Force and Torque Equilibria

An object subjected to forces \((F_1, F_2, \cdots, F_N)\), applied at different locations on the object, is in static equilibrium if the net force \(\sum_{n=1}^N F_n\) and the net torque \(\sum_{n=1}^N \tau_n\) on the object both vanish. Assuming that the forces are planar (say in the x-y plane), then the conditions of static equilibrium become

\[
\sum_{n=1}^N F_{nx} = 0 = \sum_{n=1}^N F_{ny},
\]

\[
\sum_{n=1}^N \tau_{nz} = 0,
\]

where the torque produced by an \((x, y)\)-planar force is directed along the z-axis.

Statics Problems

As an example, we consider the case of a uniform beam (B) of length \(L\) and mass \(m_B\) attached on a wall at point \(O\) (see Figure below) with the help of a cable (assumed massless) so that the beam is hanging horizontally while the cable makes an angle \(\theta\) with respect to the horizontal. Next, attached to the beam at a distance \(\alpha L\) from the wall, we place a weight (W) of mass \(m_W\).
This problem requires that the tension $T$ in the cable and the normal force

$$F_N = F_N (\cos \varphi \hat{x} + \sin \varphi \hat{y})$$

provided by the wall be calculated by using the three static-equilibrium conditions (1) and (2). First, the condition for force equilibrium in the $x$-direction requires that

$$F_N \cos \varphi = T \cos \theta,$$

while the condition for force equilibrium in the $y$-direction requires that

$$F_N \sin \varphi + T \sin \theta = (m_B + m_W) g.$$ 

Next, the condition for torque equilibrium requires that

$$TL \sin \theta = m_B g \frac{L}{2} + m_W g \alpha L,$$

where the torque exerted by the weight of the beam is calculated by placing the full mass of the beam at its center of mass located at its center (at a distance $L/2$ from point $O$ on the wall).

From the last equation, we find the tension

$$T = \left( \frac{1}{2} m_B + \alpha m_W \right) \frac{g}{\sin \theta},$$

while the amplitude of the normal force $F_N$ and its angle $\varphi$ are now determined from the two equations

$$F_N \cos \varphi = \left( \frac{1}{2} m_B + \alpha m_W \right) g \cot \theta,$$

$$F_N \sin \varphi = \left[ \frac{1}{2} m_B + (1 - \alpha) m_W \right] g.$$ 

From these equations, we solve for the angle $\varphi$ as

$$\tan \varphi = \tan \theta \cdot \left[ \frac{m_B + 2(1 - \alpha) m_W}{m_B + 2\alpha m_W} \right],$$

while the magnitude $F_N$ is solved as

$$F_N = g \sqrt{\left[ \frac{m_B}{2} + (1 - \alpha) m_W \right]^2 + \left[ \frac{m_B}{2} + \alpha m_W \right]^2 \cot^2 \theta}.$$ 

Note that if $m_W = 0$, we find $\varphi = \theta$ and $F_N = T = m_B g/(2 \sin \theta)$.

Test your knowledge: Problems 3 & 5 of Chapter 10