Geometric Optics & Image Formation

Chapter 34 – sections 1-3.

• Summary: Light as an Electromagnetic Wave

A light wave propagating along the $x$-axis with wavelength $\lambda$ and frequency $f$ is an electromagnetic wave represented by oscillating electric and magnetic fields

$$E = E_0 \cos(kx - \omega t) \hat{e} \quad \text{and} \quad B = B_0 \cos(kx - \omega t) \hat{b},$$

where $E_0$ and $B_0$ denote the magnitudes of the electric and magnetic fields, respectively, the unit vector $\hat{e}$ is known as the polarization of the electromagnetic wave (with $\hat{b} = \hat{x} \times \hat{e}$), while $k = 2\pi/\lambda$ is the wave number and $\omega = 2\pi f$ is the wave angular frequency.

The speed of light $c = \lambda f = \omega/k$ in free space (vacuum) is defined as

$$c = \frac{E_0}{B_0} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.99792458 \times 10^8 \text{ m/s}.$$ 

(Note that for most calculations, we use $c \simeq 3.0 \times 10^8$ m/s.) The speed of light in matter is defined as

$$v = \frac{c}{n} \leq c,$$

where $n \geq 1$ is called the index of refraction, which is known to be a function of wavelength $\lambda$ and the condition $dn(\lambda)/d\lambda \neq 0$ leads to the phenomena of light dispersion.

A light wave can either be represented in terms of the ray picture or the wave front picture. In the ray picture, we focus our attention on the wave vector $\mathbf{k} = k \hat{k}$, where $\hat{k} = \hat{e} \times \hat{b}$ is the unit vector pointing in the direction of propagation (see lecture notes below). In the wave front picture, on the other hand, we focus our attention on the wave crests associated with consecutive electric and magnetic maxima (separated by one wavelength $\lambda$).
• **Ray Picture of Light & Law of Reflection**

In a simple sense, the ray picture of light makes use of the experimental fact that light travels in a straight line when it propagates in a uniform medium. We use the ray picture to demonstrate the Law of Reflection, whereby an incident ray hitting a smooth reflecting surface at an angle of incidence $\theta_i$ (measured from the normal to the surface) is reflected at the surface and a reflected ray is seen departing the surface at an angle of reflection $\theta_r$ (again measured from the normal to the surface).

![Ray Diagram](image)

According to the Law of Reflection, the angle of reflection $\theta_r$ is equal to the angle of incidence $\theta_i$ (see Figure above)

\[
\text{Law of Reflection : } \theta_r = \theta_i.
\]

• **Image Formation by Mirror**

An image is a reproduction of an object via light. If light rays go through the image then it is a real image; otherwise it is a virtual image. Note that only a real image can be recorded on photographic paper.

The simplest way to form an image is with a plane mirror (see Figure below).
Here, an object of height $h_o$ is placed at a distance $d_o$ from a plane mirror. A light ray starting from the top of the object (point $O'$) hits the mirror at point $B'$ and, based on the Law of Reflection, the reflected ray reaches the observer (at point $A$). A light ray starting from the bottom of the object (point $O$) hits the mirror at point $B$ and, based on the Law of Reflection, the reflected ray reaches the observer (at point $A$). From the observer’s point of view, however, the received rays appear to come from the virtual image (with top at point $I'$ and bottom at point $I$) with a height $h_i$ located at a distance $d_i$ behind the plane mirror. From the Law of Reflection, we find that

$$d_i = d_o \quad \text{and} \quad h_i = h_o.$$ 

Images can appear distorted when the mirror surface is curved. For example, we consider the case of a spherical concave (convex) mirror, with the reflecting surface on the inner (outer) surface of a sphere of radius $R$ (at point $C$). In the top Figure below, we see that a narrow bundle of parallel rays converge to a focal point $F$ located at a distance $f = R/2$ in front of a concave mirror. In the bottom Figure, on the other hand, we see that a narrow bundle of parallel rays diverge from a focal point $F$ located at a distance $f = R/2$ behind a convex mirror.

