_0/e = 0.38 \text{ \AA} \)). When a polar molecule (with electric
dipole $p_0$) is placed in an external *uniform* electric field, the molecule experiences a torque $N = p_0 \times E$, which causes the dipole $p_0$ to align itself with the electric field $E$ (see Figure below).

Hence, on the one hand, the net force on an electric dipole due to a uniform electric field vanishes. On the other hand, when the electric field is *nonuniform*, the net electric force on the dipole is

$$F = F_+ + F_- = q \Delta E = p_0 \cdot \nabla E = -p_0 \cdot \nabla \nabla \Phi = \nabla E \cdot p_0 = \nabla (E \cdot p_0).$$

An electric dipole also acquires potential energy

$$U = -p_0 \cdot E$$

when it is exposed to an external electric field. This expression is obtained by calculating the work done by the electric field $E = E \hat{x}$ when the electric dipole $p = p (\cos \theta \hat{x} + \sin \theta \hat{y})$ in moved from $\theta = \frac{\pi}{2}$ to an arbitrary angle $\theta$. Using the expression for the torque

$$N = p \times E = -p E \sin \theta \hat{z},$$

we find

$$W = \int_{\pi/2}^{\theta} N(\theta') d\theta' = pE \cos \theta = U(\pi/2) - U(\theta).$$
### 3.2 Electric Displacement

#### 3.2.1 Electric Field of a Polarized Object

A polarized object is one in which the dipole moment per unit volume, called the polarization \( \mathbf{P} \), is not zero. Since the electric potential of a single dipole moment \( \mathbf{p} \) (located at the origin) is

\[
\Phi(r) = \frac{\mathbf{p} \cdot \hat{r}}{4\pi \varepsilon_0 r^2},
\]

the total electric potential due to a polarized object is

\[
\Phi(r) = \frac{1}{4\pi \varepsilon_0} \int_V \mathbf{P}(r') \cdot \left( \frac{r - r'}{|r - r'|^3} \right) d\tau'.
\] (48)

Using the identity

\[
\frac{r - r'}{|r - r'|^3} = \nabla' \left( \frac{1}{|r - r'|} \right),
\]

and integrating by parts, the electric potential (48) becomes

\[
\Phi(r) = \oint_{\partial V} \frac{\mathbf{P}(r') \cdot \hat{n}}{4\pi \varepsilon_0 |r - r'|} d\alpha' - \int_V \frac{\nabla' \cdot \mathbf{P}(r')}{4\pi \varepsilon_0 |r - r'|} d\tau'
\] 

\[
= \oint_{\partial V} \frac{\sigma_b(r_S) d\alpha'}{4\pi \varepsilon_0 |r - r_S|} + \int_V \frac{\rho_b(r')}{4\pi \varepsilon_0 |r - r'|} d\tau',
\] (49)

where \( \sigma_b(r_S) = \mathbf{P}(r_S) \cdot \hat{n} \) denotes the surface charge density and \( \rho_b(r') = -\nabla' \cdot \mathbf{P}(r') \) denotes the volume charge density. Note that

\[
q_b = \oint_{\partial V} \sigma_b d\alpha + \int_V \rho_b d\tau = \oint_{\partial V} \mathbf{P} \cdot d\alpha - \int_V \nabla \cdot \mathbf{P} d\tau = 0,
\]

the net bound charge is zero (which immediately follows from the charge conservation law).

For example, we consider the electric field due to a uniformly polarized sphere of radius \( R \). Choosing the \( z \)-axis along the polarization vector \( \mathbf{P} \), where the volume charge density is \( \rho_b = 0 \), we find the surface charge density

\[
\sigma_b = \mathbf{P} \cdot \hat{r} = P \cos \theta'.
\]

Here, using spherical coordinates for \( \mathbf{r} \) and \( \mathbf{r}_S \), we find

\[
|r - r_S| = \sqrt{r^2 + R^2 - 2rR \cos \gamma},
\]

where \( \gamma \) is the three-dimensional angle between \( r \) and \( r_S \), defined

\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi').
\]
Next, we use the Legendre-polynomial generating function

\[
\frac{1}{\sqrt{1 + \eta^2 - 2\eta \cos \gamma}} = \sum_{n=0}^{\infty} \eta^n P_n(\cos \gamma),
\]

where \(\eta < 1\) and, thus,

\[
\frac{1}{\sqrt{r^2 + R^2 - 2rR \cos \gamma}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^n P_n(\cos \gamma),
\]

where \((r_<, r_>) = (r, R)\) inside the polarized sphere and \((r_<, r_>) = (R, r)\) outside the polarized sphere. Hence, the electric potential is

\[
\Phi(r) = \frac{P R^2}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^n S_n(\theta),
\]

which follows from the cylindrical symmetry of the problem, where

\[
S_n(\theta) = \int_0^\pi \sin \theta' \, d\theta' \int_0^{2\pi} \, d\phi' \cos \theta' \, P_n(\cos \gamma) = \left(\frac{4\pi}{3} \cos \theta\right) \delta_{n1},
\]

so that

\[
\Phi(r, \theta) = \frac{P \cos \theta}{3\varepsilon_0} \left\{ \begin{array}{ll}
  r & \text{(for } r < R) \\
  \frac{R^3}{r^2} & \text{(for } r > R)
\end{array} \right.
\]

(50)

Note that the electric field inside the polarized sphere is

\[
E_{in} = -\nabla \Phi_{in} = -\frac{P}{3\varepsilon_0},
\]

while the electric potential outside the sphere is

\[
\Phi_{out} = \frac{\mathbf{p} \cdot \hat{r}}{4\pi \varepsilon_0 r^2}, \quad \text{with} \quad \mathbf{p} = \frac{4\pi}{3} R^3 \mathbf{P},
\]

i.e., the electric field outside the uniformly polarized sphere is given by the expression for the electric field of a pure dipole.

### 3.2.2 Electric Displacement

Within a dielectric, the charge density \(\rho\) is the sum of the density \(\rho_f\) of free charges present inside the dielectric and the density \(\rho_b = -\nabla \cdot \mathbf{P}\) of bound charges generated by the polarization vector \(\mathbf{P}\). Using Gauss’s Law \(\varepsilon_0 \nabla \cdot \mathbf{E} = \rho\), where \(\mathbf{E}\) denotes the total electric field, we define the electric displacement \(\mathbf{D}\) in terms of \(\mathbf{E}\) and \(\mathbf{P}\) as

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P},
\]

(51)
so that Gauss’s Law is now written as
\[ \nabla \cdot \mathbf{D} = \varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho - \rho_b = \rho_f \]
or
\[ \oint_{\partial V} \mathbf{D} \cdot d\mathbf{a} = (q_V)_f, \]
where \((q_V)_f\) is the net free enclosed by the volume \(V\). This version of Gauss’s Law implies that the electric displacement vector \(\mathbf{D}\) starts or ends only on free charges while the electric field \(\mathbf{E}\) starts or ends on any charge. Note that the dipole moment of the bound-charge distribution
\[ \int_V \rho_b r \, d\tau = -\int_V r \nabla \cdot \mathbf{P} \, d\tau = \int_V \mathbf{P} \, d\tau \]
is defined as the volume integral of the polarization vector.

Consider, for example, a dielectric shell of inner radius \(a\) and outer radius \(b\) with a \textit{frozen-in} polarization \(\mathbf{P} = P(r) \hat{r}\) (within the shell) and we assume that there are no free charges in the problem. The bound-charge density is
\[ \rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 P(r) \right), \]
while the surface bound-charge density is
\[ \sigma_b = \mathbf{P} \cdot \hat{n} = \begin{cases} P(b) \quad (r = b \text{ and } \hat{n} = \hat{r}) \\ -P(a) \quad (r = a \text{ and } \hat{n} = -\hat{r}) \end{cases} \]
According to Gauss’s Law, the electric field
\[ \mathbf{E} = \frac{q_V(r) \hat{r}}{4\pi \varepsilon_0 r^2}, \]
vanishes outside the dielectric shell (i.e., \(r < a\) and \(r > b\)) since \(q_V = 0\) everywhere outside the dielectric. Inside the dielectric shell \((a < r < b)\), however, we find
\[ q_V(r) = -P(a) 4\pi a^2 + \int_a^r \left[ -\frac{1}{s^2} \frac{d}{ds} \left( s^2 P(s) \right) \right] 4\pi s^2 ds = -4\pi r^2 P(r), \]
and, therefore, the electric field inside the dielectric shell is
\[ \mathbf{E} = -\frac{P(r)}{\varepsilon_0} \hat{r} = -\frac{\mathbf{P}}{\varepsilon_0}. \]
This result naturally follows from Gauss’s Law for the electric displacement \(\mathbf{D}\) in the absence of free charges:
\[ \oint_{\partial V} \mathbf{D} \cdot d\mathbf{a} = 0 \Rightarrow \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = 0. \]
Note that, in general, the electric displacement $D$ is not \textit{curl-free} even though the electric field $E$ is, since

$$\nabla \times D = \epsilon_0 \nabla \times E + \nabla \times P = \nabla \times P,$$

and $\nabla \times P$ may not vanish. Moreover, the differential form of Gauss’s Law for the electric displacement is $\nabla \cdot D = \rho_f$, the solution for $D(r)$ is NOT the Coulomb’s Law solution

$$D_C(r) = \frac{1}{4\pi} \int_V \left( \frac{r - r'}{|r - r'|^3} \right) \rho_f(r') \, dr'.$$

Note: for a uniformly polarized sphere

$$E_{in} = -\frac{P}{3\epsilon_0} \quad \text{and} \quad D_{in} = \frac{2P}{3\epsilon_0}.$$

### 3.2.3 Boundary Conditions

Gauss’s Law for the electric displacement implies that the jump condition across a surface carrying the free-charge surface density $\sigma_f = \sigma - P \cdot \hat{n}$ is

$$(D_+ - D_-) \cdot \hat{n} = \sigma_f,$$

where $\hat{n}$ is a unit vector perpendicular to the surface, i.e., a surface density of free charges produces a discontinuity in the electric displacement across the surface. The discontinuity in the parallel component of the electric displacement, on the other hand, is associated with the discontinuity in the parallel component of the polarization. Hence,

$$D_+ - D_- = \sigma_f \hat{n} + \hat{n} \times [(P_+ - P_-) \times \hat{n}],$$

follows from $D_\pm = \epsilon_0 E_\pm + P_\pm$ and $(E_+ - E_-) = (\sigma/\epsilon_0) \hat{n}$, with

$$\hat{n} \times [(P_+ - P_-) \times \hat{n}] = (P_+ - P_-) - [(P_+ - P_-) \cdot \hat{n}] \hat{n} = (P_+ - P_-) + \sigma_b \hat{n}.$$

### 3.3 Linear Dielectrics

#### 3.3.1 Electric Susceptibility

In a \textit{linear} dielectric medium, the polarization vector $P$ is proportional to the electric field $E$:

$$P = \epsilon_0 \chi E,$$  \hspace{1cm} (52)

where the dimensionless parameter $\chi$ is called the electric susceptibility. Here, the susceptibility depends on the microscopic structure of the dielectric medium and the linear
relationship between $P$ and $E$ holds if $E$ is not too strong. In general, we note that the polarization vector $P(E)$ can be Taylor-expanded in powers of the electric field $E$:

$$P(E) = E^i \left( \frac{\partial P}{\partial E^i} \right)_0 + \frac{E^i E^j}{2} \left( \frac{\partial^2 P}{\partial E^i \partial E^j} \right)_0 + \cdots = \varepsilon_0 \left( \chi^{(1)} \cdot E + \frac{1}{2} E \cdot \chi^{(2)} \cdot E + \cdots \right),$$

where the linear susceptibility tensor $\chi^{(1)}$ is defined (in an isotropic dielectric medium) as

$$\varepsilon_0 \chi^{(1)} = \varepsilon_0 \chi I,$$

while the higher-order terms define the nonlinear electric susceptibility tensor $\chi^{(2)}$ (which plays a crucial in the nonlinear optics of materials such as lithium niobate, LiNbO$_3$, and potassium dihydrogenphosphate, KH$_2$PO$_4$ or KDP); since nonlinear susceptibilities are typically less than $(\chi^{(2)})_{max} = 10^{-9}$ m/V, only intense electric fields, $|E| > \chi^{(1)} \times 10^9$ V/m, can be used to investigate nonlinear optical effects.

