Parametrized Tutte polynomials of graphs and matroids

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Abstract

We generalize and unify results on parametrized and coloured Tutte polynomials of graphs and matroids due to Zaslavsky [*Trans. Amer. Math. Soc.* **334** (1992), 317-347] and Bollobás and Riordan [*Combin. Prob. Comput.* **8** (1999), 45-93]. We give a generalized Zaslavsky-Bollobás-Riordan theorem that characterizes parametrized contraction-deletion functions on minor-closed classes of matroids, as well as the modifications necessary to apply the discussion to classes of graphs. We observe that these parametrized Tutte polynomials need not satisfy analogues of all the familiar properties of the classical Tutte polynomial. For instance they are not generally given by corank-nullity formulas, and they need not "nicely" reflect the structure of series-parallel connections.

1. Introduction

The Tutte polynomial is a well known invariant of graphs and matroids, associated with many different properties including flows, vertex colorings and reliability; it has found applications in the theory of electrical circuits, in knot theory, and in physics. (We refer the reader to [1, 7, 14, 15, 16, 17] for background on graphs, matroids and the Tutte polynomial.) Several of these applications involve graphs whose edges have been weighted to represent quantities of interest; for instance, when calculating the reliability of a graph the weight of an edge might indicate the probability that the edge successfully transmits information from one of its end-vertices to the other.

It is an irony that the Tutte polynomial – a single invariant which unites what might seem to be unrelated ideas like colorings, flows and reliability – has several different, seemingly incompatible weighted versions [2, 4, 5, 9, 10, 18, 19]. It turns out that most of these weighted versions are essentially equivalent; they determine each other through evaluations and changes of variables. (N.b. Although equivalent, these weighted versions

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of the Tutte polynomial have different properties, which can make it more or less convenient to consider one or another in a given context.) In [2] and [19], however, Bollobás, Riordan and Zaslavsky introduce more general weighted versions of the Tutte polynomial. Our purpose is to reconcile these two important papers.

The discussions in [2] and [19] differ in several significant ways. The invariants considered in [2] lie in commutative rings, for instance, while those of [19] are restricted to fields; also the invariants of [19] are *strong* (see the definition below) while those of [2] need not be. On the other hand, the discussion of [2] is focused on invariants defined for all graphs (or all matroids) while the invariants considered in [19] need only be defined on minor-closed classes of matroids. Moreover, the matroid invariants of [2] and [19] must be 0 or 1 for the empty matroid, though the graph invariants considered in [2] are not similarly limited. In order to unify the two discussions we introduce definitions which are sufficiently general to include all the cases considered in both [2] and [19].

In Section 2 we state these definitions, prove several propositions and present our central result, the generalized Zaslavsky-Bollobás-Riordan theorem for matroids. The modifications necessary to apply these ideas to graphs rather than matroids are presented in Section 3. In Section 4 we present a proof of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids, modeled closely on that of Theorem 3.3 in [19]. The reader who is familiar with the classical Tutte polynomial may be surprised to learn that even though the invariants we consider have subset expansions, they need not have corank-nullity formulas; details are provided in Section 5. In Section 6 we briefly describe the classification of parametrized Tutte polynomials with values in fields, à la [19], and in Section 7 we briefly discuss the effects of series/parallel connections on parametrized Tutte polynomials.

2. Definitions and results for matroids

Definition 2.1. *Let U be a class, and let R be a commutative ring. Then an R-*parametrization *of U consists of four* parameter functions *$x, y, X, Y : U \to R$, denoted $e \mapsto x_e, y_e, X_e$ and Y_e.*

When it is unnecessary to specify R we simply refer to U as *parametrized.*

Definition 2.2. *Let U be a class. Then a parametrization of U is a* colouring *if for every $e_1 \in U$ there are $e_2, e_3 \in U$ with $e_1 \neq e_2 \neq e_3 \neq e_1$ such that $x_{e_1} = x_{e_2} = x_{e_3}$, $y_{e_1} = y_{e_2} = y_{e_3}$, $X_{e_1} = X_{e_2} = X_{e_3}$, and*

$Y_{e_1} = Y_{e_2} = Y_{e_3}$.

Definition 2.3. *Let $\mathcal{M}$ be a minor-closed class of matroids defined on subsets of an R-parametrized class U. Then a* parametrized Tutte polynomial on $\mathcal{M}$ *is a function $T : \mathcal{M} \to R$ which satisfies the following: $T(M) = X_e T(M/e)$ for any coloop (or "isthmus") e of $M \in \mathcal{M}$, $T(M) = Y_e T(M - e)$ for any loop e of $M \in \mathcal{M}$, and $T(M) = y_e T(M - e) + x_e T(M/e)$ for any other element e of $M \in \mathcal{M}$.*

Before proceeding we make several observations.

