Vector Algebra

Textbook Reference: Chapter 3 – sections 1-6 & Appendix A.

- Definition of a Vector
  - A vector \( \mathbf{v} \) is determined in terms of its magnitude \( v = |\mathbf{v}| \) and its direction \( \hat{\mathbf{v}} = \mathbf{v}/v \).
  - In two-dimensional space, a vector \( \mathbf{v} \) is written as
    \[
    \mathbf{v} = v_x \hat{x} + v_y \hat{y},
    \]
    where \( \hat{x} = (1, 0) \) and \( \hat{y} = (0, 1) \) are unit vectors in the directions of increase of \( x \) and \( y \), respectively, and \( (v_x, v_y) \) denotes its components.
  - In terms of its components \( (v_x, v_y) \), the magnitude of the vector \( \mathbf{v} \) is
    \[
    v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2},
    \]
    while its direction is
    \[
    \hat{\mathbf{v}} = \frac{v_x}{\sqrt{v_x^2 + v_y^2}} \hat{x} + \frac{v_y}{\sqrt{v_x^2 + v_y^2}} \hat{y}.
    \]
    Note: The magnitude of a vector is always positive.
  - The direction unit vector \( \hat{\mathbf{v}} \) can also be represented in terms of the direction angle \( \theta \) as
    \[
    \hat{\mathbf{v}} = \cos \theta \hat{x} + \sin \theta \hat{y}.
    \]

- Vector Algebra
  - Multiplication of a Vector \( \mathbf{v} \) by a Scalar \( \alpha \)
    \[
    \alpha \mathbf{v} = (\alpha v_x) \hat{x} + (\alpha v_y) \hat{y} \rightarrow \begin{cases} 
    |\alpha \mathbf{v}| = \sqrt{(\alpha v_x)^2 + (\alpha v_y)^2} = |\alpha| |\mathbf{v}| \\
    \alpha \hat{\mathbf{v}} = \left(\alpha/|\alpha|\right) \hat{\mathbf{v}}
    \end{cases}
    \]
  - Vector addition \( \mathbf{w} = \mathbf{u} + \mathbf{v} \)
    \[
    \mathbf{w} = w_x \hat{x} + w_y \hat{y} = (u_x + v_x) \hat{x} + (u_y + v_y) \hat{y}
    \]
Figure 1: Cartesian $\mathbf{v} = v_x \hat{x} + v_y \hat{y}$ and polar $\mathbf{v} = v (\cos \theta \hat{x} + \sin \theta \hat{y})$ vector decompositions.

Figure 2: Vector addition $\mathbf{w} = \mathbf{u} + \mathbf{v}$. 
Example: \( \mathbf{u} = u \hat{x} \) and \( \mathbf{v} = v (\cos \varphi \hat{x} + \sin \varphi \hat{y}) \)

\[
w = \sqrt{u^2 + v^2 + 2uv \cos \varphi} \quad \text{and} \quad \tan \theta = \frac{v \sin \varphi}{u + v \cos \varphi}
\]

Note: Direction angle \( \theta \) is defined as follows

\[
\theta = \begin{cases} 
\arctan(w_y/w_x) & \text{if } w_x > 0 \\
\pi + \arctan(w_y/w_x) & \text{if } w_x < 0 
\end{cases}
\]

Test your knowledge: Problems 4-7 & 12 of Chapter 3