In a linear dielectric medium, the electric displacement $D$, therefore, becomes

$$D = \varepsilon_0 E + P = \varepsilon_0 (1 + \chi) E = \varepsilon E,$$

where $\varepsilon$ denotes the permittivity of the medium. The ratio $\varepsilon_r = \varepsilon / \varepsilon_0$ defines the relative permittivity, or dielectric constant, of the dielectric medium.$^3$

Consider, for example, a metal sphere of radius $a$, and carrying a positive charge $Q$, surrounded by a spherical shell (up to radius $b$) composed of a linear dielectric medium of permittivity $\varepsilon$. Inside the metal sphere (region I: $r < a$), all fields vanish: $E_I = P_I = D_I = 0$. The electric displacement $D$ outside the metal sphere (region II: $a < r < b$ and region III: $r > b$) is $D = Q/(4\pi r^2) \hat{r}$, since $Q$ is the only free charge present. Outside the dielectric shell (region III: $r > b$), where there is no polarization, we find

$$E_{III} = \frac{D_{III}}{\varepsilon_0} = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{r},$$

as expected. Lastly, inside the dielectric shell (region II: $a < r < b$), we find

$$E_{II} = \frac{D_{II}}{\varepsilon} = \frac{Q}{4\pi \varepsilon r^2} \hat{r},$$

where we have used $D_{III} = D_{II}$. We, thus, calculate the electric potential (relative to infinity) in each region by using the formula

$$\Phi(r) = - \int_{\infty}^{r} E \cdot dl,$$

$^3$Values of dielectric constant range from 1 (exactly) for free space (by definition), to values slightly greater than one for many gases (at 20° C and 1 atm), to values greater than one for most liquids and solids (e.g., 5.7 for diamond, 5.9 for NaCl, 33.0 for methanol, and 80.1 for liquid water).
3 POLARIZATION

to find: in region III \((r > b)\)

\[
\Phi_{III}(r) = \frac{Q}{4\pi\epsilon_0 r},
\]

in region II \((a < r < b)\)

\[
\Phi_{II}(r) = \frac{Q}{4\pi\epsilon_0 b} + \frac{Q}{4\pi\epsilon} \left(\frac{1}{r} - \frac{1}{b}\right),
\]

and in region I \((r < a)\)

\[
\Phi_I(r) = \frac{Q}{4\pi\epsilon_0 b} + \frac{Q}{4\pi\epsilon} \left(\frac{1}{a} - \frac{1}{b}\right).
\]

As an interesting limiting case, we consider the limits \(a \to 0\) and \(b \to \infty\) corresponding to the case of a single charge \(Q\) immersed in a dielectric medium of permittivity \(\epsilon\). The electric potential

\[
\Phi(r) = Q/(4\pi\epsilon r) = (\epsilon_0/\epsilon) \Phi_0(r)
\]

is, therefore, reduced by a factor \(\epsilon_0/\epsilon\) from its free-space value \(\Phi_0(r) = Q/(4\pi\epsilon_0 r)\).

The polarization vector in region II is

\[
P = \epsilon_0\chi \mathbf{E}_{II} = \left(\frac{\chi}{1+\chi}\right) \frac{Q}{4\pi r^2} \hat{r} = P(r) \hat{r},
\]

from which we obtain the bound-charge volume density

\[
\rho_b = -\nabla \cdot P = -\frac{1}{r^2} \frac{d}{dr} \left(r^2 P(r)\right) = 0,
\]

and the bound-charge surface densities at \(r = a\) (where \(\hat{n} = -\hat{r}\)) and \(r = b\) (where \(\hat{n} = \hat{r}\))

\[
(\sigma_b)_a = -P(a) \quad \text{and} \quad (\sigma_b)_b = P(b).
\]

Here, we note that the net surface bound-charges are

\[
(q_b)_a = -\left(\frac{\chi}{1+\chi}\right) Q = -(q_b)_b,
\]

i.e., the net surface bound-charge on the inner surface is negative, as expected, and its magnitude is \(|(q_b)_a| = [(\chi/(1+\chi))] Q < Q\), which implies that a dielectric is a poor conductor. In the ideal-conductor limit \((\chi \to \infty)\), we find \((q_b)_a = -Q = -(q_b)_b\), as expected.

In general, the bound-charge volume density \(\rho_b\) in an homogeneous linear dielectric medium is proportional to the free-charge volume density \(\rho_f\):

\[
\rho_b = -\nabla \cdot P = -\nabla \cdot \left[\left(\frac{\chi}{1+\chi}\right) \mathbf{D}\right] = -\left(\frac{\chi}{1+\chi}\right) \rho_f,
\]
and, hence, inside an homogeneous linear dielectric medium, we find the net charge volume
density \( \rho = \rho_b + \rho_f = (\epsilon_0/\epsilon) \rho_f \), which vanishes if \( \rho_f = 0 \). For an inhomogeneous linear
dielectric medium without free charges (\( \nabla \cdot \mathbf{D} = 0 \)), on the other hand, the bound-charge
volume density is

\[
\rho_b = -\nabla \cdot \left( \frac{\chi}{1+\chi} \mathbf{D} \right) = -\frac{\mathbf{D} \cdot \nabla \chi}{(1+\chi)^2} = -\epsilon_0 \mathbf{E} \cdot \nabla \ln \epsilon,
\]

where \( \epsilon = \epsilon_0 (1 + \chi) \).

We consider the example of a free charge \( q \) placed at the center of dielectric sphere of
radius \( R \) and linear susceptibility \( \chi \). First, Gauss’s Law for the electric displacement vector
yields \( \mathbf{D} = q/(4\pi r^2) \hat{r} \) everywhere in space. Next, the inside electric field (\( r < R \)) is

\[
\mathbf{E}_{in} = \frac{\mathbf{D}}{\epsilon} = \frac{q \hat{r}}{4\pi \epsilon r^2},
\]

while the outside electric field (\( r > R \)) is

\[
\mathbf{E}_{out} = \frac{\mathbf{D}}{\epsilon_0} = \frac{q \hat{r}}{4\pi \epsilon_0 r^2}.
\]

From the inside electric field, we now obtain the polarization vector

\[
\mathbf{P} = \epsilon_0 \chi \mathbf{E}_{in} = \left( \frac{\chi}{1+\chi} \right) \frac{q \hat{r}}{4\pi r^2} \equiv -q \left( \frac{\chi}{1+\chi} \right) \nabla \left( \frac{1}{4\pi |\mathbf{r}|} \right),
\]

from which we obtain the bound-charge volume density

\[
\rho_b = -\nabla \cdot \mathbf{P} = \left( \frac{q \chi}{1+\chi} \right) \nabla^2 \left( \frac{1}{4\pi |\mathbf{r}|} \right) = - \left( \frac{\chi}{1+\chi} \right) \rho_f,
\]

where \( \rho_f = q \delta^3(\mathbf{r}) \) denotes the free-charge volume distribution. The bound-charge surface
density, on the other hand, is

\[
\sigma_b = \mathbf{P} \cdot \hat{n} = \left( \frac{\chi}{1+\chi} \right) \frac{q}{4\pi R^2},
\]

so that the net bound-charge is \( q_0 = -q\chi/(1+\chi) + q\chi/(1+\chi) = 0 \). Note that the net charge
\( q_0 \) located at the center of the sphere is the sum of the free charge \( q \) and the contribution
\(-q\chi/(1+\chi)\) from the bound-charge volume density: \( q_0 = q/(1+\chi) < q \) and that, in the
perfect-conductor limit (\( \chi \to \infty \)), the center charge \( q_0 \to 0 \) while the surface bound-charge
\( q_R \to q \), as expected.
3.3.2 Boundary-Value Problems with Linear Dielectrics

Boundary conditions involving two media of different dielectric properties are expressed in terms of the following potential relations

$$\Phi_{\text{below}} = \Phi_{\text{above}} \quad \text{and} \quad \epsilon_{\text{above}} \frac{\partial \Phi_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial \Phi_{\text{below}}}{\partial n} = -\sigma_f, \quad (54)$$

where $\partial/\partial n$ denotes the component of the gradient operator perpendicular to the boundary interface between the two dielectric media and $\sigma_f$ denotes the free-charge surface density at the interface. Consider, for example, the problem of determining the inside and outside potentials associated with a dielectric sphere of radius $R$ and linear susceptibility $\chi$ immersed in an external uniform electric field $E_0 = E_0 \hat{z}$. The azimuthal spherical symmetry of the problem dictates that the inside and outside potentials be expressed as Legendre-polynomial expansions

$$\Phi_{\text{in}}(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell} r^\ell P_\ell(\cos \theta),$$

$$\Phi_{\text{out}}(r, \theta) = -E_0 r \cos \theta + \sum_{\ell=0}^{\infty} B_{\ell} \frac{R^{2\ell+1}}{r^{\ell+1}} P_\ell(\cos \theta),$$

where the contribution $-E_0 r \cos \theta$ in the outside potential ensures that the outside electric field matches the external electric field $E_0$ for $r \gg R$ and the Legendre coefficients $A_{\ell}$ and $B_{\ell}$ will now be chosen to satisfy the boundary conditions (54) in the absence of surface free-charges ($\sigma_f = 0$). The continuity of the electric potentials across the interface implies that

$$B_{\ell} = \begin{cases} A_{\ell} & \text{(for } \ell \neq 1) \\ A_1 + E_0 & \text{(for } \ell = 1) \end{cases}$$

Matching the perpendicular gradients, on the other hand, yields the relations

$$[(2\ell + 1) + \ell \chi] A_{\ell} = \begin{cases} 0 & \text{(for } \ell \neq 1) \\ -3E_0 & \text{(for } \ell = 1) \end{cases}$$

Hence, the inside and outside solutions for the electric potential are

$$\Phi_{\text{in}}(r, \theta) = -\left(\frac{3}{3 + \chi}\right) E_0 r \cos \theta,$$

$$\Phi_{\text{out}}(r, \theta) = -E_0 \left[ r - \left(\frac{\chi}{3 + \chi}\right) \frac{R^3}{r^2} \right] \cos \theta,$$

which implies that the inside electric field

$$E_{\text{in}} = \left(\frac{3}{3 + \chi}\right) E_0$$
is also uniform, while the outside electric field

$$E_{\text{out}} = E_0 + E_0 \left( \frac{\chi}{3 + \chi} \right) \frac{R^3}{r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right),$$

combines the uniform external electric field with the electric field due to an induced electric dipole moment. To see this, we note that the polarization vector $\mathbf{P}$ inside the dielectric sphere is

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E}_{\text{in}} = \epsilon_0 \left( \frac{3\chi}{3 + \chi} \right) \mathbf{E}_0,$$

so that the bound-charge surface density at the surface of the dielectric sphere is

$$\sigma_b = \left( \frac{3\chi}{3 + \chi} \right) \epsilon_0 E_0 \cos \theta.$$

The electric dipole moment associated with the bound-charge density, on the other hand, is

$$\mathbf{p}_b = R^3 \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} \, d\varphi \, \sigma_b \, \hat{r} = \left( \frac{\chi}{3 + \chi} \right) 4\pi R^3 \epsilon_0 E_0 \hat{z},$$

so that the electric field associated with this electric dipole

$$\mathbf{E}_b = \frac{\mathbf{p}_b}{4\pi \epsilon_0 r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right),$$

which yields the identity $\mathbf{E}_b \equiv \mathbf{E}_{\text{out}} - \mathbf{E}_0$ as expected.

As another example of boundary-value problems involving linear dielectrics, we consider the problem of determining the force on a point charge $q$ located at a distance $h$ from a linear dielectric medium of susceptibility $\chi$ (located below the $z = 0$ plane). The presence of the free charge $q$ above the dielectric medium will polarize it thereby producing volume and surface bound-charge densities. Since the volume free-charge density $\rho_f$ inside the dielectric medium is zero, the volume bound-charge density $\rho_b$ is also zero. The surface bound-charge density, however, is defined as

$$\sigma_b = \mathbf{P} \cdot \hat{z} = P_z = \epsilon_0 \chi E_z^{(-)},$$

where $E_z^{(-)}$ is the $z$-component of the total electric field just inside the surface of the linear dielectric medium at $z = 0$. This field is the sum of the field produced by the free charge $q$ and the field produced by the surface charge distribution $\sigma_b$:

$$E_z^{(-)} = -\frac{q h}{4\pi \epsilon_0 (r^2 + h^2)^{3/2}} - \frac{\sigma_b}{2 \epsilon_0} = \frac{\sigma_b}{\epsilon_0 \chi},$$

from which we solve for $\sigma_b$:

$$\sigma_b = -\frac{1}{2\pi} \left( \frac{\chi}{\chi + 2} \right) \frac{q h}{(r^2 + h^2)^{3/2}}.$$
This result implies that the total bound charge is
\[ q_b = \int_0^\infty \sigma_b 2\pi r \, dr = -\left( \frac{\chi}{\chi + 2} \right) \frac{q}{\int_0^\infty \frac{rh \, dr}{(r^2 + h^2)^2}} = -\left( \frac{\chi}{\chi + 2} \right) q. \]

The electric field evaluated at the field point \( r = h \hat{z} \) due to the surface bound-charge density \( \sigma_b \) is
\[
E = \frac{1}{4\pi\epsilon_0} \int_0^\infty r' \, dr' \int_0^{2\pi} d\theta' \left[ -\frac{1}{2\pi} \left( \frac{\chi}{\chi + 2} \right) \frac{qh}{(r'^2 + h^2)^2} \right] \left( \frac{r - r'}{|r - r'|^3} \right)
\]
\[
= -\frac{qh^2 \hat{z}}{4\pi\epsilon_0} \left( \frac{\chi}{\chi + 2} \right) \int_0^\infty \frac{r' \, dr'}{(r'^2 + h^2)^2} = -\frac{q \hat{z}}{4\pi\epsilon_0 (2h)^2} \left( \frac{\chi}{\chi + 2} \right),
\]
hence, the electric field can be written in terms of the net bound charge \( q_b < 0 \) as
\[
E = \frac{q_b \hat{z}}{4\pi\epsilon_0 (2h)^2},
\]
as if \( q_b \) was located at a distance \( h \) below the surface of the dielectric medium (at \( z = 0 \)). The net force on the free charge \( q \) is, therefore, attractive and its magnitude is
\[
|F| = q |E| = \frac{q^2}{16\pi\epsilon_0 h^2} \left( \frac{\chi}{\chi + 2} \right).
\]

### 3.3.3 Energy in Dielectric Systems

The energy associated with a linear dielectric medium is
\[
W = \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} \, d\tau.
\]
To prove this result, we are reminded that the energy of the dielectric system should be equal to the work done in assembling the dielectric system. We begin by considering a fixed dielectric medium in the absence of free charges. Next, we introduce an incremental free charge \( \Delta \rho_f \) into the dielectric medium, which immediately polarizes the medium and introduces bound-charges. An electric potential \( \Phi \) is created and, therefore, the incremental energy is
\[
\Delta W = \int_V \Delta \rho_f \Phi \, d\tau = \int_V (\nabla \cdot \Delta \mathbf{D}) \Phi \, d\tau = \int_V \Delta \mathbf{D} \cdot \mathbf{E} \, d\tau,
\]
where we have used the definition \( \nabla \cdot \Delta \mathbf{D} = \Delta \rho_f \) and applied the Divergence Theorem for an infinite volume. Lastly, since \( \mathbf{D} = \epsilon \mathbf{E} \) in a linear dielectric medium, we find
\[
\Delta \mathbf{D} \cdot \mathbf{E} = \epsilon \Delta \mathbf{E} \cdot \mathbf{E} = \frac{\epsilon}{2} \Delta (\mathbf{E})^2 = \Delta \left( \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \right),
\]
and thus
\[
\Delta W = \Delta \left( \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} \, d\tau \right) \rightarrow W = \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} \, d\tau.
\]
4 Magnetostatics I

4.1 Magnetic Fields

Magnetic field lines connect points of opposite polarities. Magnetic field lines are either directed inward toward a south (magnetic) pole or directed outward away from a north (magnetic) pole. For example, the Figure below shows the magnetic field around a bar magnet where the south (S) and north (N) poles are at opposite ends of the bar.

