Linear Momentum and Collisions


- **Linear Momentum and Newton’s Second and Third Laws**

  Consider two objects of masses \( m_1 \) and \( m_2 \) attached together with a spring of constant \( k \). Through the compressed spring, the object of mass \( m_1 \) exerts a force \( F_{1\rightarrow 2} \) on the object of mass \( m_2 \) while, in return, the object of mass \( m_2 \) exerts a force \( F_{2\rightarrow 1} \) on the object of mass \( m_1 \).

  ![Diagram of two objects and a spring](image)

  The two objects are initially at rest. When the spring is released, the two objects move apart from each other with opposing velocities \( \mathbf{v}_1 \) (to the right) and \( \mathbf{v}_2 \) (to the left). These final velocities are obtained as a result of the respective (averaged) forces experienced by the objects

  \[
  F_{1\rightarrow 2} = m_2 \frac{\Delta v_2}{\Delta t} = - m_1 \frac{\Delta v_1}{\Delta t} = - F_{2\rightarrow 1},
  \]

  where we have made use of Newton’s Second and Third Laws of motion. Since the time intervals \( \Delta t \) over which the two objects *interact* with each other are identical, Newton’s Third Law implies that

  \[
  \Delta (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2) = 0,
  \]

  which means that the quantity

  \[
  \mathbf{P} = \sum_i \mathbf{p}_i = \sum_i m_i \mathbf{v}_i
  \]
is conserved throughout the process. The vector quantity \( \mathbf{P} \) is called the total linear momentum (or total momentum), and Newton’s Second Law is now expressed as

\[
\mathbf{F}_{\text{net}} = \sum_i \mathbf{F}_i = \sum_i \frac{d\mathbf{p}_i}{dt} = \frac{d\mathbf{P}}{dt}.
\]

Hence, if the (external) net force acting on a system of particles is zero, then the total momentum of the system is conserved; this is the Law of Conservation of Momentum.

**Collisions and Impulse**

When two objects, moving with respective momentum \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \), collide, the objects transfer momentum to each other. The change in momentum \( \Delta \mathbf{p}_1 \) for particle 1 is obtained integrating the force \( \mathbf{F}_{2\rightarrow1} \) through the collision interval \( \Delta t \):

\[
\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_{2\rightarrow1}(t) \, dt = \mathbf{J}_{2\rightarrow1},
\]

where the collision begins at time \( t_i \) and ends at time \( t_f = t_i + \Delta t \) and \( \mathbf{J}_{2\rightarrow1} \) denotes the impulse received from particle 2. The impulse is clearly defined as the area under the force-versus-time curve (see Figure below).

From this definition, we may easily extract an average force of impact \( \mathbf{F} \) defined as

\[
\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t}.
\]

For example, when a 1 kg ball hits a vertical wall with velocity \( \mathbf{v}_i = (10 \text{ m/s}) \hat{x} \) and comes to rest on the wall in a time of 0.01 sec, the wall has exerted an average force of impact of

\[
\mathbf{F} = \frac{1 \text{ kg} \ (0 \text{ m/s} - 10 \text{ m/s})}{0.01 \text{ s}} \hat{x} = -(1,000 \text{ N}) \hat{x}.
\]
If the ball bounces back with velocity \( v_f = -(10 \text{ m/s}) \hat{x} \), however, the average force of impact is now \( \overline{F} = -(2,000 \text{ N}) \hat{x} \). Note that the ball exerts a force of impact \textbf{on} the wall of equal magnitude but in the opposite direction.

For a given change in momentum \( \Delta p \), we find that the relation
\[
\Delta p = \overline{F} \Delta t
\]
implies that a short collision time is associated with a large force of impact while a long collision time is associated with a small force of impact.