1. The name *parametrized Tutte polynomial on $\mathcal{M}$* is technically incorrect because it is U that is parametrized, not T. We thank the careful reader for reading *Tutte polynomial on the minor-closed class $\mathcal{M}$ of matroids defined on subsets of the R-parametrized class U* when we write *parametrized Tutte polynomial on $\mathcal{M}$*.

2. $T(M)$ is not generally invariant under matroid isomorphisms which are not parameter-preserving.

3. The parameters X_e and Y_e do not generally coincide with the values of T on the two matroids whose sole element is e; for instance if the only element of M is a loop e then $T(M) = Y_e T(\emptyset)$, not Y_e. (Here $\emptyset$ denotes both the empty set and the unique matroid on the empty set.)

4. A parametrized Tutte polynomial T is completely determined by the parameter functions x, y, X, Y and the value of $T(\emptyset)$. We call $T(\emptyset)$ the *initial value* of T, and sometimes denote it α.

5. Suppose R is the polynomial ring $\mathbb{Z}[u, v]$ and every $e \in E$ has $x_e = 1$, $y_e = 1$, $X_e = u+1$ and $Y_e = v+1$. Then the classical Tutte polynomial (the *dichromate* of [14]) satisfies Definition 2.3. (The indeterminates u, v are often replaced by $x - 1$ and $y - 1$, respectively.)

6. The reader familiar with [19] will already have noted that our notation is closer to that of [2]: the *parameter-point-value quadruple* denoted $(a_e, b_e; x_e, y_e)$ in [19] is $(y_e, x_e; X_e T(\emptyset), Y_e T(\emptyset))$ here.

7. We focus on matroids rather than graphs in this section, and discuss graphs in Section 3.

If we are given a parametrized Tutte polynomial on $\mathcal{M}$ then we may use Definition 2.3 to recursively calculate $T(M)$ for every $M \in \mathcal{M}$. Indeed, if M is a matroid on the set $E(M) = E$ then there are $|E|!$ different calculations of $T(M)$ corresponding to the different orders in which the elements of E may be removed through contraction and deletion; there are even more calculations which do not consistently follow linear orders on E. The well-definedness of T requires that these calculations all have the same result, for every M. A simple consequence is the following.

Proposition 2.4. *Suppose a parametrized Tutte polynomial is defined on $\mathcal{M}$. If e is an element of $M \in \mathcal{M}$ then there are $r, s \in R$ with $T(M) = (rX_e + sY_e) \cdot T(\emptyset)$.*

Proposition 2.4 is proven by observing that $T(M)$ may be calculated by applying Definition 2.3 to delete and contract all the elements of E other than e; then e appears as a loop or coloop in every part of the calculation. Note that the equality $T(M) = rX_eT(\emptyset) + sY_eT(\emptyset)$ is fundamentally different from the equality $T(M) = x_eT(M/e) + y_eT(M - e)$ in that $T(M - e)$ and $T(M/e)$ are uniquely determined by M and e, while $rT(\emptyset)$ and $sT(\emptyset)$ are not.

Proposition 2.5. *Suppose a parametrized Tutte polynomial is defined on $\mathcal{M}$. Then $T(\emptyset)T(M_1 \oplus M_2) = T(M_1)T(M_2)$ for every direct sum $M_1 \oplus M_2$ in $\mathcal{M}$.*

Proof. Certainly if $M_1 = \emptyset$ then $T(M_1)T(M_2) = T(\emptyset)T(\emptyset \oplus M_2) = T(\emptyset)T(M_1 \oplus M_2)$. If $M_1 \neq \emptyset$ we may choose any element e of M_1, and assume inductively that $T(\emptyset)T((M_1/e) \oplus M_2) = T(M_1/e)T(M_2)$ and $T(\emptyset)T((M_1-e)\oplus M_2) = T(M_1-e)T(M_2)$. Then Definition 2.3 implies that $T(\emptyset)T(M_1\oplus M_2) = T(M_1)T(M_2)$. ■

Following [19], we call a parametrized Tutte polynomial a *strong Tutte function* if $T(M_1 \oplus M_2) = T(M_1)T(M_2)$ for every direct sum $M_1 \oplus M_2$ in $\mathcal{M}$.

Corollary 2.6. *A parametrized Tutte polynomial is a strong Tutte function if and only if $T(\emptyset)$ is idempotent, i.e., $T(\emptyset) = T(\emptyset)^2$.*

Proof. Certainly if $T(\emptyset \oplus \emptyset) = T(\emptyset)T(\emptyset)$ then $T(\emptyset)$ must be idempotent, because $\emptyset \oplus \emptyset = \emptyset$.