The analysis of image formation by curved mirrors leads to the mirror equation

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f},$$
which relates the distance \(d_o = OA\) from the object to the mirror, the distance \(d_i = IA\) from the image to the mirror, and the focal length \(f = FA\) (see Figure below). A simple application of trigonometry implies that the ratio of the height \(h_i\) of the image to the height \(h_o\) of the object is

\[
\frac{h_i}{h_o} = \frac{d_i}{d_o} = \frac{f}{d_o - f}.
\]

Here, a parallel ray starting from the top (point \(O'\)) of the object hits the curved mirror at point \(B'\) and, by definition, the reflected ray goes through the focal point \(F\) and travels further to point \(I'\). On the other hand, a ray starting from the top of the object traveling through the focal point \(F\) until it hits the mirror at point \(C'\) where, by definition, the parallel reflected ray travels to point \(I'\). Notice that point \(I'\) is a point where multiple reflected rays that started at point \(O'\) converge.

As can be seen in the Figure above, the image is a real image since light rays actually go through the image although the image is seen to be inverted. We define the lateral magnification \(m\) as

\[
m = \frac{h_i}{h_o} = \frac{-f}{d_o - f},
\]

which states that the image is inverted if \(d_o > f\) \((m < 0)\) while the image is upright if \(d_o < f\) \((m > 0)\). In the latter case, the image is in fact a virtual image since the image is formed on the other side of the mirror.
Lastly, we note that image formation with a spherical convex mirror always leads to upright virtual images since \( f < 0 \) for a convex mirror and, thus, the lateral magnification is

\[
0 < m = \frac{|f|}{d_o + |f|} < 1,
\]
i.e., the image is smaller than the object.

- **Law of Refraction**

  Image formation can also proceed through the process of light *refraction* in which an incident ray propagating in medium 1 (with index of refraction \( n_1 \)) hits the boundary between medium 1 and medium 2 (with index of refraction \( n_2 \)) at an angle of incidence \( \theta_i \) (measured from the normal to the boundary). At the boundary (see Figure below), a reflected ray propagates in medium 1 at an angle of reflection \( \theta_r = \theta_i \) (according to the Law of Reflection) and a transmitted ray is propagating at an angle of transmission \( \theta_t \) (measured from the normal to the boundary).

![Light Refraction between two media (\( n_1 < n_2 \))](image)

The Law of Refraction (Snell’s Law) states that

\[
n_1 \sin \theta_i = n_2 \sin \theta_t \quad \Rightarrow \quad \theta_t = \arcsin \left( \frac{n_1}{n_2} \sin \theta_i \right) \quad \Rightarrow \quad \begin{cases} \theta_t < \theta_i & \text{(if } n_1 < n_2) \\ \theta_t > \theta_i & \text{(if } n_1 > n_2) \end{cases}
\]

The process of refraction corresponds to the fact that the transmitted ray is propagating along a direction different from the incident direction.
• Image Formation by a Thin Lens & Thin-Lens Equation

A lens is made of material with index of refraction \( n \) whose shape is formed by the intersection of two spherical surfaces of radii \( R_1 \) and \( R_2 \). The resulting focal length \( f \) is determined by the Lensmaker’s equation

\[
\frac{1}{f} = (n - 1) \left( \frac{1}{R_1} + \frac{1}{R_2} \right).
\]

Here, the two radii are considered positive if the two surfaces are outwardly convex (see Figure below) and the focal length is the same on both sides of the lens.

The focal length \( f \) is positive if both surfaces are outwardly convex or the focal length is negative if both surfaces are outwardly concave; in the first case, the lens is said to be a converging lens while in the second case, the lens is said to be a diverging lens. A lens is said to be thin if the maximum thickness of the lens is much smaller than the radii \( R_1 \) and \( R_2 \).

The two Figures below show the process of image formation with a thin converging lens.
The two cases show what happens when the object is placed beyond the focal point $F$ (top Figure), which produces an inverted/real image, or between the focal point and the lens (bottom Figure), which produces an upright/virtual image. When an object (of height $h_o$) is placed at a distance $d_o$ from a thin lens of focal length $f$ (the lens is converging when $f > 0$ or diverging when $f < 0$), an image is formed at a distance $d_i$ (on the other side of the lens) defined as

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \Rightarrow d_i = \frac{d_o f}{d_o - f}.$$

The height $h_i$ of the image is defined as

$$h_i = m \, h_o \quad \text{with} \quad m = -\frac{d_i}{d_o} = -\frac{f}{d_o - f}.$$

The two Figures below show the process of image formation with a thin diverging lens.

Note, here, that the image formed by a diverging lens is upright but virtual.