- **Elastic and Inelastic Collisions**

  All collisions conserve the total momentum of the colliding particles. For collisions involving two particles (labeled 1 and 2), we then find that the total momentum before the collision \( \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \) and the total momentum after the collision \( \mathbf{P}' = \mathbf{p}'_1 + \mathbf{p}'_2 \) (a prime is used to denote quantities after the collision) are equal

\[
\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2 \rightarrow \Delta \mathbf{p}_1 = - \Delta \mathbf{p}_2.
\]

An elastic collision is one in which the total kinetic energy of the colliding particles is also conserved:
\[
K = K_1 + K_2 = K'_1 + K'_2 = K'.
\]
An inelastic collision is one in which only momentum is conserved. All collisions in which particles stick to each other after the collision are inelastic. For example, when a particle of mass \( m_1 \) moving at speed \( v \) collides with a second particle of mass \( m_2 \) (initially at rest) and the two particles remain together after the collision, the conservation of momentum dictates that
\[
m_1 v = (m_1 + m_2) v',
\]
where \( v' \) is the speed of the two-particle system of combined mass \( m_1 + m_2 \). The kinetic energy before the collision is \( K = \frac{1}{2} m_1 v^2 \) while the kinetic energy after the collision is
\[
K' = \frac{1}{2} (m_1 + m_2) v'^2 = \frac{1}{2} (m_1 + m_2) \left( \frac{m_1 v}{m_1 + m_2} \right)^2 = \left( \frac{m_1}{m_1 + m_2} \right) K.
\]
Hence, since \( K' < K \), kinetic energy was lost in the course of the sticking collision; most likely the energy was converted into thermal energy, sound energy, and other forms of energy.

- **Elastic Collisions in One Dimension**

To investigate elastic collisions in one dimension, we consider the generic scenario in which the conditions before and after the collision are as follows; here, we shall assume that
an object moving to the right is moving with positive velocity. Before the collision, object 1 (mass \( m_1 \)) is moving to the right with velocity \( v_1 \) while object 2 (mass \( m_2 \)) is moving to the left with velocity \( -v_2 \). After the collision, object 1 is moving to the left with velocity \( -v'_1 \) while object 2 is moving to the right with velocity \( v'_2 \) (see Figure below).

![Before Collision](image1)

Before Collision

![After Collision](image2)

After Collision

Conservation of momentum implies that

\[
m_1 v_1 - m_2 v_2 = -m_1 v'_1 + m_2 v'_2,
\]

or, introducing the mass ratio \( \alpha = m_1/m_2 \) and the velocity changes \( \Delta v_1 = -v'_1 - v_1 < 0 \) and \( \Delta v_2 = v'_2 - (-v_2) > 0 \), we find

\[
\Delta v_2 = -\alpha \Delta v_1 \quad \rightarrow \quad \begin{cases} 
  v'_1 = -v_1 - \Delta v_1 \\
  v'_2 = -v_2 - \alpha \Delta v_1
\end{cases}
\]

Conservation of kinetic energy, on the other hand, implies that

\[
\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 = \frac{m_1}{2} v'_1^2 + \frac{m_2}{2} v'_2^2 \quad \rightarrow \quad \alpha v_1^2 + v_2^2 = \alpha (v_1 + \Delta v_1)^2 + (v_2 + \alpha \Delta v_1)^2.
\]

By expanding the right side of the second equation and cancelling terms on the left side, we obtain

\[
0 = \alpha \Delta v_1 [2(v_1 + v_2) + \Delta v_1 (1 + \alpha)],
\]

whose solution is either \( \Delta v_1 = 0 \) (i.e., nothing happened) or

\[
\Delta v_1 = -\frac{2(v_1 + v_2)}{(1 + \alpha)}.
\]
From this solution, we obtain the speeds after the collision

\[ v_1' = -v_1 - \Delta v_1 = \frac{-v_1 (1 + \alpha) + 2(v_1 + v_2)}{(1 + \alpha)} = \frac{v_1 (1 - \alpha) + 2v_2}{(1 + \alpha)} \]

\[ v_2' = -v_2 - \alpha \Delta v_1 = \frac{-v_2 (1 + \alpha) + 2\alpha (v_1 + v_2)}{(1 + \alpha)} = \frac{2\alpha v_1 + v_2 (\alpha - 1)}{(1 + \alpha)} \]

By restoring masses, we find

\[ v_1' = \frac{v_1 (m_2 - m_1) + 2m_2 v_2}{(m_1 + m_2)} \quad \text{and} \quad v_2' = \frac{2m_1 v_1 - v_2 (m_2 - m_1)}{(m_1 + m_2)}. \]

If the object 2 is initially at rest, for example, these equations simplify to

\[ v_1' = \frac{v_1 (m_2 - m_1)}{(m_1 + m_2)} \quad \text{and} \quad v_2' = \frac{2m_1 v_1}{(m_1 + m_2)}, \]

and, hence, object 1 \textit{rebounds} only if \( m_2 > m_1 \). If \( m_2 = m_1 \), however, we find \( v_1' = 0 \) (object 1 is at rest after the collision) and \( v_2' = v_1 \).