In perfect analogy with the attraction or repulsion of electric charges, it is an experimental fact that magnetic poles of opposite polarities attract each other while poles of like polarities repel each other. Note that the strength of the magnetic field increases in the vicinity of a magnetic pole (as is seen by the increase in the density of lines near the north and south poles of a bar magnet).

4.2 Lorentz Force Law

When a charged particle of mass $m$ and charge $q$ is traveling in a magnetic field $B$ with velocity $v$, it experiences a force

$$F = qv \times B,$$

known as the Lorentz Force Law. The Lorentz force on a charged particle is, therefore, zero if the particle is either at rest ($v = 0$) or moving along the magnetic field ($v \times B = 0$). If the charged particle is moving with a velocity $v$ perpendicular to $B$ (i.e., $v \cdot B = 0$),
it moves about a magnetic field line (assuming that the field is uniform) along a circular path of radius $R$. Note that the SI unit for the magnetic field is the Tesla (abbreviated T) defined as

$$[B] = \frac{[F]}{[q][v]} = \frac{N}{C \cdot m/s} = T.$$  

Consider, for example, a charged particle ($q > 0$) moving in the $(x, y)$-plane in a uniform magnetic field $\mathbf{B} = B \hat{z}$ (see Figure below). Here, the magnetic force $\mathbf{F} = q \mathbf{v} \times \mathbf{B}$ provides the necessary centripetal force needed to keep the particle on its circular path.

Let $\mathbf{R} = R (\cos \theta \hat{x} - \sin \theta \hat{y})$ denote the instantaneous position of the particle after it has moved an angular displacement $\theta$ (as measured clockwise from the $x$-axis). The velocity is now calculated as

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = -R \Omega (\sin \theta \hat{x} + \cos \theta \hat{y}) = \Omega \mathbf{R} \times \hat{z},$$

where $\Omega = d\theta/dt$ is the gyration frequency (or gyrofrequency). From the acceleration $\mathbf{a} = \mathbf{F}/m$, we find

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \Omega \mathbf{v} \times \hat{z} = \frac{q}{m} \mathbf{v} \times \mathbf{B},$$

so that the gyrofrequency is

$$\Omega = \frac{qB}{m}$$

and the radius of gyration (or gyroradius) is

$$R = \frac{v}{\Omega} = \frac{mv}{qB}.$$
Note that the Lorentz force cannot be involved in doing work since it cannot be used to increase the kinetic energy of the particle:

$$\frac{dK}{dt} = \mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{v} \cdot (q \mathbf{v} \times \mathbf{B}) = 0.$$  

When a charged particle is moving in a constant electric field $\mathbf{E} = E \hat{x}$ perpendicular to a constant magnetic field $\mathbf{B} = B \hat{z}$, the components of the force equation are written as

$$\ddot{x} = \Omega (v_E + \dot{y}), \quad \ddot{y} = -\Omega \dot{x}, \quad \ddot{z} = 0,$$

where $v_E = E/B$ is a quantity with units of velocity. One readily sees that the motion along the $z$-axis (i.e., parallel to the magnetic field) is completely decoupled from the motion on the $(x,y)$-plane. Hence, if we set $\dot{z} = 0$, the force equations described the coupled motion of the charged particle, we can integrate Eqs. (55) to obtain

$$\dot{x} = \dot{x}_0 + \Omega (v_E t + y - y_0) \quad \text{and} \quad \dot{y} = \dot{y}_0 - \Omega (x - x_0),$$

where $\mathbf{v}_0 = (\dot{x}_0, \dot{y}_0)$ denotes the initial velocity. Substituting this solution back into Eqs. (55), we find the uncoupled equations

$$\ddot{x} + \Omega^2 x = \Omega (v_E + \dot{y}_0 + \Omega x_0),$$
$$\ddot{y} + \Omega^2 y = -\Omega (\dot{x}_0 + \Omega v_E t - \Omega y_0).$$

The solution to these equations, subject to the initial conditions $(x_0, y_0)$ and $(\dot{x}_0, \dot{y}_0)$, is given as

$$x(t) = x_0 + \frac{1}{\Omega} \left[ (v_E + \dot{y}_0) (1 - \cos \Omega t) + \dot{x}_0 \sin \Omega t \right],$$
$$y(t) = y_0 + \frac{1}{\Omega} \left[ \dot{x}_0 (\cos \Omega t - 1) + \dot{y}_0 \sin \Omega t + v_E (\sin \Omega t - \Omega t) \right].$$

For example, if the particle starts at $(x_0, y_0) = (0, 0)$ and $(\dot{x}_0, \dot{y}_0) = (0, 0)$, we find the parametric representation of a cycloid curve (see Figure below):

$$x(t) = R_E (1 - \cos \Omega t) \quad \text{and} \quad y(t) = R_E (\sin \Omega t - \Omega t),$$

where $2R_E = 2v_E/\Omega$ denotes the maximum extension along the $x$-axis (even though the electric field is directed along the $x$-axis), which the particle reaches at $t = (2n + 1)\pi/\Omega$ (for $n = 0, 1, \ldots$) while at $t = 2n\pi/\Omega$ (for $n = 1, \ldots$), the particle returns to $x = 0$ after having traveled a distance $2n\pi R_E$ along the negative $y$-axis.

### 4.3 Currents

When a small electric-charge linear density $\lambda$ travels with velocity $\mathbf{v}$ along a curve $\Gamma$ (see Figure below), we say that electric current $\mathbf{I} = \lambda \mathbf{v}$ is flowing along curve $\Gamma$. 

Figure 24: Cycloid orbit of a charged particle

Figure 25: Electric current
The unit of electric current (or current) is \([I] = (C/m) \cdot (m/s) = C/s = A\), known as Ampere. Note that the infinitesimal force \(d\mathbf{F}\) on an infinitesimal charge \(dq\) traveling with velocity \(\mathbf{v}\) in a magnetic field \(\mathbf{B}\) is given by the Lorentz Force Law \(d\mathbf{F} = dq \mathbf{v} \times \mathbf{B}\) so that the net force on the current-carrying curve \(\Gamma\) is

\[
\mathbf{F} = \int_{\Gamma} (\mathbf{v} \times \mathbf{B}) \lambda \, d\ell = I \int_{\Gamma} d\mathbf{l} \times \mathbf{B},
\]

where \(I\) denotes the magnitude of the electric current assumed, here, to be constant along curve \(\Gamma\).

When currents flow on two-dimensional surfaces or in three-dimensional volumes, we define the surface current density \(\mathbf{K}\) and volume current density \(\mathbf{J}\) as

\[
\mathbf{K} = \sigma \mathbf{v} \quad \text{and} \quad \mathbf{J} = \rho \mathbf{v},
\]

where \(\sigma = dq/da\) and \(\rho = dq/d\tau\) are the surface and volume charge densities, respectively. Hence, the magnetic force for surface and volume current distributions are

\[
\mathbf{F} = \int_{S} \mathbf{K} \times \mathbf{B} \, da \quad \text{or} \quad \int_{V} \mathbf{J} \times \mathbf{B} \, d\tau.
\]

Note that the total current \(I\) flowing across a surface \(S\) is

\[
I = \int_{S} \mathbf{J} \cdot da.
\]

In particular, the rate at which charge leaves a volume \(V\) is given as

\[
\oint_{\partial V} \mathbf{J} \cdot da = \int_{V} \nabla \cdot \mathbf{J} \, d\tau,
\]

upon making use of the Divergence Theorem. This charge loss can be expressed as

\[
\frac{dq}{dt} = - \frac{d}{dt} \left( \int_{V} \rho \, d\tau \right) = - \int_{V} \frac{\partial \rho}{\partial t} \, d\tau,
\]

and by applying the Conservation Law of Electric Charge, we find

\[
0 = \int_{V} \left( \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) \, d\tau \rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,
\]

which is called the **continuity** equation for charge flow (in analogy with the continuity equation for mass flow).
4.4 Biot-Savart Law for Steady Currents

When a steady current flows in a wire, its magnitude $I$ is a constant (otherwise charge would pile up somewhere) and, since $\partial \rho / \partial t = 0$, the charge continuity equation implies that $\nabla \cdot \mathbf{J} = 0$. In this case, the infinitesimal magnetic field $d\mathbf{B}$ at a field point $\mathbf{r}$ due to an infinitesimal line current $I \, dl'$ at a source point $\mathbf{r}'$ is given by the Biot-Savart Law

$$ d\mathbf{B}(\mathbf{r}, \mathbf{r}') = \frac{\mu_0 I}{4\pi} \, dl' \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right), $$

where $\mu_0 = 4\pi \times 10^{-7} \, \text{T} \cdot \text{m} \cdot \text{A}^{-1}$ is the permeability of free space. The net magnetic field generated by a line current distribution along a path $\Gamma$ (i.e., current $I$ flowing through a wire) is, therefore, given as

$$ \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\Gamma} \, dl' \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right), \quad (58) $$

where the infinitesimal vector $dl'$ is tangent to the curve $\Gamma$. When currents flow on a surface $S$ or in a volume $V$, the magnetic field is now calculated as

$$ \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{S} \mathbf{K}(\mathbf{r}', \mathbf{r}) \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \, d\mathbf{a}' \quad \text{or} \quad \frac{\mu_0 I}{4\pi} \int_{V} \mathbf{J}(\mathbf{r}') \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \, d\mathbf{r}', $$

respectively.
5 Magnetostatics II – Ampère’s Law

5.1 Divergence and Curl of Magnetic Fields

We begin with the magnetic field due to a time-independent volume current distribution \( J(r) \) (i.e., \( \nabla \cdot J = 0 \)):

\[
B(r) = \frac{\mu_0}{4\pi} \int_V J(r') \times \left( \frac{r - r'}{|r - r'|^3} \right) d\tau'.
\]

Using the identity

\[
\left( \frac{r - r'}{|r - r'|^3} \right) = -\nabla \left( \frac{1}{|r - r'|} \right),
\]

we find

\[
B(r) = \nabla \times \left( \frac{\mu_0}{4\pi} \int_V \frac{J(r') d\tau'}{|r - r'|} \right) = \nabla \times A(r), \tag{59}
\]

where

\[
A(r) = \frac{\mu_0}{4\pi} \int_V \frac{J(r') d\tau'}{|r - r'|} \tag{60}
\]

is known as the magnetic vector potential. From this expression, the divergence of the magnetic field yields

\[
\nabla \cdot B(r) = \nabla \cdot \nabla \times A(r) = 0.
\]

An application of the Divergence Theorem

\[
\int_V \nabla \cdot B \, d\tau = \oint_{\partial V} B \cdot d\mathbf{a} = 0,
\]

therefore, shows that the net magnetic flux leaving a closed surface is zero (i.e., the number of field lines entering the volume is equal to the number of field lines leaving it).

The curl of the magnetic field

\[
\nabla \times B = \nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A
\]

is divided into two parts. The first part \( \nabla (\nabla \cdot A) \) is calculated from Eq. (60) by, first, computing \( \nabla \cdot A \):

\[
\nabla \cdot A = \frac{\mu_0}{4\pi} \int_V J(r') \cdot \nabla \left( \frac{1}{|r - r'|} \right) d\tau' = -\frac{\mu_0}{4\pi} \int_V J(r') \cdot \nabla' \left( \frac{1}{|r - r'|} \right) d\tau'
\]

\[
= \frac{\mu_0}{4\pi} \left( \int_V \nabla' \cdot J(r') \frac{d\tau'}{|r - r'|} - \oint_{\partial V} \frac{J \cdot d\mathbf{a}'}{|r - r'|} \right).
\]

By using the time-independent charge conservation law \( \nabla \cdot J = 0 \), the first term in \( \nabla \cdot A \) vanishes and, assuming the the current density \( J \) vanishes on the surface \( \partial V \), the second term also vanishes. Hence, the term \( \nabla (\nabla \cdot A) \) vanishes.
The second part \(-\nabla^2 \mathbf{A}\) is calculated as
\[
-\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \int_V \mathbf{J}(\mathbf{r}') \nabla^2 \left( \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \right) d\tau' = \mu_0 \int_V \mathbf{J}(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}') d\tau' = \mu_0 \mathbf{J}(\mathbf{r}),
\]
where we have made use of \(\nabla^2 (|\mathbf{r} - \mathbf{r}'|^{-1}) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}')\), and, therefore, the curl of \(\mathbf{B}\) is
\[
\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}),
\]
which is consistent with the time-independent charge conservation law \(\nabla \cdot \mathbf{J} = 0\).

### 5.2 Ampère’s Law

Eq. (61) is called the differential form of Ampère’s Law. The integral form of Ampère’s Law makes use of Stokes’s Theorem:
\[
\int_S \nabla \times \mathbf{B} \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_S,
\]
where \(I_S\) is the current flowing through the surface in the positive direction.

