Waves


- **Characteristics of Wave Motion**

  Like an oscillation, a wave has an amplitude $D$, a frequency $f = \omega / (2\pi)$, and a period $T = 1/f$. In addition, however, since a wave is periodic in both space and time, a wave has a wavelength $\lambda = 2\pi / k$ (where $k$ is called the wavenumber) and it travels with wave speed

  $$v = \lambda f = \frac{\lambda}{T} = \frac{\omega}{k}.$$  

  Note that waves appear either as continuous waves, as wave packets, or as pulses. In the latter case, the pulse can be described in terms of an amplitude and a pulse speed (which is identical to its wave speed if the pulse were a continuous wave).

  Waves can be divided into two different types: transverse waves or longitudinal waves. A transverse wave has its wave-displacement axis perpendicular to its propagation axis while a longitudinal wave has its wave-displacement axis parallel to its propagation axis (see Figure below).
Examples of transverse waves include waves at the surface of water and light waves (with minima and maxima referred to as throughs and crests, respectively), while examples of longitudinal waves include sound waves (with minima and maxima referred to as compressions and expansions, respectively).

Wave speeds for transverse and longitudinal waves depend on properties of the medium in which they propagate. Since wave speed depends on the frequency $f$ of the wave, then the wave speed depends on the restoring-force and inertial properties of the medium. For a transverse wave travelling on a stretched string, for example, the restoring force is provided by the tension $F_T(N)$ in the string while inertia is represented by the linear mass density $\mu(kg/m)$ of the string. From simple dimensional analysis, we find that the wave speed for a transverse wave on a stretched string is $v = \sqrt{F_T/\mu}$, as might be expected on physical grounds. For a longitudinal wave such a sound travelling in a solid, on the other hand, the restoring force is provided by the elastic modulus $E(N/m^2)$ of the material and inertia is represented by the mass density $\rho(kg/m^3)$. Once again from simple dimensional analysis, the wave speed of a sound wave travelling in a solid is $v = \sqrt{E/\rho}$.

**Energy Transported by a Wave**

By simple analogy with oscillations ($E = \frac{1}{2} m \omega^2 D^2$), travelling waves also possess energy and because they travel, this wave energy can be transported through space. Indeed, by expressing mass $m$ as

$$m = \rho V = \rho (A \ell) = \rho A v t,$$

which denotes the amount of mass transported across an area $A$ in time $t$, where $\rho$ is the mass density and $A$ is the cross-sectional area through which the wave travels. Hence, the average rate $\overline{P} = E/t$ at which energy is transported is defined as

$$\overline{P} = \frac{1}{2} (\rho A v) \omega^2 D^2 = 2\pi^2 (\rho A v) f^2 D^2.$$

Lastly, we define the **intensity** $I$ of the wave as the average power transported by the wave across unit area transverse to its propagation axis:

$$I = \frac{\overline{P}}{A} = 2\pi^2 (\rho v) f^2 D^2,$$

i.e., the intensity of a wave depends on the square of its amplitude $D$ and its frequency $f$ and is linearly proportional to its wave speed.

We now note that, for a continuous wave, the rate of wave generation is constant and is proportional to the average wave power $\overline{P}$ and, thus, as the wave propagates outward away from its source, it is spread over a progressively larger area $A$ and, therefore, the wave intensity is inversely proportional to the area $A$. For a wave produced by a point source and travelling in three dimensions, the area $A = 4\pi r^2$ is the area of the surface of a sphere.
of radius \( r \) (note that \( v = dr/dt \)) and, thus the intensity of a spherical wave decreases with the inverse square of the distance to the source:

\[
I \propto \frac{1}{r^2} \rightarrow \frac{I_1}{I_2} = \left( \frac{r_2}{r_1} \right)^2.
\]

Associated with a decrease in wave intensity, the wave amplitude also decreases with distance to the source (i.e., \( A \propto 1/r \) for a spherical wave).

- **Solution for a Travelling Wave**

  The mathematical description of a travelling wave involves the function \( D(x, t) \) representing the wave displacement at location \( x \) at time \( t \) defined as

  \[
  D(x, t) = D_0 \sin \left( \frac{2\pi x}{\lambda} \pm \frac{2\pi t}{T} \right) = D_0 \sin(kx \pm \omega t),
  \]

  where the \((+)-sign\) refers to a wave travelling to the left (towards negative \( x \)-values) and the \((-)-sign\) refers to a wave travelling to the right (towards positive \( x \)-values). Note that, if we introduce the wave phase \( \Phi(x, t) = kx - \omega t \), we find that the condition of constant phase

  \[
  \frac{d\Phi}{dt} = k \frac{dx}{dt} \pm \omega = 0 \quad \rightarrow \quad \frac{dx}{dt} = \mp \frac{\omega}{k},
  \]

  shows that the wave speed (also called phase speed) is the speed with which a fixed point on the wave is moving (e.g., think of a surfer riding the wave).

- **Wave Properties**

  All waves have a certain set of properties they have in common; a more detailed account of wave properties (e.g., diffraction) will be presented next Semester in the context of light waves. First, all waves can be reflected at boundaries and can be transmitted from one medium to another medium while undergoing refraction. Next, all waves can experience constructive and destructive interference, which is analysed through the Principle of Superposition. As an example of the subtle interplay of reflection and interference effects, we mention the resonant interference involving counter-propagating waves leading to the formation of standing waves.

  Test your knowledge: Problems 1, 3, & 36 of Chapter 15