GLOBAL BEHAVIOR OF SOLUTIONS OF

\[ x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}} \]

J. FEUER\textsuperscript{1}, E. J. JANOWSKI\textsuperscript{2}, G. LADAS\textsuperscript{2,3} and C. TEIXEIRA\textsuperscript{2}

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Contact author: G. LADAS

\textsuperscript{1}Department of Mathematics and Computer Science, Merrimack College, North Andover, MA. 01845, USA.
\textsuperscript{2}Department of Mathematics, University of Rhode Island, Kingston, R.I. 02881-0816, USA.
\textsuperscript{3}E Mail: gladas@math.uri.edu
Telephone: (401)874-5592
Fax: (401)874-4617
ABSTRACT

We investigate the asymptotic behavior, the oscillatory character and the periodic nature of solutions of the difference equation

\[ x_{n+1} = \frac{\max \{ x_n, A \} }{x_n x_{n-1}}, \quad n = 0, 1, \ldots \]

where \( A \) is a real parameter and the initial conditions are arbitrary nonzero real numbers.

**Key Words:** asymptotic behavior, invariant, invariant region, periodic solutions, oscillatory solutions.

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GLOBAL BEHAVIOR OF SOLUTIONS OF $x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}}$

1. INTRODUCTION

Our aim in this paper is to investigate the difference equation

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots \quad (1.1.1)$$

where $A$ is a real parameter and the initial conditions are nonzero real numbers. In particular, we study the asymptotic behavior, the oscillatory character, and the periodic nature of its solutions.

Observe that when $A = 0$, every nontrivial solution of Eq(1.1.1) is periodic with prime period four.

It is interesting to note that when $A > 0$, the change of variables

$$x_n = \begin{cases} A^{1+y_n} & \text{if } A > 1 \\ e^{y_n} & \text{if } A = 1 \\ A^{-1+y_n} & \text{if } A < 1 \end{cases}$$

reduces Eq(1.1.1) to the piecewise linear difference equation

$$y_{n+1} = \frac{1}{2} |y_n| - \frac{1}{2}y_n - y_{n-1} + \delta, \quad n = 0, 1, \ldots \quad (1.1.2)$$

where

$$\delta = \begin{cases} -2 & \text{if } A > 1 \\ 0 & \text{if } A = 1 \\ 2 & \text{if } A < 1 \end{cases}$$

which is a special case of the Lozi Map (see [7]).

We will show that when $A = 1$, every positive solution of Eq(1.1.1) is periodic with period 7.

When $A$ is a positive real number, we will show that Eq(1.1.1) possesses an invariant (see [3],[5]) and then use it to establish that every positive solution is bounded and persists. No proof has yet been found to show this result without using the invariant [6]. We will exhibit invariant regions in the first quadrant [4] inside of which every solution is periodic of period 3 or 4. This, in particular, implies that the equilibrium of Eq(1.1.1) is stable but not asymptotically stable.

By using semicycle [1] analysis, we will show that every positive
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solution is strictly oscillatory about the positive equilibrium and that a solution which lies outside of the invariant region achieves its upper bound in every positive semicycle (except possibly the first). We also give a detailed description of the solutions of Eq(1.1.1) which are outside of the first quadrant.

When $A$ is a negative real number, we will use a change of variables to show that the solutions of Eq(1.1.1) in the third quadrant have the same behavior as the solutions of Eq(1.1.1) in the first quadrant when $A > 0$. Finally, we describe the behavior of the solutions of Eq(1.1.1) in the remaining quadrants.

2. THE CASE OF POSITIVE $A$

In the following sections we will assume $A$ is a positive real number.

2.1. THE CASE $A = 1$

**Theorem 2.1.1** Assume $A = 1$. If $x_{-1}, x_0 \in (0, \infty)$, then every nontrivial solution of Eq(1.1.1) is periodic with prime period 7.