Note that, in the absence of volume currents (\(\mathbf{J} = 0\)), Ampère’s Law becomes \(\nabla \times \mathbf{B} = 0\) and, thus, the magnetic field can be written in terms of a magnetic scalar potential \(\mathbf{B} = -\nabla \Phi_M\). On the other hand, the divergence condition \(\nabla \cdot \mathbf{B} = 0\) implies that \(\nabla^2 \Phi_M = 0\), i.e., the magnetic scalar potential must satisfy Laplace’s equation.
6 Magnetostatics III – Magnetic Vector Potential

6.1 Vector Potential

The magnetic field $B$ was written previously as $B(r) = \nabla \times A(r)$, where the magnetic vector potential is defined in terms of the current density $J(r)$ as

$$A(r) = \frac{\mu_0}{4\pi} \int_V \frac{J(r') \, d\tau'}{|r - r'|}.$$ 

From this definition, Ampère’s Law becomes

$$\nabla^2 A(r) = -\mu_0 J(r),$$

whose solution is given by Eq. (60). Note that, like the magnetic field $B$, the divergence of the magnetic vector potential vanishes: $\nabla \cdot A = 0$.

6.2 Example I: Current Loop

As a first application of the vector-potential formalism, we consider the case of the vector potential due to a single-turn current loop of radius $R$ carrying current $I$ (see Figure below).

The current density in this case is expressed in terms of the source point

$$r' = R \left( \cos \varphi' \hat{x} + \sin \varphi' \hat{y} \right)$$
as
\[ \mathbf{J}(r') = \frac{I}{R} \delta(\cos \theta') \delta(r' - R) \left( -\sin \varphi' \hat{x} + \cos \varphi' \hat{y} \right), \]
and, by azimuthal symmetry, the vector potential has to be independent of the azimuthal angle \( \varphi \) so that we may place the field point on the \((x,z)\)-plane:
\[ r = r \left( \sin \theta \hat{x} + \cos \theta \hat{z} \right). \]
Hence, we find
\[ |r - r'| = \sqrt{r'^2 + R^2 - 2rR \sin \theta \cos \varphi'}, \]
and Eq. (60) becomes
\[ \mathbf{A}(r, \theta) = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{(-\sin \varphi' \hat{x} + \cos \varphi' \hat{y}) \, d\varphi'}{\sqrt{r'^2 + R^2 - 2rR \sin \theta \cos \varphi'}}. \]
Since the integration is symmetric about \( \varphi' = 0 \), only the \( y \)-component of the vector potential \( \mathbf{A} \) remains, which represents the component in the \( \varphi \)-direction, i.e., \( \mathbf{A}(r, \theta) = A_\varphi(r, \theta) \hat{\varphi} \), where
\[ A_\varphi(r, \theta) = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{\cos \varphi' \, d\varphi'}{\sqrt{r'^2 + R^2 - 2rR \sin \theta \cos \varphi'}}. \tag{63} \]
Although we are not interested in evaluating this integral explicitly, we can look at two interesting limits: \( r \gg R \) and \( r \ll R \). In both limits (with \( r_\leq/r_\geq = R/r \) or \( r/R \)), we use the expansion
\[ \left( r^2 + R^2 - 2rR \sin \theta \cos \varphi' \right)^{-1/2} = \frac{1}{r_\geq} \left( 1 - \frac{1}{2} \frac{r_\leq^2}{r_\geq^2} + \cdots \right), \]
to find
\[ A_\varphi(r, \theta) \approx \frac{\mu_0 I}{4} \left( \frac{R r_\leq}{r_\geq^2} \right) \sin \theta. \tag{64} \]
The vector potential in the far region \((r \gg R)\) is, therefore, given by the approximate expression
\[ A_\varphi(r, \theta) \approx \frac{\mu_0 I \pi R^2}{4\pi} \frac{\sin \theta}{r^2}, \quad \text{for} \quad r \gg R, \tag{65} \]
while in the near region \((r \ll R)\), we find
\[ A_\varphi(r, \theta) \approx \frac{\mu_0 I}{4} \left( \frac{r}{R} \right) \sin \theta, \quad \text{for} \quad r \ll R. \tag{66} \]
In both cases, we easily check that $\nabla \cdot \mathbf{A} = 0$, since $A_{\psi}$ is independent of the azimuthal angle $\psi$. We now obtain approximate expressions for the magnetic field from the vector potential $\mathbf{A} = A_{\psi} \hat{\phi}$:

$$\mathbf{B} = \hat{r} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_{\varphi}) - \hat{\theta} \frac{1}{r} \frac{\partial}{\partial r} (r A_{\varphi}).$$

In the far region, for example, the magnetic field is

$$\mathbf{B} \simeq \frac{\mu_0 |\mathbf{m}|}{4\pi r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) = \frac{\mu_0}{4\pi r^3} \left[ 3 (\mathbf{m} \cdot \hat{r}) \hat{r} - \mathbf{m} \right], \quad (67)$$

where $\mathbf{m} = I \mathbf{a} = (I \pi R^2) \hat{z}$ denotes the magnetic dipole moment (here, the direction of the area vector $\mathbf{a}$ obeys the right-hand rule).

### 6.3 Example II: Spinning Charged Sphere

Next, we consider the case of a spinning charged sphere of radius $R$ and carrying a uniform surface charge density $\sigma$ spinning with constant angular velocity $\omega$. To set up the problem, we assume that the field point is on the $z$-axis: $\mathbf{r} = r \hat{z}$ and that the rotation axis is on the $(x, z)$-plane and makes an angle $\psi$ with respect to the $z$-axis: $\mathbf{\omega} = \omega (\sin \psi \hat{x} + \cos \psi \hat{z})$ (see Figure below).

An arbitrary point $\mathbf{r}'$ on the surface of the sphere is moving with tangential velocity

$$\mathbf{v} = \mathbf{\omega} \times \mathbf{r}' = \omega R \left[ \sin \psi (\sin \theta' \sin \varphi' \hat{z} - \cos \theta' \hat{y}) + \cos \psi \sin \theta' (\cos \varphi' \hat{y} - \sin \varphi' \hat{x}) \right],$$
and, therefore, the surface current density is $K = \sigma \mathbf{v}$. With this surface current density and 

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r'^2 + R^2 - 2rR \cos \theta'},$$

the vector potential is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_0^\pi R^2 \sin \theta' \, d\theta' \int_0^{2\pi} \frac{\sigma \mathbf{v}(\theta', \varphi') \, d\varphi'}{\sqrt{r'^2 + R^2 - 2rR \cos \theta'}}$$

$$= -\hat{y} \left[ \frac{\mu_0}{2} \sigma \omega R^3 \sin \psi \int_0^\pi \frac{\sin \theta' \cos \theta' \, d\theta'}{\sqrt{r'^2 + R^2 - 2rR \cos \theta'}} \right].$$

Note that the vector potential is in the direction of $\mathbf{\omega} \times \mathbf{r} = -\omega r \sin \psi \hat{y}$ and that the integral can be written as

$$\int_0^\pi \frac{\sin \theta' \cos \theta' \, d\theta'}{\sqrt{r'^2 + R^2 - 2rR \cos \theta'}} = \frac{1}{\sqrt{rR}} \int_1^{-1} \frac{u \, du}{\sqrt{\eta + \eta^{-1} - 2u}} = \frac{2}{3} \left( rR \eta^3 \right)^{-1/2},$$

where $\eta = r_>/r_<$. Hence, the vector potential inside and outside the sphere is

$$\mathbf{A} = \frac{\mu_0}{3} \sigma R (\mathbf{\omega} \times \mathbf{r}) \begin{cases} 1 & \text{(inside sphere)} \\ R^3/r^3 & \text{(outside sphere)} \end{cases}$$

as viewed in the frame in which the field point is on the $z$-axis and the angular velocity $\mathbf{\omega}$ is in the $(x, z)$-plane. In the frame in which $\mathbf{\omega}$ is directed along the $z$-axis and the field position is arbitrary, on the other hand, we find $\mathbf{\omega} \times \mathbf{r} = \omega r \sin \theta \hat{\varphi}$ and, thus, the vector potential $\mathbf{A} = A_\varphi(r, \theta) \hat{\varphi}$ has a single azimuthal component given by

$$A_\varphi(r, \theta) = \frac{\mu_0}{3} \sigma \omega R^2 \begin{cases} (r/R) \sin \theta & \text{(inside sphere)} \\ (R/r)^2 \sin \theta & \text{(outside sphere)} \end{cases}$$

The magnetic field in this case is expressed in terms of the vector potential as

$$\mathbf{B} = \frac{\hat{r}}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\varphi) - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} (r A_\varphi),$$

and, thus, the magnetic field inside the spinning charged sphere is

$$\mathbf{B}_{in} = \frac{2\mu_0}{3} \sigma R \mathbf{\omega},$$

i.e., it is constant. The magnetic field outside the spinning charged sphere, on the other hand, is

$$\mathbf{B}_{out} = \frac{\mu_0}{4\pi} \frac{|\mathbf{m}|}{r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right),$$

where the magnitude of the magnetic dipole moment for the spinning charged sphere is

$$|\mathbf{m}| = \sigma \cdot \omega R \cdot \frac{4\pi}{3} R^3.$$
6.4 Magnetostatic Boundary Conditions

From the divergenceless condition \( \nabla \cdot \mathbf{B} = 0 \) for the magnetic field, we find that the perpendicular component of the magnetic field is continuous across a surface carrying current density \( \mathbf{K} \), i.e.,

\[
\hat{n} \cdot (\mathbf{B}_+ - \mathbf{B}_-) = 0,
\]

where \( \hat{n} \) is the unit vector perpendicular to the surface (with \( \hat{n} \cdot \mathbf{K} = 0 \)). Ampère’s Law applied to an infinitesimally thin volume enclosing the surface, on the other hand, implies that the parallel components of the magnetic field and discontinuous across the surface:

\[
\hat{n} \times (\mathbf{B}_+ - \mathbf{B}_-) = \mu_0 \mathbf{K}.
\]

Combining these two results, we find

\[
\mathbf{B}_+ - \mathbf{B}_- = \mu_0 (\mathbf{K} \times \hat{n}).
\]

The vector potential, on the other hand, satisfies the boundary conditions

\[
\mathbf{A}_+ = \mathbf{A}_- \quad \text{and} \quad \frac{\partial \mathbf{A}_+}{\partial n} - \frac{\partial \mathbf{A}_-}{\partial n} = -\mu_0 \mathbf{K}.
\]

7 Magnetization I

7.1 Force and Torque on a Magnetic Dipole

A magnetic dipole \( \mathbf{m} \) experiences a torque when exposed to an external magnetic field. We show this by looking, first, at the torque on a rectangular current loop (of sides \( a \) and \( b \)) in a uniform magnetic field \( \mathbf{B} \). Assuming that the magnetic field is directed along the \( z \)-axis, \( \mathbf{B} = B\hat{z} \), and the current is oriented in such a way that its magnetic dipole moment is

\[
\mathbf{m} = I ab \left( \sin \theta \, \hat{y} + \cos \theta \, \hat{z} \right),
\]

where the rectangular loop carries current \( I \) as shown in the Figure below.

The forces on the four segments \((i), (ii), (iii), \) and \((iv)\) are, respectively, given as

\[
\begin{align*}
\mathbf{F}_b^+ &= Ib B \cos \theta \, \hat{x}, \\
\mathbf{F}_a^+ &= Ia B \hat{y}, \\
\mathbf{F}_b^- &= -Ib B \cos \theta \, \hat{x}, \\
\mathbf{F}_a^- &= -Ia B \hat{y},
\end{align*}
\]

and are shown in the Figure above. Note that, whereas the forces \( \mathbf{F}_b^+ \) do not generate torque on the rectangular loop since they are planar forces (i.e., \( \mathbf{F}_b^+ \cdot \mathbf{m} = 0 \)), the forces
$\mathbf{F}_{a}^{\pm}$ are nonplanar forces and, therefore, generate torque:

$$\mathbf{N} = \mathbf{r}_{a}^{\pm} \times \mathbf{F}_{a}^{\pm} + \mathbf{r}_{a}^{-} \times \mathbf{F}_{a}^{-} = Ia B \left[ \frac{b}{2} (\cos \theta \hat{y} - \sin \theta \hat{z}) \times \hat{y} + \frac{b}{2} (-\cos \theta \hat{y} + \sin \theta \hat{z}) \times (-\hat{y}) \right] = I (a b) \sin \theta \hat{x} = \mathbf{m} \times \mathbf{B}.$$ 

Although it is clear that the net force on the current loop is zero in a uniform magnetic field, it is expressed as

$$\mathbf{F}_{\text{net}} = -\nabla U_B = \nabla (\mathbf{m} \cdot \mathbf{B}),$$

where

$$U_B = -\mathbf{m} \cdot \mathbf{B}$$

denotes the potential energy of a magnetic dipole in an external magnetic field, in analogy with the potential energy $U_E = -\mathbf{p} \cdot \mathbf{E}$ of an electric dipole in an electric field. Hence, we note that a magnetic dipole $\mathbf{m}$ exposed to a magnetic field $\mathbf{B}$ experiences a torque $\mathbf{N} = \mathbf{m} \times \mathbf{B}$, which causes it to align itself with the magnetic field in order to minimize its magnetic potential energy $U_B = -\mathbf{m} \cdot \mathbf{B}$.