Suppose conversely that $T(\emptyset)$ is idempotent, and suppose $M_1 \oplus M_2$ is a direct sum in $\mathcal{M}$. If e is an element of $M_1 \oplus M_2$ then by Proposition 2.4, $T(M_1 \oplus M_2) = (rX_e + sY_e) \cdot T(\emptyset)$ for some $r, s \in R$ and hence $T(\emptyset)T(M_1 \oplus M_2) = (rX_e + sY_e) \cdot T(\emptyset)^2 = (rX_e + sY_e) \cdot T(\emptyset) = T(M_1 \oplus M_2)$. It follows from Proposition 2.5 that $T(M_1 \oplus M_2) = T(M_1)T(M_2)$. ■

As noted above, if we are given a parametrized Tutte polynomial T on $\mathcal{M}$ then well-definedness requires that for every $M \in \mathcal{M}$, all the different calculations of $T(M)$ using Definition 2.3 have the same result.

Consequently the well-definedness of T is, in effect, a large set of constraints on the various parameters x_e, y_e, X_e and Y_e. It turns out that three very simple matroid structures in $\mathcal{M}$ generate the entire set of constraints: the *digons* (which have two elements in parallel), *triangles* (which have three elements in a circuit) and *triads* (which have three elements, all parallel).

Definition 2.7. *(a) Two elements of U are* digonal in $\mathcal{M}$ *if they constitute a circuit in some element of $\mathcal{M}$.*

(b) Three elements of U are triangular in $\mathcal{M}$ *if they constitute a circuit in some element of $\mathcal{M}$.*

(c) Three elements of U are triadic in $\mathcal{M}$ *if they are mutually parallel in some element of $\mathcal{M}$.*

When we say *two elements* or *three elements* we mean that the elements must be pairwise distinct. As $\mathcal{M}$ is minor-closed, $e_1, e_2 \in U$ are digonal in $\mathcal{M}$ if and only if $\mathcal{M}$ contains the digon with elements e_1 and e_2; similarly $e_1, e_2, e_3 \in U$ are triangular (or triadic) in $\mathcal{M}$ if and only if $\mathcal{M}$ contains the triangle (or triad) on $\{e_1, e_2, e_3\}$. Note also that if e_1, e_2, e_3 are triadic or triangular in $\mathcal{M}$ then they are pairwise digonal.

Our central theorem is the following generalization of results of [2] and [19]. It is clearly a synthesis, rather than an original result; moreover its proof, discussed in Section 4 below, requires no new ideas.

Theorem 2.8 (The Generalized Zaslavsky-Bollobás-Riordan Theorem for Matroids). *Let R be a commutative ring, let $\mathcal{M}$ be a minor-closed class of matroids defined on subsets of an R-parametrized class U, and let $\alpha \in R$. Then there is a parametrized Tutte polynomial on $\mathcal{M}$ with $T(\emptyset) = \alpha$ if and only if the following identities are satisfied.*

(a) Whenever e_1 and e_2 are digonal in $\mathcal{M}$,

$$\alpha \cdot (x_{e_1} Y_{e_2} + y_{e_1} X_{e_2}) = \alpha \cdot (x_{e_2} Y_{e_1} + y_{e_2} X_{e_1}).$$

(b) Whenever e_1, e_2 and e_3 are triangular in $\mathcal{M}$,

$$\alpha \cdot X_{e_3} \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2}) = \alpha \cdot X_{e_3} \cdot (Y_{e_1} x_{e_2} + x_{e_1} y_{e_2}).$$

(c) Whenever e_1, e_2 and e_3 are triadic in $\mathcal{M}$,

$$\alpha \cdot Y_{e_3} \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2}) = \alpha \cdot Y_{e_3} \cdot (Y_{e_1} x_{e_2} + x_{e_1} y_{e_2}).$$

Let $\mathcal{M}_{(3)} = \{M \in \mathcal{M}$ with no more than three elements$\}$. Because the identities (a), (b) and (c) are necessary for a parametrized Tutte polynomial to be well defined on $\mathcal{M}_{(3)}$, the following analogue of Theorem 3.3 of [19] is an immediate consequence of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids.