**Proof.** Let $x_{-1} = \alpha$ and $x_0 = \beta$. The proof is a consequence of the following analysis:

- **Case 1:** $\alpha \geq 1$, $\beta \geq 1$ and $\alpha \leq \beta$. The solution is $\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, \frac{\beta}{\alpha}, 1\}$.
- **Case 2:** $\alpha \geq 1$, $\beta \geq 1$ and $\alpha > \beta$. The solution is $\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, \frac{\beta}{\alpha}, \frac{1}{\beta}\}$.
- **Case 3:** $\alpha \geq 1$, $\beta < 1$ and $\alpha \beta < 1$. The solution is $\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, 1\}$.
- **Case 4:** $\alpha > 1$, $\beta < 1$ and $\alpha \beta \geq 1$. The solution is $\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, \frac{1}{\beta}\}$.
- **Case 5:** $\alpha < 1$, $\beta \geq 1$ and $\alpha \beta \geq 1$. The solution is $\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, 1\}$.
- **Case 6:** $\alpha < 1$, $\beta \geq 1$ and $\alpha \beta < 1$. The solution is $\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, \frac{1}{\alpha}, \frac{1}{\beta}\}$.
- **Case 7:** $\alpha < 1$, $\beta < 1$ and $\alpha \beta < 1$. The solution is $\{\alpha, \beta, \frac{1}{\alpha}, \frac{\alpha}{\beta}, \frac{1}{\alpha}, \frac{1}{\alpha \beta}\}$.

$\square$
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2.2. BOUNDEDNESS AND PERSISTENCE OF SOLUTIONS

We will first show that Eq(1.1.1) possesses an invariant.

**Theorem 2.2.1** Let \( A > 0 \). Then Eq(1.1.1) possesses the invariant

\[
I_n = \max\{A, x_{n-1}, x_n\} \max\{1, \frac{1}{x_n x_{n-1}}\} = \text{constant}, \quad (2.2.1)
\]

for all \( n \geq 0 \).

**Proof.** Indeed,

\[
I_{n+1} = \max\{A, x_n, x_{n+1}\} \max\{1, \frac{1}{x_{n+1} x_n}\}
\]

\[
= \max\left\{A, x_n, \frac{\max\{x_n, A\}}{x_n x_{n-1}}\right\} \max\left\{1, \frac{1}{\max\{x_{n-1}, A\}}\right\}
\]

\[
= \max\left\{\max\{x_n, A\}, \frac{\max\{x_n, A\}}{x_n x_{n-1}}\right\} \frac{\max\{x_{n-1}, x_n, A\}}{\max\{x_n, A\}}
\]

\[
= \max\{1, \frac{1}{x_n x_{n-1}}\} \max\{x_{n-1}, x_n, A\}
\]

\[
= I_n.
\]

\[\square\]

**Corollary 2.2.1** Let \( A > 0 \). Then every positive solution of Eq(1.1.1) is bounded and persists. Furthermore, if \( \{x_n\} \) is a positive solution of Eq(1.1.1), then

\[x_n \in \left[ \frac{1}{I_0}, I_0 \right] \quad \text{for } n = 1, 2, \ldots .\]


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2.3. INVARIANT REGIONS

Define the sets,

$$R_1 = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x, y \in \left[\frac{1}{A}, A\right] \text{ and } xy \geq 1\}$$

and

$$R_2 = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x, y \in [A, \frac{1}{A}]\}.$$

Theorem 2.3.1

(a) Suppose $A > 1$. Then $R_1$ is an invariant region. Furthermore, if $\{x_n\}$ is a nontrivial solution of Eq (1.1.1) with $(x_{-1}, x_0) \in R_1$, then $\{x_n\}$ is periodic with prime period three.

(b) Suppose $0 < A < 1$. Then $R_2$ is an invariant region. Furthermore, if $\{x_n\}$ is a nontrivial solution of Eq (1.1.1) with $(x_{-1}, x_0) \in R_2$, then $\{x_n\}$ is periodic with prime period four.

Proof. We will prove (a). The proof of (b) follows similarly and will be omitted. Suppose $(x_{-1}, x_0) \in R_1$. Then

$$x_1 = \frac{\max\{x_0, A\}}{x_0 x_{-1}} = \frac{A}{x_0 x_{-1}}$$

and clearly $x_1 \in \left[\frac{1}{A}, A\right]$. Also note that

$$x_0 x_1 = \frac{A}{x_{-1}} \geq \frac{A}{A} = 1.$$

Hence, $(x_0, x_1) \in R_1$. To complete the proof, observe that

$$x_2 = \frac{\max\{x_1, A\}}{x_0 x_1} = \frac{A}{x_0 x_1} = x_{-1}$$

and

$$x_3 = \frac{\max\{x_2, A\}}{x_1 x_2} = \frac{A}{x_1 x_2} = x_0.$$

Therefore, the solution is periodic with period three and $(x_1, x_2) \in R_1$ since $x_2 x_1 \geq 1$. \qed
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Lemma 2.3.1 Eq(1.1.1) has a unique positive equilibrium $\overline{x}$. Furthermore, 

$$\overline{x} = \begin{cases} 1 & \text{if } 1 \geq A > 0 \\ \sqrt[\sqrt{A}] & \text{if } A > 1. \end{cases}$$