### 7.2 Effect of a Magnetic Field on Atomic Orbits

In the simplest classical model for an electron orbit (with charge $-e$) about a positively-charged nucleus (with charge $Ze$), the electron undergoes uniform circular motion with
radius \( r_e \) and tangential velocity \( v_e = \omega_e r_e \) determined by the centripetal force

\[
\frac{Z e^2}{4\pi\varepsilon_0 r_e^2} = \frac{m_e v_e^2}{r_e} \rightarrow v_e = \omega_e r_e,
\]

(68)

generated by the electrostatic attractive force between the electron and the nucleus, where

\[
\omega_e = \sqrt{\frac{Z e^2}{4\pi\varepsilon_0 r_e^3}}.
\]

Assuming that the circular motion of the electron takes place on the \((x,y)\)-plane, the electron orbit can be interpreted as a small magnetic dipole

\[
\mathbf{m} = I_e \pi r_e^2 \hat{z} = \left( -\frac{e\omega_e}{2\pi} \right) \pi r_e^2 \hat{z} = -\frac{e}{2} v_e r_e \hat{z},
\]

(69)

where the negative sign comes from the electron’s negative charge. Note that if we make use of the expression for the orbital angular momentum

\[
\mathbf{L} = r_e \times m_e v_e = m_e v_e r_e \hat{z}
\]

for the electron’s circular orbit, we may write the magnetic dipole moment as

\[
\mathbf{m} = -\frac{e}{2 m_e} \mathbf{L}.
\]

The introduction of an external magnetic field \( \mathbf{B} = B \hat{z} \) changes the centripetal-force equation (68) to become

\[
\frac{Z e^2}{4\pi\varepsilon_0 r_e^2} + e \mathbf{v}_e \mathbf{B} = \frac{m_e v_e^2}{r_e} \rightarrow e \mathbf{v}_e \mathbf{B} = \frac{m_e}{r_e} \left( \mathbf{v}_e^2 - v_e^2 \right),
\]

(70)

where we assume that the electron speeds up \((\mathbf{v}_e > v_e)\) but remains on the same orbit \((r_e = r_e)\). Assuming that \( |\Delta v_e| = |\mathbf{v}_e - v_e| \ll v_e \) is small (i.e., \( \Omega = eB/m_e \ll \omega_e \)), we easily find from Eq. (70)

\[
\Delta v_e = \frac{e r_e B}{2 m_e},
\]

and, thus, the magnetic dipole moment (69) changes by an amount

\[
\Delta \mathbf{m} = -\frac{e}{2} \Delta v_e r_e \hat{z} = -\frac{e^2 r_e^2}{4 m_e} \mathbf{B}.
\]

Note that this change is equivalent to a change in orbital angular momentum

\[
\Delta \mathbf{L} = r_e \times (-e \mathbf{A}) = \frac{e}{2} r_e^2 \mathbf{B} \rightarrow \Delta \mathbf{m} = -\frac{e}{2 m_e} \Delta \mathbf{L},
\]

where \( \mathbf{A} = \frac{1}{2} r_e \times \mathbf{B} \). Here, we note that the change in magnetic dipole moment is opposite to the direction of \( \mathbf{B} \) – this is the source of diamagnetism.\(^4\)

\(^4\)The classical discussion presented here must be replaced by a fully quantum-mechanical discussion of diamagnetism.
In the presence of a magnetic field $B$, thus, matter becomes magnetized, i.e., the state of magnetic polarization of matter is represented by a finite magnetic dipole moment per unit volume known as magnetization and denoted as $M$. Hence, we see that the magnetization vector $M$ of diamagnetic materials is aligned anti-parallel to the magnetic field $B$.

While diamagnetism is a universal phenomenon, however, it is typically weaker than paramagnetism, in which paramagnetic material acquire a magnetization vector $M$ parallel to $B$.

As can be seen in the Figure below, most atoms are slightly diamagnetic, some atoms are paramagnetic (with magnetization properties far stronger than diamagnetic atoms), and some atoms (the triumvirate group of Iron, Nickel, and Cobalt) are ferromagnetic.

In particular, note the relationship between the presence of $d$ – orbital electrons and paramagnetism and ferromagnetism. To investigate further this relationship, we write down the valence-electron configuration for Scandium through Manganese:

- Scandium (Sc) $3d^4s^2$
- Titanium (Ti) $3d^24s^2$
- Vanadium (V) $3d^34s^2$
- Chromium (Cr) $3d^54s$
- Manganese (Mn) $3d^54s^2$

and note that paramagnetic properties of these atoms is strongest when the number of...
valence electrons is odd.

Much of paramagnetism can be explained classically by the fact that the torque experienced by a magnetic dipole \( \mathbf{m} \) in an external field \( \mathbf{B} \) forces the magnetic dipole to line up with the magnetic field (so that its potential energy reaches a minimum value \( U_B = -\mathbf{m} \cdot \mathbf{B} < 0 \)).

### 7.4 Field of a Magnetized Object

Since the magnetic vector potential \( \mathbf{A}(\mathbf{r}) \) of a single magnetic dipole \( \mathbf{m} \) located at the source point \( \mathbf{r}' \) is expressed as

\[
\mathbf{A}(\mathbf{r}; \mathbf{r}') = \frac{\mu_0}{4\pi} \mathbf{m} \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right),
\]

the vector potential associated with a magnetized object is expressed as

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{M}(\mathbf{r}') \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\tau'.
\] (71)

Using the identity

\[
\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right),
\]

we obtain after integration by parts

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{M}(\mathbf{r}') \times \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d\tau' \\
= \frac{\mu_0}{4\pi} \left\{ \int_V \left[ \nabla' \times \mathbf{M}(\mathbf{r}') \right] d\tau' + \oint_{\partial V} \frac{\mathbf{M}(\mathbf{r}') \times d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} \right\}.
\]

Hence, by defining the volume bound current density \( \mathbf{J}_b \) and the surface bound current density \( \mathbf{K}_b \) as

\[
\mathbf{J}_b = \nabla \times \mathbf{M} \quad \text{and} \quad \mathbf{K}_b = \mathbf{M} \times \mathbf{n},
\] (72)

the magnetic vector potential of a magnetized object is expressed in terms of volume and surface bound currents as

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{J}_b(\mathbf{r}') d\tau' + \frac{\mu_0}{4\pi} \oint_{\partial V} \frac{\mathbf{K}_b(\mathbf{r}') d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|}.
\] (73)

Notice the striking similarity with the electric case:

\[
\rho_b = \nabla \cdot \mathbf{P} \quad \leftrightarrow \quad \mathbf{J}_b = \nabla \times \mathbf{M}
\]

\[
\sigma_b = \mathbf{P} \cdot \mathbf{n} \quad \leftrightarrow \quad \mathbf{K}_b = \mathbf{M} \times \mathbf{n}
\]
and that the volume bound current $J_b$ is explicitly divergenceless: $\nabla \cdot J_b = \nabla \cdot \nabla \times M = 0$.

For example, the magnetic field of a uniformly magnetized sphere (with radius $R$ and magnetization $M = M \hat{z}$) is calculated from the bound currents

$$J_b = \nabla \times M = 0 \quad \text{and} \quad K_b = M \times \hat{n} = M \sin \theta \hat{\varphi}.$$  

Noting that this surface current density is analogous to the case of a rotating charged sphere, with $\sigma R \omega \rightarrow M$, we find the vector potential

$$A(r) = \frac{\mu_0}{3} M \times r \begin{cases} 1 & \text{(r < R inside sphere)} \\ (R/r)^3 & \text{(r > R outside sphere)} \end{cases}$$

and the magnetic field inside the magnetized sphere

$$B_{in} = \nabla \times A_{in} = \frac{2\mu_0}{3} M$$

is constant, while the magnetic field outside the sphere is that of a pure magnetic dipole.

## 8 Magnetization II – Linear and Nonlinear Magnetic Media

### 8.1 Auxiliary Field $H$

We write the total current density flowing through matter as

$$J = J_f + J_h,$$

where $J_f$ denotes the free-current density and $J_h = \nabla \times M$ denotes the bound-current density expressed in terms of the curl of the magnetization vector $M$. By substituting this definition into Ampère’s Law $\nabla \times B = \mu_0 J$ and introducing the auxiliary field $H$ defined as

$$H = \frac{B}{\mu_0} - M \quad \text{or} \quad B = \mu_0 (H + M),$$

Ampère’s Law becomes

$$\nabla \times H = J_f \leftrightarrow \oint_{\partial S} H \cdot d\ell = (I_f)_{S},$$

where $(I_f)_{S}$ denotes the total free current flowing through the surface $S$.

For example, we consider again the case of the uniformly magnetized sphere, where the magnetic field inside the sphere is $B_{in} = \frac{2}{3} \mu_0 M$ and, thus, the auxiliary field is

$$H_{in} = \frac{B_{in}}{\mu_0} - M = -\frac{1}{3} M.$$
Since the magnetization vector vanishes outside the magnetized sphere, we also find \( H_{\text{out}} = B_{\text{out}} / \mu_0 \).

If the free-current density is zero everywhere inside a magnetized object with nonuniform magnetization, we find that the auxiliary field \( H \) must be expressed in terms of a magnetic scalar potential: \( H = -\nabla \Phi \), so that the relation \( B = \mu_0 (M - \nabla \Phi) \) implies that

\[
\nabla \cdot B = 0 = \mu_0 \nabla \cdot (M - \nabla \Phi) \rightarrow \nabla^2 \Phi = -\rho_M = \nabla \cdot M, \tag{75}
\]

where \( \rho_M = -\nabla \cdot M \) defines an effective magnetic charge density. Note here that the methods developed in solving the electrostatic Poisson’s equation can be used to solve this magnetostatic Poisson’s equation.

### 8.2 Boundary Conditions

The two equations (74) and (75) are, respectively, associated with the following boundary conditions

\[
\hat{n} \times (H_+ - H_-) = K_f
\]

\[
\hat{n} \cdot (H_+ - H_-) = -\hat{n} \cdot (M_+ - M_-),
\]

where \( \hat{n} \) is the unit vector perpendicular to the surface on which the surface free-current density \( K_f \) flows.

For example, in the case of a uniformly magnetized sphere \( (M = M \hat{z}) \), where \( \rho_M = -\nabla \cdot M = 0 \) and \( H = -\nabla \Phi \rightarrow \nabla^2 \Phi = 0 \), the solution of the magnetostatic Poisson’s equation for the magnetic scalar potential \( \Phi \) is in the form

\[
\Phi_M(r, \theta) = \begin{cases} 
\sum_{\ell} A_\ell r^\ell P_\ell(\cos \theta) & (r < R \text{ inside sphere}) \\
\sum_{\ell} B_\ell r^{-(\ell+1)} P_\ell(\cos \theta) & (r > R \text{ outside sphere})
\end{cases}
\]

Here, the Legendre coefficients \( A_\ell \) and \( B_\ell \) are determined from the boundary conditions at the surface of the sphere \( (r = R) \):

\[
\Phi_M^{(+)}(R, \theta) = \Phi_M^{(-)}(R, \cos \theta) \quad \text{and} \quad \frac{\partial \Phi_M^{(-)}(r, \theta)}{\partial r} \bigg|_{r=R} - \frac{\partial \Phi_M^{(+)}(r, \theta)}{\partial r} \bigg|_{r=R} = M \cos \theta,
\]

where \( M \cdot \hat{n} = M \hat{z} \cdot \hat{r} = M \cos \theta \) and \( \Phi_M^{(+)} \) denote the outside/inside magnetic scalar potentials. The first boundary condition implies that \( B_\ell = A_\ell R^{2\ell+1} \) for all values of \( \ell \) while the second boundary condition yields

\[
M \cos \theta = \sum_\ell P_\ell(\cos \theta) \left[ \ell A_\ell R^{\ell-1} + (\ell + 1) B_\ell R^{-(\ell+2)} \right] = \sum_\ell P_\ell(\cos \theta)(2\ell + 1) A_\ell R^{\ell-1},
\]
where we have substituted the first result in the final expression. The two boundary conditions, therefore, require that \( A_\ell = 0 \) for all \( \ell \neq 1 \) while \( A_1 = \frac{1}{3} M \) so that the magnetic scalar potential for a uniformly magnetized sphere is

\[
\Phi_M(r, \theta) = \begin{cases} 
\frac{M}{3} \cos \theta & (r < R \text{ inside sphere}) \\
R^3/r^2 & (r > R \text{ outside sphere}) 
\end{cases}
\]

The auxiliary field \( \mathbf{H}_{in} = -\nabla \Phi_M \) inside the magnetized sphere is

\[
\mathbf{H}_{in} = -\nabla \left( \frac{M}{3} z \right) = -\frac{1}{3} \mathbf{M},
\]

exactly as calculated previously.

### 8.3 Linear and Nonlinear Media

In a linear magnetic medium, the magnetization vector \( \mathbf{M} \) is linearly proportional to the auxiliary field \( \mathbf{H} \):

\[
\mathbf{M} = \chi_m \mathbf{H},
\]

where the dimensionless constant \( \chi_m \) is known as the magnetic susceptibility; by definition, it is positive for paramagnetic materials and negative for diamagnetic materials. By combining this relation with the definition of the magnetic field \( \mathbf{B} \) in terms of \( \mathbf{M} \) and \( \mathbf{H} \), we find

\[
\mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H},
\]

where \( \mu \) is called the permeability of the linear magnetic medium.

For example, we consider the case of a solenoid (with \( n \) turns per unit length) carrying current \( I \) and surrounding a cylinder with uniform magnetic susceptibility \( \chi_m \) (see Figure below).

Using Ampère’s Law (74), we easily find that the auxiliary field inside the cylinder is \( \mathbf{H} = nI \mathbf{\hat{z}} \) so that the magnetic field inside the cylinder is

\[
\mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu_0 nI (1 + \chi_m) \mathbf{\hat{z}}.
\]

Next, since the magnetization vector \( \mathbf{M} = \chi_m nI \mathbf{\hat{z}} \) is uniform, there are no volume bound-currents \( \mathbf{J}_b = \nabla \times \mathbf{M} = 0 \) flowing inside the cylinder but surface bound-currents are present

\[
\mathbf{K}_b = \mathbf{M} \times \mathbf{\hat{r}} = \chi_m nI \mathbf{\hat{\varphi}},
\]

i.e., surface bound-currents flow in the same direction as \( I \) for a paramagnetic cylinder \((\chi_m > 0)\) or in opposite to \( I \) for a diamagnetic cylinder \((\chi_m < 0)\). Note that, in general, there is a relation between volume free-currents and volume bound-currents

\[
\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f,
\]
i.e., volume bound-currents flow in the same direction as volume free-currents for a paramagnetic cylinder ($\chi_m > 0$) or in opposite to volume free-currents for a diamagnetic cylinder ($\chi_m < 0$).