Note that an explicit solution for the problem of elastic collisions in one dimension was obtained: given the mass ratio \( \alpha = m_1/m_2 \) and the velocities \( v_1 \) and \( v_2 \) before the collision, the velocities \( v_1' \) and \( v_2' \) could be determined uniquely. This is because we had two unknowns (in one dimension) and two equations (the conservation laws of momentum and kinetic energy).

\section*{Collisions in Two Dimensions}

An explicit solution for the problem of elastic collisions in two dimensions cannot be obtained. Indeed, the four components \( v_1' = (v_{1x}', v_{1y}') \) and \( v_2' = (v_{2x}', v_{2y}') \) cannot be determined from the mass ratio \( \alpha = m_1/m_2 \) and the velocities \( v_1 \) and \( v_2 \) before the collision since the conservation of momentum and kinetic energy only give us \( 2 + 1 = 3 \) equations. Hence, one velocity component after the collision must be measured in order for the remaining three components to be determined uniquely from the conditions before the collision.

A simple way to proceed is to assume that the direction of motion one of the two colliding objects (say object 1) define the \( \hat{x} \)-direction, so that the momentum of particle 1 before the collision is simply \( p_1 = p_{1x} \hat{\mathbf{x}} \). The momentum of particle 2 before the collision can now be expressed as

\[ p_2 = p_{2x} \cos \theta_2 \hat{\mathbf{x}} + p_{2y} \sin \theta_2 \hat{\mathbf{y}}, \]

where the angle \( \theta_2 \) is measured from the \( x \)-axis as shown in the Figure below.
The momenta of particles 1 and 2 after the collision are also expressed

\[ p'_1 = p'_1 (\cos \theta'_1 \hat{x} + \sin \theta'_1 \hat{y}) \]
\[ p'_2 = p'_2 (\cos \theta'_2 \hat{x} + \sin \theta'_2 \hat{y}), \]

where the angles \( \theta'_1 \) and \( \theta'_2 \) are also measured from the \( x \)-axis (see Figure above). Conservation of momentum is now simultaneously applied in the \( x \)- and \( y \)-directions:

\[
\begin{align*}
p_1 + p_2 \cos \theta_2 &= p'_1 \cos \theta'_1 + p'_2 \cos \theta'_2 \\
p_2 \sin \theta_2 &= p'_1 \sin \theta'_1 + p'_2 \sin \theta'_2
\end{align*}
\]

Conservation of kinetic energy, on the other hand, yields

\[
\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p'_1^2}{2m_1} + \frac{p'_2^2}{2m_2}.
\]

Once again, the four unknowns \( (p'_1, \theta'_1; p'_2, \theta'_2) \) cannot be solved uniquely from the initial conditions \( (p_1; \ p_2, \theta_2) \) and the mass ratio \( \alpha = m_1/m_2 \) unless one of the unknowns is measured experimentally (e.g., the deflection angle \( \theta'_1 \)).

**Center of Mass and its Linear Momentum**

An important consequence of the conservation law of momentum in any collision is that the momentum of the center of mass is unperturbed by the collision. To prove this statement, we begin with the definition of position \( \mathbf{R}_{CM} \) of the center of mass (CM) of two particles

\[
\mathbf{R}_{CM} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}.
\]
Next, we introduce the velocity $V_{CM}$ of the center of mass

$$V_{CM} = \frac{dR_{CM}}{dt} = \frac{\sum_i m_i v_i}{\sum_i m_i} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2},$$

which implies that the momentum of the center of mass, defined as $P_{CM} = (\sum_i m_i) V_{CM}$, is also expressed as $P_{CM} = P$, i.e., the total momentum of a system of colliding particles is equal to the momentum of its center of mass. Consequently, since the total momentum $P$ is conserved by collisions, the momentum of the center of mass $P_{CM}$ is also conserved by collisions.

Test your knowledge: Problems 9, 22 & 33 of Chapter 9