It is interesting to note that if $A > 1$, then $(\overline{x}, \overline{x}) \in R_1$ and if $0 < A < 1$, then $(\overline{x}, \overline{x}) \in R_2$. Thus the following result is clear.

Corollary 2.3.1 Assume that $A > 0$. Then the equilibrium $\overline{x}$ of Eq(1.1.1) is stable.

2.4. BEHAVIOR OF SOLUTIONS IN THE FIRST QUADRANT

Throughout this section we shall assume that the initial conditions $x_{-1}$ and $x_0$ of Eq(1.1.1) are positive and outside the invariant regions. The following two lemmas characterize these solutions for $A > 1$ by giving partial explicit form solutions.

Define the set, 

$$R = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x < A \text{ and } y \leq A^{-1}\}.$$ 

Lemma 2.4.1 Let $\{x_n\}$ be a solution of Eq(1.1.1) such that $(x_{N-1}, x_N) \in R$ for some nonnegative integer $N$. Let $l$ be a positive integer such that 

$$A^{-1-2l} < x_N \leq A^{1-2l}.$$ 

Then for $k = 0, 1, \ldots, l - 1,$

$$x_{7k+N-1} = \frac{x_{N-1}}{A^{2k}} < A^{1-2k}$$

$$x_{7k+N} = A^{2k}x_N \leq \frac{1}{A}$$

$$x_{7k+N+1} = \frac{A}{x_{N-1}x_N} > A$$
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\[
x_{7k+N+2} = \frac{1}{A^{2k}x_N} \geq A
\]
\[
x_{7k+N+3} = \frac{x_{N-1}x_N}{A} < \frac{1}{A}
\]
\[
x_{7k+N+4} = \frac{A^{2k+2}}{x_{N-1}} > A
\]
\[
x_{7k+N+5} = \frac{A}{x_{N-1}x_N} > A.
\]

Furthermore,
\[
x_{7l+N-1} = \frac{x_{N-1}}{A^{2l}} < \frac{1}{A}
\]
and
\[
x_{7l+N} = A^{2l}x_N \in \left(\frac{1}{A}, A\right].
\]

**Proof.** Proof is by induction and will be omitted. \(\square\)

Note that $x_{7l+N-1}x_{7l+N} < 1$. Hence a solution satisfying the hypothesis of the previous lemma will now satisfy the hypothesis of the following lemma. Its proof is also by induction and will be omitted.

Define the set,
\[
C = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x < A \text{ and } A^{-1} < y \leq A \text{ and } xy < 1\}.
\]

**Lemma 2.4.2** Let \(\{x_n\}\) be a solution of Eq(1.1.1) such that \((x_{N-1}, x_N) \in C\) for some nonnegative integer \(N\). Let \(m\) be a positive integer such that
\[
\frac{1}{A} < x_N(x_{N-1}x_N)^{m-1} \leq A \quad \text{and} \quad x_N(x_{N-1}x_N)^m \leq \frac{1}{A}.
\]

Then for \(k = 0, 1, \ldots, m-1,\)
\[
x_{3k+N} = (x_Nx_{N-1})^k x_N \in \left(\frac{1}{A}, A\right]
\]
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\[
x_{3k+N+1} = \frac{A}{x_{N-1} x_N} > A
\]

\[
x_{3k+N+2} = \frac{1}{x_N (x_N x_{N-1})^k} \in \left[ \frac{1}{A}, A \right).
\]

Furthermore,

\[
x_{3m+N-1} = \frac{x_{N-1}}{(x_N x_{N-1})^m} \in \left[ \frac{1}{A}, A \right)
\]

and

\[
x_{3m+N} = (x_N x_{N-1})^m x_N \leq \frac{1}{A}.
\]

Note that the solution now satisfies the hypothesis of lemma 2.4.1.