8.4 Ferromagnetism

Ferromagnetic materials (e.g., Iron, Nickel, or Cobalt) do not require external magnetic fields to maintain a permanent state of magnetization. Under normal conditions, ferromagnetic materials do not exhibit permanent magnetization because ferromagnetic properties are distributed over small domains inside which the magnetization vector has a preferred direction; hence, we expect the mean magnetization vector to be zero.

When the ferromagnetic material is exposed to an external magnetic field (i.e., produced by wrapping the material with a wire coil in which current $I$ flows), the competition between the domains is dominated by the domain with its magnetization vector parallel to the external magnetic field $\mathbf{B}(I)$ (as a result of the torque experienced by all magnetic dipoles). As a result of this magnetic interaction, the mean magnetization $M$ begins to grow from zero as the current $I$ increases (see point $a$ in the Figure below).

Once all domain magnetizations have aligned themselves to the external magnetic field $\mathbf{B}(I)$, the mean magnetization reaches a saturation point (point $b$) beyond which a further increase in current $I$ does not lead to an increase in magnetization. Because of the magnetic interaction between the domains, a reduction in the current $I$ does not lead to a return to the zero-magnetization point $a$; instead, a partial reduction is observed until the current $I$
Figure 33: Hysteresis phenomenon

returns to zero and the mean magnetization reaches a state of permanent magnetization (point \(c\)). The ferromagnetic material is now a permanent magnet.

The polarity of the permanent magnet can be reversed by, first, having negative current flow in the wire coil until the saturation point \(e\) is reached when the mean domain-magnetization has gone from \(M_{\text{sat}}\) to \(-M_{\text{sat}}\). Next, by returning the negative current to zero, we obtain a permanent magnet with opposite polarity (point \(f\)). The hysteresis loop is completed by having positive current flow through the wire coil again until we reach the saturation point \(b\). Here, we note that the points \(d\) and \(g\) in the Figure above correspond to points in which the mean magnetization of the ferromagnetic material is zero even though current is flowing in the wire coil.

Lastly, we note that the magnetization vector in a ferromagnet is much larger than its auxiliary field and, thus, the magnetic field \(B = \mu_0 (H + M) \simeq \mu_0 M\) can be very large compared to \(\mu_0 H\). Hence, powerful electromagnets are often built by wrapping coil around a solid iron core.

9 Electrodynamics

9.1 Ohmic and Hall Electric Fields

When a charged particle (of mass \(m\) and charge \(q\)) is moving in combined electric and magnetic fields \(E\) and \(B\) in the presence of collisions (which occur at a mean frequency \(\nu\),
the equation of motion for the charged particle is

\[ m \frac{dv}{dt} = q \left( E + v \times B \right) - \nu m v, \]

where collisions are expected to oppose the acceleration produced by the electric and magnetic (Lorentz) forces. Under equilibrium conditions (i.e., in steady state with constant velocity, \( dv/dt = 0 \)), the particle is moving with a drift velocity \( v_d \) obtained as the solution of the equation:

\[
\begin{pmatrix}
\nu & -\Omega & 0 \\
\Omega & \nu & 0 \\
0 & 0 & \nu
\end{pmatrix}
\begin{pmatrix}
v_{dx} \\
v_{dy} \\
v_{dz}
\end{pmatrix}
= \frac{q}{m}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
\rightarrow v_d = \mu \cdot E,
\]

where \( \Omega = q B / m \) denotes the gyrofrequency of the charged particle (here, we have assumed \( B = B \hat{z} \)) and the mobility tensor \( \mu \) is defined as the 3 x 3 matrix

\[
\mu = \frac{q}{m \nu}
\begin{pmatrix}
(1 + \alpha^2)^{-1} & \alpha (1 + \alpha^2)^{-1} & 0 \\
-\alpha (1 + \alpha^2)^{-1} & (1 + \alpha^2)^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( \alpha = \Omega / \nu \) denotes the ratio of the gyrofrequency to the collision frequency. The charge current density associated with these steady-state conditions (assuming a single charge-carrier species) can be defined as

\[
J = qn v_d = qn \mu \cdot E = \sigma \cdot E,
\]

where \( n \) denotes the volume density of charge carriers and we have introduced the conductivity tensor

\[
\sigma = qn \mu = \sigma_0
\begin{pmatrix}
(1 + \alpha^2)^{-1} & \alpha (1 + \alpha^2)^{-1} & 0 \\
-\alpha (1 + \alpha^2)^{-1} & (1 + \alpha^2)^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
\sigma_0 = \frac{nq^2}{m \nu}
\]

denoting the Ohmic conductivity. In the absence of magnetic field \( (B = 0 = \alpha) \), the conductivity tensor is \( \sigma = \sigma_0 I \) and Eq. (76) becomes Ohm’s Law: \( J = \sigma_0 E \). Here, the Ohmic conductivity is inversely proportional to the mass of the charge carrier and directly proportional to the square of its charge; hence, the largest conductivities are associated with electron charge carriers.
When a charge current $\mathbf{J}$ flows in a conductor slab in the presence of a magnetic field $\mathbf{B} = B\hat{z}$, the steady-state electric field

$$
\mathbf{E} = \frac{m}{n q^2} \begin{pmatrix}
\nu & -\Omega & 0 \\
\Omega & \nu & 0 \\
0 & 0 & \nu
\end{pmatrix} \cdot \begin{pmatrix}
J_x \\
J_y \\
J_z
\end{pmatrix}
$$

is divided into its Ohmic component $\mathbf{E}_O = \sigma_0^{-1} \mathbf{J}$ and its Hall component

$$
\mathbf{E}_H = \frac{B}{n q} \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \cdot \begin{pmatrix}
J_x \\
J_y \\
J_z
\end{pmatrix} = \mathbf{B} \times \frac{\mathbf{J}}{q n} = \mathbf{B} \times \mathbf{v}_d.
$$

In the Figure below, the Ohmic and Hall field components are shown for the case $\mathbf{J} = J\hat{x}$ and positive charge carriers.

Here, we note that the Hall field $\mathbf{E}_H = -d\Phi_H/dy$ is perpendicular to both $\mathbf{J}$ and $\mathbf{B}$ and is inversely proportional to the charge $q$ of the charge carriers (including its sign) and their density $n$. In addition, there is a steady-state potential difference $V_H = \Delta \Phi_H$ between the left side and the right side of the conductor slab.

### 9.2 Ohm’s Law

If we denote the current flowing through the bottom and top surfaces as $I = JA$, where $A$ is the cross-sectional area of the conductor slab, the Ohmic field $\mathbf{E}_O = -d\Phi_O/dx$ is
associated with a potential difference $V_0 = \Delta \Phi_0$ between the bottom and top surfaces (separated by a distance $L$), so that we obtain Ohm’s Law:

$$V_0 = E_0 L = \frac{L}{A \sigma_0} I = R_0 I,$$

where $R_0 = \rho_0 L/A$ denotes the Ohmic resistance of the conductor slab, with $\rho_0 = \sigma_0^{-1}$ denoting its resistivity (a perfect conductor has zero resistivity or infinite conductivity). Note that only the Ohmic field is involved in energy dissipation, since $\mathbf{E}_H \cdot \mathbf{J} = 0$, and the dissipated power is

$$P = \int_V \mathbf{E} \cdot \mathbf{J} \, d\tau = \sigma_0^{-1} J_a^2 A L = R_0 I^2,$$

which is also called the Joule Heating Law.

In general, the Ohmic resistance (or simply resistance) depends on the geometry of the conductor. For example, consider the current $I$ flowing between two concentric metal spherical shells (of radius $a$ and $b > a$) with a potential difference $V = V_a - V_b$ and separated by a medium with finite conductivity $\sigma$. On the one hand, according to Coulomb’s Law, the charge stored on the inner shell is

$$Q = 4\pi \varepsilon_0 \left( \frac{V_a - V_b}{1/a - 1/b} \right) = V C,$$

where $C$ denotes the capacitance of the two spheres. Using Ohm’s Law and Gauss’s Law, on the other hand, we find

$$I = \int \mathbf{J} \cdot d\mathbf{a} = \sigma \int \mathbf{E} \cdot d\mathbf{a} = \sigma \frac{Q}{\varepsilon_0} = 4\pi \sigma \left( \frac{V_a - V_b}{1/a - 1/b} \right) = \frac{V}{R},$$

from which we obtain the resistance

$$R = \frac{1}{4\pi \sigma} \left( \frac{1}{a} - \frac{1}{b} \right) = \frac{\varepsilon_0}{\sigma C}.$$

### 9.3 Electromotive Force

In a typical circuit, a current source (e.g., a battery) generates a charged current which flows throughout the entire circuit. According to Ohm’s Law, the current flow is generated by an electric field which permeates the entire circuit.

In fact, the total electric field $\mathbf{E} = \mathbf{e}_s - \nabla \Phi$ can be divided into an electric field internal to the source, labeled $\mathbf{e}_s$, and an electrostatic field $-\nabla \Phi$ that appears throughout the circuit (see Figure below).
If we integrate the electric field over the entire circuit (with terminals of the source labeled as \(a\) and \(b\) in the Figure above), we find

\[
\oint \mathbf{E} \cdot d\mathbf{l} = \oint \mathbf{e}_s \cdot d\mathbf{l} = \int_a^b \mathbf{e}_s \cdot d\mathbf{l} = \mathcal{E},
\]

which denotes the electromotive force (or emf) of the source. Because the net force on charges in an ideal circuit is zero (i.e., \(\mathbf{J} = \sigma \mathbf{E}\) is finite, in the limit \(\sigma \to \infty\), only if the net electric field \(\mathbf{E}\) is zero), we, therefore, find \(\mathbf{e}_s = \nabla \Phi\) and \(\mathcal{E} = \Phi_a - \Phi_b\) denotes the potential difference across the terminals of the source.

An additional source of emf is associated with a closed circuit loop moving in a magnetic field. If we denote

\[
\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{a}
\]

as the magnetic flux through the circuit loop, then the change in magnetic flux leads to an emf

\[
\mathcal{E} = -\frac{d\Phi_B}{dt}.
\]
10 Faraday’s Law & Maxwell’s Equations

10.1 Electromagnetic Induction

The flux rule states that a changing magnetic flux \( \Phi_B = \int_S \mathbf{B} \cdot d\mathbf{a} \) through a surface \( S \) induces an emf \( \mathcal{E} \) on the boundary \( \partial S \):

\[
\mathcal{E} = -\frac{d\Phi_B}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{E} \cdot d\mathbf{l}. \tag{77}
\]

The flux rule is also known as Faraday’s Law (in integral form). The minus sign in Faraday’s Law is associated with Lenz’s Law: the induced current on the boundary \( \partial S \) generated by the induced emf flows in a direction which opposes the change in magnetic flux.

It is worth mentioning, here, that there is an infinite number of open surfaces \( S \) with boundary \( \partial S \) but that the magnetic flux \( \int_S \mathbf{B} \cdot d\mathbf{a} \) through surface \( S \) (with boundary \( \partial S \)) is equal to the magnetic flux \( \int_{S'} \mathbf{B} \cdot d\mathbf{a} \) through surface \( S' \) (with boundary \( \partial S' \)) if the boundaries are identical \( \partial S' = \partial S \). This statement is proved by considering the volume \( V \) enclosed by the two surfaces \( S \) and \( S' \) (such that \( \partial V = S - S' \)):

\[
\int_S \mathbf{B} \cdot d\mathbf{a} - \int_{S'} \mathbf{B} \cdot d\mathbf{a} = \oint_{\partial V} \mathbf{B} \cdot d\mathbf{l} = \int_V \nabla \cdot \mathbf{B} \, d\tau = 0,
\]

which follows from the Divergence Theorem and Gauss’s Law for magnetic fields.

Returning to Faraday’s Law (77), we use Stoke’s Theorem on the last integral to obtain

\[
-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} = \int_V \nabla \times \mathbf{E} \, d\tau = 0,
\]

which becomes Faraday’s Law in differential form:

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{78}
\]

Hence, the presence of a time-dependent magnetic field induces an electric field whose curl is not zero. Consequently, this induced electric field cannot be expressed in terms of the gradient of a scalar field (since \( \nabla \times \mathbf{E} \neq 0 \)). The clue to the nature of this new electric field is provided by the representation of the magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \) in terms of the vector potential \( \mathbf{A} \):

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \frac{\partial \mathbf{A}}{\partial t} = -\nabla \times \mathbf{E} \rightarrow \nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.
\]

From the Helmholtz Decomposition Theorem for vector fields, the electric and magnetic fields are, therefore, expressed in terms of the scalar potential \( \Phi \) and vector potential \( \mathbf{A} \) as

\[
\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{79}
\]
where the new term in the electric field is called the *inductive* electric field. Note that these definitions immediately imply Faraday’s Law and Gauss’s Law for magnetic fields: \( \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \) and \( \nabla \cdot \mathbf{B} = 0 \), respectively.

