Oscillations


• Simple Harmonic Motion of a Mass on a Spring

The equation of motion for a mass \( m \) is attached to a spring of constant \( k \) is

\[
m a = m \frac{d^2x}{dt^2} = -k x,
\]

where the restoring force of the spring, \( F = -k x \), is linearly proportional to the spring displacement \( x \) from away from its equilibrium state.

Using the initial conditions \( x(t = 0) = x_0 \) and \( v(t = 0) = v_0 \), we find that the solution \( x(t) \) for this motion is an oscillatory function of time (see Figure above)

\[
x(t) = A \cos(\omega t + \varphi), \tag{1}
\]

where \( A \) denotes the amplitude of the oscillation, \( \varphi \) is the initial phase of the oscillatory motion so that

\[
x_0 = A \cos \varphi \quad \text{and} \quad v_0 = -A \omega \sin \varphi = -\omega x_0 \tan \varphi,
\]
and

\[ \omega = 2\pi f = \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \]

denotes the angular frequency (with units rad/s), while \( f = 1/T \) denotes the frequency (with units Hz = s\(^{-1}\)) defined as the inverse of the period \( T \).

Since the restoring force is linearly proportional to the displacement \( x \), the oscillatory motion (1) is called \textit{Simple Harmonic Motion} (SHM) and the mass-spring system is known as a \textit{simple harmonic oscillator} (SHO).

**Energy of a Simple Harmonic Oscillator**

The total energy for a mass-spring system

\[ E = \frac{m}{2} v^2 + \frac{k}{2} x^2 \]

can be expressed in terms of the SHM solution (1) by, first, writing the expression for the velocity

\[ v(t) = \frac{dx}{dt} = -A \omega \sin(\omega t + \varphi). \]

If we now substitute this expression into the energy expression, we find

\[ E = \frac{A^2}{2} \left[ m \omega^2 \sin^2(\omega t + \varphi) + k \cos^2(\omega t + \varphi) \right] = \frac{1}{2} \left\{ \begin{array}{c} k A^2 \\ m \omega^2 A^2 \end{array} \right\} \]

where we have used the definition \( \omega = \sqrt{k/m} \) for the angular frequency. Hence, we find that the energy is indeed a constant of the motion (it is determined solely from the initial conditions) and is directly proportional to the square of the amplitude \( A \) (i.e., by doubling the amplitude of an oscillation, we quadruple its energy).

**Simple Pendulum**

The equation of motion for a pendulum of mass \( m \) and length \( \ell \) is expressed as

\[ m\ell \frac{d^2 \theta}{dt^2} = -mg \sin \theta, \quad (2) \]

where \( \theta \) denotes the angular displacement of the pendulum away from the vertical (equilibrium) line (see Figure below).
If $\theta$ remains below approximately $15^\circ$, we may replace $\sin \theta$ with $\theta$ (where $\theta$ MUST be expressed in radians); the graph below shows the percent error $[(\theta - \sin \theta)/\sin \theta] \times 100$ as a function of $\theta$ (here expressed in degrees), and we clearly see that this simple-pendulum approximation has better than 1% accuracy for angles below $\sim 15^\circ$. 
Under the simple-pendulum approximation \((\sin \theta \simeq \theta)\), the equation of motion (2), therefore, becomes
\[
m \ell \frac{d^2 \theta}{dt^2} = -mg \theta,
\] (3)
whose solution is of the form
\[
\theta(t) = \Theta \cos(\omega t + \varphi),
\]
where \(\Theta\) and \(\varphi\) denote the amplitude and initial phase of the simple harmonic motion of the simple pendulum and the angular frequency \(\omega\) of the simple pendulum is
\[
\omega = \sqrt{\frac{g}{\ell}}.
\]
This result could have been obtained on simple dimensional grounds and it is interesting to note that the mass of the pendulum does not enter since mass is represents both inertial effects and the restoring force and, thus, it cancels out.

- **Damped Oscillations**

When a simple harmonic oscillator is exposed to dissipation, the amplitude of oscillations decreases as a function of time as the oscillation energy goes to zero (see Figure below).

As a simple model to investigate damped oscillations, we consider the case of an object of mass \(m\) attached to a spring of constant \(k\) and exposed to a dissipative force of the form \(F_{\text{diss}} = -m \nu v\), where \(\nu\) denotes an energy-dissipation rate and \(v = dx/dt\) denotes the instantaneous velocity of the block. The equation of motion becomes
\[
\frac{d^2 x}{dt^2} + \nu \frac{dx}{dt} + \omega^2 x = 0,
\]
where $\omega = \sqrt{k/m}$ denotes the undamped angular frequency. Solutions of this second-order differential equation are generally studied in a course on differential equations; in the interest of expediency, we simply introduce the following solution in the form

$$x(t) = A e^{-\nu t/2} \cos(\Omega t + \varphi),$$

where $A$ denotes the initial amplitude of the damped oscillation and the modified angular frequency

$$\Omega = \sqrt{\omega^2 - \frac{\nu^2}{4}}$$

depends on the properties of the undamped system as well as the energy-dissipation rate.

Note that, for this solution, we can consider three cases. In case I, the **underdamped** case, the energy-dissipation $\nu$ is small (i.e., $\nu < 2\omega$), and the system performs several oscillations before the amplitude decreases significantly. In case II, the **overdamped** case, the energy-dissipation $\nu$ is large (i.e., $\nu > 2\omega$), and the system cannot even undergo a single oscillation cycle (i.e., the solution is purely exponentially decreasing since $\Omega^2 < 0$). In case III, the **critical-damping** case, the energy-dissipation rate $\nu = 2\omega$ is such that $\Omega = 0$ (i.e., the motion is purely exponentially decreasing, just as in case II). In all three cases, the equilibrium state is reached exponentially, with case III exhibiting the fastest approach.

Lastly, we note that the rate with which the mechanical energy $E = \frac{1}{2}(mv^2 + kx^2)$:

$$\frac{dE}{dt} = \left( m \frac{d^2x}{dt^2} + k \, x \right) \frac{dx}{dt} = -2\nu \, K < 0$$

is both proportional to the energy-dissipation rate $\nu$ as well as the kinetic energy $K = \frac{1}{2} \, mv^2$ of the block (i.e., energy dissipation is strongest where kinetic energy is largest).

Test your knowledge: Problems 3, 27 & 39 of Chapter 14