Define the sets,

\[
S = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x \leq A^{-1} \text{ and } xy > 1\},
\]

\[
T = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: A \leq y < x\},
\]

\[
U = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x \geq A \text{ and } xy < 1\},
\]

\[
V = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: y > A \text{ and } xy \leq 1\},
\]

\[
W = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: A < x \leq y\},
\]

\[
Z = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: y < A^{-1} \text{ and } xy \geq 1\},
\]

\[
D = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: A^{-1} < x \leq A \text{ and } y > A\},
\]

\[
E = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x > A \text{ and } A^{-1} \leq y < A\}.
\]

The next result follows directly from lemma 2.4.1 and lemma 2.4.2.

**Lemma 2.4.3** Let \( \{x_n\} \) be a solution of Eq(1.1.1) such that \( (x_{N-1}, x_N) \in R \) for some nonnegative integer \( N \). Then there exists positive integers \( l \) and \( m \) such that

\[
A^{-1-2l} < x_N \leq A^{1-2l}
\]

and

\[
\frac{1}{A} < x_N (x_{N-1} x_N)^{m-1} \leq A \quad \text{and} \quad x_N (x_{N-1} x_N)^m \leq \frac{1}{A}.
\]
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Furthermore for $k = 0, 1, \ldots, l - 1$,

$$
(x_{N+7k-1}, x_{N+7k}) \in R,
(x_{N+7k}, x_{N+7k+1}) \in S,
(x_{N+7k+1}, x_{N+7k+2}) \in T,
(x_{N+7k+2}, x_{N+7k+3}) \in U,
(x_{N+7k+3}, x_{N+7k+4}) \in V,
(x_{N+7k+4}, x_{N+7k+5}) \in W,
(x_{N+7k+5}, x_{N+7k+6}) \in Z,
$$

and for $k = 0, 1, \ldots, m - 1$,

$$
(x_{N+7l+3k-1}, x_{N+7l+3k}) \in C,
(x_{N+7l+3k}, x_{N+7l+3k+1}) \in D,
(x_{N+7l+3k+1}, x_{N+7l+3k+2}) \in E,
(x_{N+7l+3m-1}, x_{N+7l+3m}) \in R.
$$

The following two lemmas characterize solutions outside the invariant region in the first quadrant for $A < 1$ by giving partial explicit form solutions. Their proofs are by induction and will be omitted. Define the set,

$$
F = \{(x, y) \in \mathbb{R}^{+} \times \mathbb{R}^{+} : xy < 1 \; \text{and} \; y \geq A^{-1}\}.
$$

**Lemma 2.4.4** Let $\{x_n\}$ be a solution of Eq(1.1.1) such that $(x_{N-1}, x_N) \in F$ for some nonnegative integer $N$. Let $l$ be a positive integer such that

$$
A^{1-2l} \leq x_N < A^{-1-2l}.
$$

Then for $k = 0, 1, \ldots, l - 1$,

$$
x_{7k+N-1} = x_{N-1} < A,
x_{7k+N} = A^{2k} x_N \geq \frac{1}{A} > A
$$
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\[
x_{7k+N+1} = \frac{1}{x_{N-1}} > \frac{1}{A} > A
\]
\[
x_{7k+N+2} = \frac{1}{A_{2k}x_N} \leq A
\]
\[
x_{7k+N+3} = A_{2k+1}x_{N-1}x_N < A_{2k+1} \leq A
\]
\[
x_{7k+N+4} = \frac{1}{x_{N-1}} > \frac{1}{A} > A
\]
\[
x_{7k+N+5} = \frac{1}{A_{2k+1}x_{N-1}x_N} > A^{-2k-1} \geq \frac{1}{A}.
\]

Furthermore,

\[
x_{7l+N-1} = x_{N-1} < A
\]

and

\[
x_{7l+N} = A^l x_N \in [A, \frac{1}{A}).
\]

Note that the solution now satisfies the hypothesis of the next lemma.