For example, when a uniform magnetic field \( \mathbf{B}(t) \) is time dependent, it induces a time-dependent nonuniform electric field \( \mathbf{E}(\mathbf{r}, t) \). To show this, we invoke the formula

\[
\mathbf{A}(\mathbf{r}, t) = -\frac{\mathbf{r}}{2} \times \mathbf{B}(t)
\]

for the vector potential of a uniform magnetic field; note, here, that

\[
\nabla \times \mathbf{A} = -\frac{\mathbf{B}(t)}{2} \cdot \nabla \mathbf{r} + \frac{\mathbf{B}(t)}{2} \nabla \cdot \mathbf{r} = -\frac{1}{2} \mathbf{B}(t) + \frac{3}{2} \mathbf{B}(t) = \mathbf{B}(t).
\]

The inductive electric field is, thus, given as

\[
\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mathbf{r}}{2} \times \frac{d\mathbf{B}(t)}{dt},
\]

and that this field is divergenceless

\[
\nabla \cdot \mathbf{E} = \frac{1}{2} \frac{d\mathbf{B}(t)}{dt} \cdot \nabla \times \mathbf{r} = 0.
\]

### 10.2 Inductance

Suppose now we are interested in calculating the magnetic flux \( \Phi_{B_2} \) through a wire loop (labeled 2) due to the magnetic field \( \mathbf{B}_1 \) produced by another wire loop (labeled 1):

\[
\Phi_{B_2} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{a}_2 = M_{1\to 2} I_1,
\]

where the magnetic flux \( \Phi_{B_2} \) must clearly be proportional to the current \( I_1 \) flowing in loop 1 and the constant of proportionality \( M_{1\to 2} \), known as the mutual inductance, is a purely geometric quantity which depends on the relative orientation and positions of the loops as well as their shapes. A similar argument leads to the formula

\[
\Phi_{B_1} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{a}_1 = M_{2\to 1} I_2.
\]

If we now replace the magnetic field \( \mathbf{B}_j = \nabla \times \mathbf{A}_j \) with the vector potential

\[
\mathbf{A}_j = \frac{\mu_0}{4\pi} I_j \oint_{\partial S_j} \frac{d\mathbf{l}_j}{|\mathbf{r} - \mathbf{r}_j|},
\]

we find

\[
\Phi_{B_k} = \oint_{\partial S_k} \mathbf{A}_j \cdot d\mathbf{l}_k = \frac{\mu_0}{4\pi} I_j \oint_{\partial S_j} \oint_{\partial S_k} \frac{d\mathbf{l}_j \cdot d\mathbf{l}_k}{|\mathbf{r}_j - \mathbf{r}_k|} = M_{j\to k} I_j.
\]
Figure 36: Mutual inductance

Hence, the mutual inductance is symmetric, $M_{1\rightarrow 2} = M_{12} = M_{2\rightarrow 1}$, and is expressed as

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{\partial S} \oint_{\partial S'} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2'}{|r - r'|} = \frac{\mu_0}{4\pi I_1 I_2} \int_V \int_{V'} d\tau d\tau' \frac{\mathbf{J}_1(r) \cdot \mathbf{J}_2(r')}{|r - r'|}.$$

For example, we consider the mutual inductance of two current loops (of radii $a$ and $b$) separated by a distance $Z$; furthermore, the two loops are parallel and share the same axis (see Figure below).

Here, the current densities are

$$\mathbf{J}_1(r) = I_1 \delta(r - a) \delta(z - Z) \mathbf{\hat{\theta}}_1(\theta) \quad \text{and} \quad \mathbf{J}_2(r') = I_2 \delta(r' - b) \delta(z') \mathbf{\hat{\theta}}_2(\theta'),$$

where $\mathbf{\hat{\theta}}_1 \cdot \mathbf{\hat{\theta}}_2 = \cos(\theta - \theta')$, so that

$$M = \frac{\mu_0}{4\pi} ab \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \frac{\cos(\theta - \theta')}{\sqrt{Z^2 + a^2 + b^2 - 2ab \cos(\theta - \theta')}}$$

$$= \frac{\mu_0}{2} \sqrt{abq} \int_0^{2\pi} \frac{\cos \chi d\chi}{\sqrt{1 - 2q \cos \chi}} = - \mu_0 \sqrt{abq} \frac{d}{dq} \left( \int_0^{\pi} \frac{\sqrt{1 - 2q \cos \chi} d\chi}{\sqrt{1 - 2q}} \right),$$

where we used the substitution $(\theta, \theta') \rightarrow (\theta, \chi = \theta - \theta')$, and integrated over $\theta$, and introduced the dimensionless quantity $q = ab/(Z^2 + a^2 + b^2)$. Note that the Taylor expansion of $\sqrt{1 - 2q \cos \chi}$ yields

$$\sqrt{1 - 2q \cos \chi} = 1 - q \cos \chi - \frac{q^2}{2} \cos^2 \chi - \frac{q^3}{2} \cos^3 \chi - \frac{5q^4}{8} \cos^4 \chi - \cdots,$$
so that
\[- \frac{d}{dq} \left( \int_0^\pi \sqrt{1 - 2q \cos \chi} \, d\chi \right) = \frac{\pi q}{2} \left( 1 + \frac{15}{8} q^2 + \cdots \right),\]
and the mutual inductance between the two circular loops is
\[M = \frac{\mu_0 \pi}{2} \sqrt{abq^3} \left( 1 + \frac{15}{8} q^2 + \cdots \right) \approx \frac{\mu_0}{2\pi} \left( \frac{\pi a^2 \cdot \pi b^2}{Z^3} \right),\]
which is valid when \(Z \gg a, b\) (with \(q \simeq ab/Z^2 \ll 1\)).

A single wire loop also experiences a self-inductance, labeled \(L\), which opposes a change in the current flowing through it:
\[\Phi_B = LI \rightarrow E = -\frac{d\Phi_B}{dt} = -L \frac{dI}{dt}.\]
Here, we note that both self-inductance \(L\) and mutual inductance \(M\) are measured in henries (SI units), abbreviated \(H\), and defined as \(H = \text{V} \cdot \text{s}\). For example, the self-inductance of a rectangular toroidal coil (height \(h\), inner radius \(a\), and outer radius \(b\)) with \(N\) turns is
\[L = \frac{N}{I} \int \mathbf{B} \cdot d\mathbf{a} = \frac{N}{I} \int_a^b dr \int_0^h dz \frac{\mu_0 NI}{2\pi r} = \frac{\mu_0}{2\pi} N^2 h \ln(b/a).\]
The self-inductance, thus, plays the role of electromagnetic inertia, i.e., the large self-inductance of an electrical circuit makes it difficult to either initiate current flow in the circuit or vary the current flowing through the circuit.

10.3 Magnetic Energy in a Current Distribution

The rate of work done in allowing a current \(I\) to flow through a circuit with self-inductance \(L\) is
\[
\frac{dW}{dt} = -\mathcal{E} I = LI \frac{dI}{dt} = \frac{d}{dt} \left( L \frac{I^2}{2} \right),
\]
and, thus, the work done in setting a current distribution is
\[
W = \frac{1}{2} \Phi_B I = \frac{I}{2} \int \mathbf{A} \cdot d\mathbf{l} = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{J} \, d\tau.
\]
If we now make use of Ampère’s Law, \(\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}\), and integrate by parts, we find
\[
W = \frac{1}{2\mu_0} \int_V \mathbf{A} \cdot \nabla \times \mathbf{B} \, d\tau = \frac{1}{2\mu_0} \left[ \int_V |\mathbf{B}|^2 \, d\tau - \int_V \nabla \cdot (\mathbf{A} \times \mathbf{B}) \, d\tau \right]
\[
= \frac{1}{2\mu_0} \left[ \int_V |\mathbf{B}|^2 \, d\tau - \oint_{\partial V} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \right].
\]
If the integration volume \(V\) is \(|\mathbb{R}^3|\), for which the magnetic field \(\mathbf{B}\) must vanish on \(\partial V\), the magnetic energy stored in a current distribution is
\[
W = \frac{1}{2} \int_{|\mathbb{R}^3|} \mu_0^{-1} |\mathbf{B}|^2 \, d\tau. \quad (80)
\]
10.4 Maxwell’s Equation

Following up on Faraday’s Law, which states that a time-dependent magnetic field generates a nonuniform electric field, we naturally ask the question: Will a time-dependent electric field generate a nonuniform magnetic field? In order to provide an answer, we begin with the following puzzle involving Ampère’s Law:

\[ 0 = \nabla \cdot \nabla \times \mathbf{B} = \mu_0 \nabla \cdot \mathbf{J} \neq 0, \]

where \( \nabla \cdot \mathbf{J} \) does not vanish for time-dependent charge and currents distributions. Hence, Ampère’s Law is incomplete for time-dependent fields.

Fortunately, from the charge conservation law and Gauss’s Law for the electric field, we find

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t}, \]

and, therefore, the solution proposed by Maxwell is to modify Ampère’s Law and write

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (81) \]

which could be called the Ampère-Maxwell Law. It states that, even in the absence of a current density \( \mathbf{J} \), a time-dependent electric field will produce a nonuniform magnetic field.

In summary, Maxwell’s equations are

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \epsilon_0^{-1} \rho \quad \text{(Gauss’s Law for } \mathbf{E}) \\
\nabla \cdot \mathbf{B} &= 0 \quad \text{(Gauss’s Law for } \mathbf{B}) \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday’s Law)} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampère-Maxwell’s Law)}
\end{align*}
\]

An important note must be made concerning the charge conservation law used above. On the one hand, the charge density \( \rho \) is known to be the sum of the free-charge density \( \rho_f \) and the bound-charge density \( \rho_b = -\nabla \cdot \mathbf{P} \) (expressed in terms of the polarization \( \mathbf{P} \)), so that

\[ \frac{\partial \rho}{\partial t} = \frac{\partial \rho_f}{\partial t} - \nabla \cdot \frac{\partial \mathbf{P}}{\partial t}. \]

On the other hand, the current density must now be the sum of the free-current density \( \mathbf{J}_f \), the bound-current density \( \mathbf{J}_b = \nabla \times \mathbf{M} \) (a divergenceless current expressed in terms of the
magnetization $\mathbf{M}$), and a polarization current $\mathbf{J}_p$ associated with the fact the polarization $\mathbf{P}$ is time dependent. Hence,

$$\nabla \cdot \mathbf{J} = \nabla \cdot \mathbf{J}_f + \nabla \cdot \mathbf{J}_p = - \frac{\partial \rho}{\partial t} = - \frac{\partial \rho_f}{\partial t} - \frac{\partial \rho_b}{\partial t},$$

and, since the free-charge and free-current distributions must satisfy the charge conservation law separately (i.e., $\nabla \cdot \mathbf{J}_f = - \partial \rho_f / \partial t$), we find

$$\nabla \cdot \mathbf{J}_p = - \frac{\partial \rho_b}{\partial t} = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} \rightarrow \mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}.$$  

From these definitions, we may now express the Ampère-Maxwell Law solely in terms of the free-current density as

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t},$$

where $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$ and $\mathbf{H} = \mu_0^{-1} \mathbf{B} - \mathbf{M} = \mu^{-1} \mathbf{B}$.

### 10.5 Energy-Momentum Transport

When we combine the electric and magnetic energies found in a distribution of charges and currents, we find

$$W = \frac{1}{2} \int_{\mathbb{R}^3} \left( \varepsilon_0 |\mathbf{E}|^2 + \mu_0^{-1} |\mathbf{B}|^2 \right) d\tau = \frac{1}{2} \int_{\mathbb{R}^3} \left( \mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B} \right) d\tau,$$

where the last expression takes into account the permittivity $\varepsilon$ and permeability $\mu$ of the medium. If we introduce the electromagnetic energy density

$$E = \frac{1}{2} \left( \varepsilon_0 |\mathbf{E}|^2 + \mu_0^{-1} |\mathbf{B}|^2 \right),$$

we find, by using Maxwell’s equations, that

$$\frac{\partial E}{\partial t} = \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mu_0^{-1} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{E} \cdot \left( \nabla \times \mu_0^{-1} \mathbf{B} - \mathbf{J} \right) - \mu_0^{-1} \mathbf{B} \cdot \nabla \times \mathbf{E},$$

which can be re-arranged as

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{S} = - \mathbf{E} \cdot \mathbf{J},$$

where $\mathbf{S} = \mathbf{E} \times \mu_0^{-1} \mathbf{B}$ represents the (Poynting) electromagnetic-energy flux. Hence, the total electromagnetic energy $E = \int_{\mathbf{V}} E \, d\tau$ is dissipated at a rate $- \mathbf{E} \cdot \mathbf{J}$.