Define the set,

\[
M = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x < A \text{ and } A \leq y < A^{-1}\}.
\]

**Lemma 2.4.5** Let \( \{x_n\} \) be a solution of Eq(1.1.1) such that \((x_{N-1}, x_N) \in M\) for some nonnegative integer \(N\). Then since

\[
\frac{A}{x_{N-1}} > 1,
\]

there exists a positive integer \(m\) such that

\[
A \leq \left(\frac{A}{x_{N-1}}\right)^{m-1} x_N < \frac{1}{A} \quad \text{and} \quad \left(\frac{A}{x_{N-1}}\right)^m x_N \geq \frac{1}{A}.
\]

Then for \(k = 0, 1, \ldots, m - 1\),

\[
x_{4k+N} = \frac{A^k}{x_{N-1}} x_N \in [A, \frac{1}{A}).
\]
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\[
x_{4k+N+1} = \frac{1}{x_{N-1}} > \frac{1}{A} > A
\]

\[
x_{4k+N+2} = \frac{x_{N-1}^k}{A^k x_N} \in (A, \frac{1}{A}]
\]

\[
x_{4k+N+3} = x_{N-1} < A
\]

Furthermore,

\[
x_{4m+N-1} = x_{N-1} < A
\]

and

\[
x_{4m+N} = \frac{x_{N-1}^m}{x_N} \geq \frac{1}{A}.
\]

The solution now satisfies the hypothesis of lemma 2.4.4.

Define the sets,

\[
G = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x \geq A^{-1} \text{ and } y > x\},
\]

\[
H = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: xy > 1 \text{ and } y \leq A\},
\]

\[
I = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x \leq A \text{ and } y < A\},
\]

\[
J = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x < A \text{ and } xy \geq 1\},
\]

\[
K = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: A^{-1} < y \leq x\},
\]

\[
L = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x > A^{-1} \text{ and } xy \leq 1\},
\]

\[
O = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: A \leq x < A^{-1} \text{ and } y > A^{-1}\},
\]

\[
P = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x > A^{-1} \text{ and } A < y \leq A^{-1}\},
\]

\[
Q = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: A < x \leq A^{-1} \text{ and } y < A\}.
\]

The next result follows directly from lemma 2.4.4 and lemma 2.4.5.

**Lemma 2.4.6** Let \(\{x_n\}\) be a solution of Eq (1.1.1) such that \((x_{N-1}, x_N) \in F\) for some nonnegative integer \(N\). Then there exists positive integers \(l\) and \(m\) such that

\[ A^{1-2l} \leq x_N < A^{-1-2l} \]

and

\[ A \leq x_N \left(\frac{A}{x_{N-1}}\right)^{m-1} < \frac{1}{A} \text{ and } x_N \left(\frac{A}{x_{N-1}}\right)^m \geq \frac{1}{A}.\]
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Furthermore for $k = 0, 1, \ldots, l - 1$,

\[
\begin{align*}
(x_{N+7k-1}, x_{N+7k}) & \in F, \\
(x_{N+7k}, x_{N+7k+1}) & \in G, \\
(x_{N+7k+1}, x_{N+7k+2}) & \in H, \\
(x_{N+7k+2}, x_{N+7k+3}) & \in I, \\
(x_{N+7k+3}, x_{N+7k+4}) & \in J, \\
(x_{N+7k+4}, x_{N+7k+5}) & \in K, \\
(x_{N+7k+5}, x_{N+7k+6}) & \in L,
\end{align*}
\]

and for $k = 0, 1, \ldots, m - 1$,

\[
\begin{align*}
(x_{N+7l+4k-1}, x_{N+7l+4k}) & \in M, \\
(x_{N+7l+4k}, x_{N+7l+4k+1}) & \in O, \\
(x_{N+7l+4k+1}, x_{N+7l+4k+2}) & \in P, \\
(x_{N+7l+4k+1}, x_{N+7l+4k+2}) & \in Q, \\
(x_{N+7l+4m-1}, x_{N+7l+4m}) & \in F.
\end{align*}
\]

From an analysis of the semicycles of Eq(1.1.1) we obtain the following lemmas whose proofs are straight forward and will be omitted.

**Lemma 2.4.7** No nontrivial solutions of Eq(1.1.1) may eventually become identically equal to the equilibrium $\bar{x}$.

**Lemma 2.4.8** Assume that $A > 0$. Let $\{x_n\}$ be a nontrivial solution of Eq.(1.1.1). Then the following are true:

(a) If $A \leq 1$, every positive semicycle (except for possibly the first) has 2 or 3 terms. Furthermore, if it has 3 terms, then the first and the third are equal to the equilibrium $\bar{x}$.

(b) If $A > 1$, every positive semicycle has at most 2 terms.

(c) Every negative semicycle has at most 2 terms.