Since the electromagnetic field carries energy, it also carries momentum, with momentum density defined as $\mathbf{\Pi} = \varepsilon_0 \mathbf{E} \times \mathbf{B}$. Hence, using Maxwell’s equations once again, we find

$$\frac{\partial \mathbf{\Pi}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \varepsilon_0 \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} = \left( \mu_0^{-1} \nabla \times \mathbf{B} - \mathbf{J} \right) \times \mathbf{B} - \varepsilon_0 \mathbf{E} \times \nabla \times \mathbf{E},$$
which can be re-arranged as
\[
\frac{\partial \Pi}{\partial t} + \nabla \cdot T = - \rho E - J \times B, \tag{83}
\]
where the momentum-stress tensor for the electromagnetic field is defined as
\[ T = E I - \left( \epsilon_0 EE + \mu_0^{-1} BB \right). \]

### 10.6 Electromagnetic Waves in Free Space

An important limit of Maxwell’s equations is provided by the case of electromagnetic fields in free space (defined by the absence of charge and current densities). In free space, Maxwell’s equations are thus
\[
\nabla \cdot E = 0 = \nabla \cdot B
\]
\[
\nabla \times E = - \frac{\partial B}{\partial t} \quad \text{and} \quad \nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t},
\]
where we note that \( \mu_0 \epsilon_0 = 1/c^2 \) (\( c \) denotes the speed of light). By taking the curl of Faraday’s Law and using the Ampère-Maxwell Law, we find
\[
\nabla \times \nabla \times E = - \nabla \times \frac{\partial B}{\partial t} = - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2},
\]
which becomes
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E = 0,
\]
after using the vector identity
\[
\nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E = - \nabla^2 E,
\]
where the second equality follows from Gauss’s Law for \( E \) in free space. Note that the magnetic field satisfies a similar equation
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) B = 0.
\]
The plane-wave solutions for the electric and magnetic fields are
\[
\begin{pmatrix}
E(x, t) \\
B(x, t)
\end{pmatrix} = \begin{pmatrix}
\tilde{E} \\
\tilde{B}
\end{pmatrix} e^{i(k \cdot x - \omega t)} + \begin{pmatrix}
\tilde{E}^* \\
\tilde{B}^*
\end{pmatrix} e^{-i(k \cdot x - \omega t)},
\]
where * denotes the complex conjugate and the electromagnetic waves satisfy the dispersion relation \( \omega^2 = (|k|c)^2 \), i.e., electromagnetic waves are light waves. Lastly, we find that the time-averaged energy density and Poynting flux are
\[
\overline{E} = \frac{\epsilon_0}{2} \left( 1 + \frac{|k|^2 c^2}{\omega^2} \right) |\tilde{E}|^2 = \epsilon_0 |\tilde{E}|^2 \quad \text{and} \quad \overline{S} = \frac{k c^2}{\omega} \overline{E} = c^2 \overline{\Pi}.
A Vector Calculus Primer

A.1 Differential Calculus

- **Gradient Operator** $\nabla f$
  \[ \nabla f(x, y, z) = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \]

- **Divergence Operator** $\nabla \cdot \mathbf{A}$
  \[ \nabla \cdot \mathbf{A}(x, y, z) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]

- **Laplacian Operator** $\nabla^2 f = \nabla \cdot \nabla f$
  \[ \nabla^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

- **Curl Operator** $\nabla \times \mathbf{A}$
  \[ \nabla \times \mathbf{A}(x, y, z) = \hat{x} \left( \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \right) + \hat{y} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \hat{z} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \]

- **Vector Identities**
  \[ \nabla (fg) = (\nabla f)g + f(\nabla g) \]
  \[ \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \]
  \[ \nabla \times (\mathbf{A} \times \mathbf{B}) = [\mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{A} \cdot \nabla \mathbf{B}] - [\mathbf{B} \nabla \cdot \mathbf{A} - \mathbf{B} \cdot \nabla \mathbf{A}] \]
  \[ \mathbf{A} \times \nabla \times \mathbf{B} = (\nabla \mathbf{B}) \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \]
  \[ \nabla (\mathbf{A} \cdot \mathbf{B}) = (\nabla \mathbf{A}) \cdot \mathbf{B} + (\nabla \mathbf{B}) \cdot \mathbf{A} \]
  \[ \nabla^2 \mathbf{A} = \nabla (\nabla \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \]
  \[ \nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{A}) \mathbf{B} + \mathbf{A} \cdot \nabla \mathbf{B} \]
  \[ \nabla \times \nabla f = 0 = \nabla \cdot \nabla \times \mathbf{A} \]

- **Special Identities**
  \[ \nabla \cdot \mathbf{r} = 3 \]
  \[ \nabla \times \mathbf{r} = 0 \]
  \[ \nabla r^n = n r^{n-2} \mathbf{r} \]

- **Helmholtz Theorem for General Vector Field $\mathbf{F}$**
  \[ \mathbf{F} = -\nabla V + \nabla \times \mathbf{A} \]
  \[ \nabla \cdot \mathbf{F} = -\nabla^2 V \]
  \[ \nabla \times \mathbf{F} = \nabla \times \nabla \times \mathbf{A} \]
A.2 Integral Calculus

◦ Line Integrals
\[ \int_{P}^{Q} d\mathbf{l} \cdot \mathbf{A} = \int_{\sigma_{P}}^{\sigma_{Q}} \left( A_{x} \frac{dx}{d\sigma} + A_{y} \frac{dy}{d\sigma} + A_{z} \frac{dz}{d\sigma} \right) d\sigma \]
\[ \int_{P}^{Q} d\mathbf{l} \cdot \nabla f = f_{Q} - f_{P} \]
\[ \oint_{\partial S} d\mathbf{l} \cdot \nabla f = 0 \]

◦ Surface Integrals
\[ \int_{S} d\mathbf{a} \cdot \mathbf{A} = \int \left( A_{x} dy dz + A_{y} dz dx + A_{z} dx dy \right) \]
\[ \int_{S} d\mathbf{a} \times \nabla f = \oint_{\partial S} d\mathbf{l} \quad \text{(Stokes' Theorem)} \]
\[ \int_{S} d\mathbf{a} \cdot (\nabla f \times \nabla g) = \oint_{\partial S} f \nabla g \cdot d\mathbf{l} = - \oint_{\partial S} g \nabla f \cdot d\mathbf{l} \]
\[ \oint_{\partial V} d\mathbf{a} \cdot \nabla \times \mathbf{A} = 0 \]

◦ Volume Integrals
\[ \int_{V} d\tau \nabla \cdot \mathbf{A} = \oint_{\partial V} d\mathbf{a} \cdot \mathbf{A} \quad \text{(Divergence Theorem)} \]
\[ \int_{V} d\tau \nabla f = \oint_{\partial V} d\mathbf{a} f \]
\[ \int_{V} d\tau \nabla \times \mathbf{A} = \oint_{\partial V} d\mathbf{a} \times \mathbf{A} \]
\[ \int_{V} d\tau \left( f \nabla^{2} g - g \nabla^{2} f \right) = \oint_{\partial V} d\mathbf{a} \cdot (f \nabla g - g \nabla f) \]

A.3 Curvilinear Coordinates

◦ Transformation from Cartesian coordinates \((x, y, z)\) to curvilinear coordinates \((u^1, u^2, u^3)\)
\[ \frac{\partial \mathbf{r}}{\partial u^i} \text{ and } \nabla u^j \text{ are orthogonal} \rightarrow \frac{\partial \mathbf{r}}{\partial u^i} \cdot \nabla u^j = \delta^j_i \]

◦ Metric tensor \(g\)
\[ g_{ij} = \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j} \rightarrow ds^2 = dx^2 + dy^2 + dz^2 = \sum_{i,j} du^i g_{ij} du^j \]
A VECTOR CALCULUS PRIMER

- **Jacobian $\mathcal{J}$**
  \[ \mathcal{J} = \sqrt{\det g} = \left| \frac{\partial \mathbf{r}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2} \times \frac{\partial \mathbf{r}}{\partial u^3} \right| \rightarrow d\tau = dx \, dy \, dz = \mathcal{J} \, du^1 \wedge du^2 \wedge du^3 \]

- **Inverse metric tensor $g^{-1}$**
  \[ g^{ij} = (g^{-1})^{ij} = \nabla u^i \cdot \nabla u^j \rightarrow \mathcal{J}^{-1} = \sqrt{\det g^{-1}} = |\nabla u^1 \cdot \nabla u^2 \times \nabla u^3| \]

- **Line element $dl$ and Surface element $da$**
  \[ dl = \sum_i \frac{\partial \mathbf{r}}{\partial u^i} \, du^i \]
  and
  \[ da = \frac{1}{2} \sum_{i,j} \left( \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} \right) \, du^i \wedge du^j = \sum_{i,j,k} \frac{\mathcal{J}}{2} \epsilon_{ijk} \, du^i \wedge du^j \wedge \nabla u^k \]
  \[ = \mathcal{J} \left( du^1 \wedge du^2 \nabla u^3 + du^2 \wedge du^3 \nabla u^1 + du^3 \wedge du^1 \nabla u^2 \right) \]

- **Gradient operator $\nabla f$**
  \[ \nabla f = \sum_i \nabla u^i \frac{\partial f}{\partial u^i} \rightarrow \frac{\partial f}{\partial u^i} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial u^i} \]

- **Divergence operator $\nabla \cdot \mathbf{A}$**
  \[ \nabla \cdot \mathbf{A} = \sum_i \frac{1}{\mathcal{J}} \frac{\partial}{\partial u^i} \left( \mathcal{J} \, \mathbf{A} \cdot \nabla u^i \right) \]

- **Curl operator $\nabla \times \mathbf{A}$**
  \[ \nabla \times \mathbf{A} = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial}{\partial u^i} \left( \mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial u^j} \right) \frac{\partial \mathbf{r}}{\partial u^k} \]

- **Laplacian operator $\nabla^2 f$**
  \[ \nabla^2 f = \nabla \cdot \nabla f = \sum_i \frac{1}{\mathcal{J}} \frac{\partial}{\partial u^i} \left( \mathcal{J} \, \nabla f \cdot \nabla u^i \right) = \sum_{i,j} \frac{1}{\mathcal{J}} \frac{\partial}{\partial u^i} \left( \mathcal{J} \, g^{ij} \frac{\partial f}{\partial u^j} \right) \]

- **Identity**
  \[ \int \frac{\partial f}{\partial x^i} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = (-1)^{i-1} \int f \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \cdots \wedge dx^n \]
\[ \oint_{\partial V} d\mathbf{a} \cdot \mathbf{A} = \oint_{\partial V} \mathbf{d} \mathbf{l} \cdot \mathbf{A} \]
On the other hand, choosing $V$ to be a sphere of radius $r$ and denoting its surface as $\partial S$, we also have

$$\oint_{\partial V} \mathbf{a} \cdot \mathbf{A} = \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\varphi \, r^2 \sin \theta \, \hat{\mathbf{r}} \cdot \frac{\mathbf{r}}{r^3} = 2\pi \int_{0}^{\pi} d\theta \, \sin \theta = 4\pi,$$

where we used the following expression for the surface element $d\mathbf{a}$ for a sphere

$$d\mathbf{a} = \left( \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right) \, d\theta \wedge d\varphi = r^2 \sin \theta \, d\theta \, d\varphi \, \hat{\mathbf{r}}.$$

We immediately notice the paradox that, according to the Divergence Theorem, we find the nonsensical result $0 = 4\pi$.

The paradox is resolved by noting that $\nabla \cdot (r^{-3} \mathbf{r}) = 0$ is valid only when $\mathbf{r} \neq 0$. To reconcile the two sides of the Divergence Theorem (B.1), we, therefore, introduce a singular function known as the delta function $\delta^3(\mathbf{r})$, defined by the identity

$$\delta^3(\mathbf{r}) = \nabla \cdot \left( \frac{\mathbf{r}}{4\pi r^3} \right), \quad \text{(B.4)}$$

with the property that $\delta^3(\mathbf{r})$ is zero when $\mathbf{r} \neq 0$ and is infinite when $\mathbf{r} = 0$. Additional properties of the delta function include

$$\int_{V} d\tau \, f(\mathbf{r}) \, \delta^3(\mathbf{r} - \mathbf{a}) = \begin{cases} f(\mathbf{a}) & \text{if } \mathbf{a} \text{ is located inside } V \\ 0 & \text{if } \mathbf{a} \text{ is located outside } V \end{cases}$$

Using the delta function $\delta^3(\mathbf{r})$, we now write Eq. (B.3) as

$$\nabla \cdot \mathbf{A} = 4\pi \, \delta^3(\mathbf{r}), \quad \text{(B.5)}$$

and, thus,

$$\int_{V} d\tau \, \nabla \cdot \mathbf{A} = 4\pi \int_{V} d\tau \, \delta^3(\mathbf{r}) = 4\pi,$$

which now satisfies the Divergence Theorem (B.1).

Note that the vector field (B.2) can also be written in terms of the gradient operator as $\mathbf{A} = -\nabla r^{-1}$ so that Eq. (B.5) becomes

$$\nabla^2 r^{-1} = -4\pi \, \delta^3(\mathbf{r}). \quad \text{(B.6)}$$

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*The delta function was first introduced in Physics by P.A.M. Dirac and was, at first, vehemently rejected by mathematicians; delta functions are now part of a branch of Mathematics known as distribution theory.*
B.2 Properties of the Delta Function

- Properties in One Dimension: Let \( f(x) \) be an arbitrary function of \( x \)

\[
\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}
\]

\[
\int_{-\infty}^{\infty} dx \ f(x) \ \delta'(x - a) = - \int_{-\infty}^{\infty} dx \ f'(x) \ \delta(x - a) = - f'(a)
\]

\[
\delta(x - a) = \frac{d}{dx} \Theta(x - a), \quad \text{where} \quad \Theta(x - a) = \begin{cases} 1 & (x > a) \\ 0 & (x < a) \end{cases}
\]

- Properties in Three Dimensions

\[
\delta^3(r - a) = \delta(x - a_x) \ \delta(y - a_y) \ \delta(z - a_z)
\]

\[
\delta^3(r) = J^{-1} \ \delta(u^1) \ \delta(u^2) \ \delta(u^3)
\]

\[
\delta^3(r - a) = \frac{1}{r^2} \ \delta(r - a) \ \delta(\cos \theta - \cos \theta_a) \ \delta(\varphi - \varphi_a)
\]

B.3 Examples

- Charge distribution on the \( z = 0 \) plane: \( \rho(r) = \sigma(x, y) \ \delta(z) \).

- Charge distribution on the surface of a sphere of radius \( a \): \( \rho(r) = \sigma(\theta, \varphi) \ \delta(r - a) \).

- Charge distribution on a circle of radius \( a \) on the \( z = 0 \) plane

\[
\rho(r) = \begin{cases} \lambda(\theta) \ \delta(z) \ \delta(r - a) & \text{in cylindrical geometry}(r, \theta, z) \\ a^{-1} \lambda(\varphi) \ \delta(\theta - \frac{\pi}{2}) \ \delta(r - a) & \text{in spherical geometry}(r, \theta, \varphi) \end{cases}
\]