As an immediate consequence of the above lemmas, we have the following result.
Theorem 2.4.1 Let $A > 0$. Then every nontrivial solution of Eq(1.1.1) is strictly oscillatory about the equilibrium $\pi$.

Our next result shows that not only are the solutions of Eq(1.1.1) outside the invariant regions bounded from above but that they actually achieve this upper bound.

Theorem 2.4.2 Assume that the initial conditions $x_{-1}$ and $x_0$ of Eq(1.1.1) are such that

$$(x_{-1}, x_0) \notin R_1 \quad \text{when} \quad A > 1$$

and

$$(x_{-1}, x_0) \notin R_2 \quad \text{when} \quad 0 < A < 1.$$ 

Then the solution of Eq(1.1.1) achieves its maximum value $I_0$ in every positive semicycle (except possibly the first).

Proof. Let $x_m$ be the largest term in a positive semicycle. We shall show the case when $A > 1$ and $x_{m+1} > \sqrt[3]{A}$. The other cases are similar and will be omitted.

Since $x_m > A$, then $x_m = \max\{A, x_m, x_{m+1}\}$. Observe that $x_m x_{m+1} > 1$. Hence $1 = \max\{1, \frac{1}{x_m x_{m+1}}\}$. Therefore,

$$I_0 = I_{m+1} = \max\{A, x_m, x_{m+1}\} \max\{1, \frac{1}{x_m x_{m+1}}\} = x_m.$$

The following theorem shows that there are periodic solutions of Eq(1.1.1) outside the invariant regions.

Theorem 2.4.3 Let $A > 0$. Assume that for $k = 2, 3, \ldots$

$$x_{-1} = A^k \quad \text{and} \quad x_0 \in \left[\frac{1}{A}, A\right]$$

or

$$x_{-1} \in \left[A^{k-1}, A^k\right] \quad \text{and} \quad x_0 = A^k.$$

(a) If $A > 1$, then the solution of Eq(1.1.1) is periodic with period

$$\begin{cases} 
7k - 1 & \text{if } k \text{ is even} \\
2k - 1 & \text{if } k \text{ is odd}
\end{cases}$$
GLOBAL BEHAVIOR OF SOLUTIONS OF $x_{n+1} = \max\{x_n, A\} / x_n$  

(b) If $A < 1$, then the solution of Eq(1.1.1) is periodic with period 

$$
\begin{cases}
7k + 1 & \text{if } k \text{ is even} \\
\frac{7k+1}{2} & \text{if } k \text{ is odd}.
\end{cases}
$$

\textbf{Proof.} One can see that with the above initial conditions the solution of Eq(1.1.1) can be exhibited in a closed form from which the result will follow. \hfill \square

\section*{2.5. SOLUTIONS OUTSIDE THE FIRST QUADRANT}

In this section we will see that the period 7 phenomenon is unique to the first quadrant. The proof of the next lemma is straightforward and will be omitted.

\textbf{Lemma 2.5.1} Let $\{x_n\}$ be a solution of Eq(1.1.1). Then the following statements are true:

(a) If $x_{-1} < 0$ and $x_0 < 0$, then $x_1 > 0$, $x_2 < 0$ and $x_3 < 0$.
(b) If $x_{-1} < 0$ and $x_0 > 0$, then $x_1 < 0$, $x_2 < 0$ and $x_3 > 0$.
(c) If $x_{-1} > 0$ and $x_0 < 0$, then $x_1 < 0$, $x_2 > 0$ and $x_3 < 0$.

From lemma 2.5.1 we see that if we wish to characterize solutions with one or both of the initial conditions negative, it suffices to consider only the case where both initial conditions are negative.

The following theorem implies that some solutions which begin in the third quadrant have subsequences diverging to negative infinity and other subsequences converging to zero.

\textbf{Theorem 2.5.1} Let $A > 0$ and let $\{x_n\}$ be a solution of Eq(1.1.1) with $x_{-1} = \alpha < 0$ and $x_0 = \beta < 0$. Assume that $\alpha\beta < 1$. Then for $k = 0, 1, \ldots$

$$
\begin{align*}
x_{3k} &= \alpha^k \beta^{k+1} \\
x_{3k+1} &= \frac{A}{\alpha \beta} \\
x_{3k+2} &= \frac{1}{\alpha^k \beta^{k+1}}.
\end{align*}
$$
Proof. For \( k = 0 \) the result holds. Now suppose that \( k > 0 \) and that our assumption holds for \( k - 1 \). We shall show that the result holds for \( k \).

\[
x_{3k} = \max\left\{ \frac{1}{\alpha^{k-1} \beta}, A \right\} = \alpha^k \beta^{k+1}
\]

\[
x_{3k+1} = \max\left\{ \alpha^k \beta^{k+1}, A \right\} = \frac{A}{\alpha \beta}
\]

\[
x_{3k+2} = \max\left\{ \frac{A}{\alpha \beta}, A \right\} = \frac{1}{\alpha^k \beta^{k+1}}.
\]

The next result shows that the remaining solutions which begin in the third quadrant are periodic of period 3.

**Theorem 2.5.2** Let \( A > 0 \) and let \( \{x_n\} \) be a solution of Eq(1.1.1) with \( x_{-1} = \alpha < 0 \) and \( x_0 = \beta < 0 \). Assume that \( \alpha \beta \geq 1 \). Then \( \{x_n\} \) is periodic with period three.

Proof. Let \( x_{-1} = \alpha \) and \( x_0 = \beta \). Then

\[
x_1 = \max\left\{ \beta, A \right\} = \frac{A}{\alpha \beta}
\]

\[
x_2 = \max\left\{ \frac{A}{\alpha \beta}, A \right\} = \alpha
\]

\[
x_3 = \max\left\{ \frac{A}{\beta}, A \right\} = \beta
\]

and the proof is complete.

3. **THE CASE OF NEGATIVE \( A \)**

In the remaining sections we will assume that \( A < 0 \). One can see that when \( A < 0 \) and \( x_{-1}, x_0 \in (\infty, 0) \) the change of variables

\[
x_n = -\frac{1}{y_n} \quad \text{and} \quad B = -\frac{1}{A}
\]
GLOBAL BEHAVIOR OF SOLUTIONS OF $x_{n+1} = \max\{\frac{x_n^A}{x_n x_{n-1}}\}$

reduces Eq(1.1.1) to

$$y_{n+1} = \frac{\max\{y_n, B\}}{y_n y_{n-1}}, \quad n = 0, 1, \ldots$$

with $y_{-1}, y_0, B \in (0, \infty)$. Hence the behavior of the solutions of Eq(1.1.1) which begin in the third quadrant when $A < 0$ can be easily deduced from our previous investigation.

3.1. SOLUTIONS OUTSIDE THE THIRD QUADRANT

Observe that Eq(1.1.1) has the unique positive equilibrium $x = 1$.

The proofs of the following theorems are straight forward and will be omitted.

**Theorem 3.1.1** Assume that $A < 0$ and that $x_{-1}, x_0 \in (0, \infty)$. Then a nontrivial solution of Eq(1.1.1) is periodic with prime period 4.

When $x_{-1}x_0 < 0$ and $A < 0$ the orbit of a solution of Eq(1.1.1) alternates between the second and fourth quadrants. Hence without loss of generality, we will assume that

$$x_{-1} < 0 \quad \text{and} \quad x_0 > 0.$$

**Theorem 3.1.2** Let $A < 0$ and let $\{x_n\}$ be a solution of Eq(1.1.1) with $x_{-1} < 0$ and $x_0 > 0$. Then the following are true:

(a) Suppose that $x_{-1} = -1$. If $A \leq -1$ and $x_0 = 1$, then $\{x_n\}$ is periodic with prime period 2. Otherwise, $\{x_n\}$ is periodic with prime period 4.

(b) Suppose that $x_{-1} \neq -1$. If $\min\{x_{-1}, \frac{1}{x_{-1}}\} \geq A$, then $\{x_n\}$ is periodic with prime period 4. Otherwise $\{x_n\}$ is unbounded and does not persist.
GLOBAL BEHAVIOR OF SOLUTIONS OF $x_{n+1} = \max\{x_n, A\} \frac{x_n}{x_{n-1}}$ 18

4. REFERENCES

1. E.A. Grove, E.J. Janowski, C.M. Kent, and G. Ladas, On the Rational Recursive Sequence $x_{n+1} = \frac{\alpha x_n + \beta}{(\gamma x_n + \delta)x_{n-1}}$, Communications on Applied Nonlinear Analysis 1 (1994), 61-72.