Corollary 2.9. *Let R be a commutative ring and let $\mathcal{M}$ be a minor-closed class of matroids defined on subsets of an R-parametrized class U. Then a parametrized Tutte polynomial on $\mathcal{M}_{(3)}$ has a unique extension to a parametrized Tutte polynomial on $\mathcal{M}$.*

Theorem 2 of [2] and Theorems 6.1 and 6.2 of [19] assume that every pair of elements of U is digonal in $\mathcal{M}$ and every triple of elements of U is both triangular and triadic in $\mathcal{M}$. Consequently the appropriate special cases of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids involve the identities (a), (b) and (c) for all choices of pairwise distinct $e_1, e_2, e_3 \in U$. Both [2] and [19] provide separate results with the additional assumption that U is coloured. Adjusting the generalized Zaslavsky-Bollobás-Riordan theorem for matroids involves only the observation that we have not required that the parameter functions x, y, X, Y be injective; equivalently, if U is coloured and every triple of elements of U is both triangular and triadic in $\mathcal{M}$ then the identities (a), (b) and (c) must hold for all choices of $e_1, e_2, e_3 \in U$, distinct or not.

As in [2], the generalized Zaslavsky-Bollobás-Riordan theorem for matroids implies that there is a most general example of a parametrized Tutte polynomial on a minor-closed class $\mathcal{M}$. For an unparametrized class U we denote by $\mathbb{Z}_U$ the ring of polynomials with integer coefficients which has four distinct indeterminates x_e, y_e, X_e, Y_e for each element of U, and also has a single separate indeterminate α. If $\mathcal{M}$ is a minor-closed class of matroids defined on U then we denote by $I^{\mathcal{M}}$ the ideal of $\mathbb{Z}_U$ generated by the following elements: whenever e_1 and e_2 are digonal in $\mathcal{M}$

$$\alpha \cdot (x_{e_1} Y_{e_2} + y_{e_1} X_{e_2} - x_{e_2} Y_{e_1} - y_{e_2} X_{e_1}) \in I^{\mathcal{M}},$$

whenever e_1, e_2 and e_3 are triangular in $\mathcal{M}$

$$\alpha \cdot X_{e_3} \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2} - Y_{e_1} x_{e_2} - x_{e_1} y_{e_2}) \in I^{\mathcal{M}},$$

and whenever e_1, e_2 and e_3 are triadic in $\mathcal{M}$

$$\alpha \cdot Y_{e_3} \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2} - Y_{e_1} x_{e_2} - x_{e_1} y_{e_2}) \in I^{\mathcal{M}}.$$

U has an obvious $(\mathbb{Z}_U/I^{\mathcal{M}})$-parametrization given by $e \mapsto x_e + I^{\mathcal{M}}, y_e + I^{\mathcal{M}}, X_e + I^{\mathcal{M}}, Y_e + I^{\mathcal{M}}$. In analogy with the discussion in [2], the generalized Zaslavsky-Bollobás-Riordan theorem for matroids immediately implies that there is a most general example of a parametrized Tutte polynomial on $\mathcal{M}$, with values in the quotient ring $\mathbb{Z}_U/I^{\mathcal{M}}$.

Corollary 2.10. *Let $\mathcal{M}$ be a minor-closed class of matroids defined on subsets of a class U. Then there is a $(\mathbb{Z}_U/I^{\mathcal{M}})$-valued parametrized Tutte polynomial $T^{\mathcal{M}}$ on $\mathcal{M}$ which has $T^{\mathcal{M}}(\emptyset) = \alpha + I^{\mathcal{M}}$. Moreover, if T is any R-parametrized Tutte polynomial on $\mathcal{M}$ then T is the composition of $T^{\mathcal{M}}$ with the homomorphism of rings with unity $\mathbb{Z}_U/I^{\mathcal{M}} \to R$ determined by the parameter functions and $\alpha + I^{\mathcal{M}} \mapsto T(\emptyset)$.*

If R is a commutative ring without unity then a multiplicative identity must be adjoined to R before the corollary can be applied.

If T is a parametrized Tutte polynomial on $\mathcal{M}$ and $M \in \mathcal{M}$ then a natural way to calculate the value of $T(M)$ is to order $E(M)$, say as $e_1, ..., e_m$, and remove the elements one at a time, through contraction and/or deletion, as specified by Definition 2.3. (We remove the elements in reverse order, e_m first, then e_{m-1}, and so on.) The result is a formula for $T(M)$ as a sum indexed by those $B \subseteq E(M)$ which have the property that no element of B is deleted as a coloop or contracted as a loop during the calculation. The summand corresponding to such a B is a product involving one factor for each $e \in E(M)$: an $e \in B$ contributes a factor of X_e or x_e according to whether e is or is not contracted as a coloop in the calculation, and an $e \notin B$ contributes a factor of Y_e or y_e according to whether or not e is deleted as a loop. Every summand also has a factor of $T(\emptyset)$.

It turns out that every such computation of $T(M)$ involves the same subsets B, namely the bases of M [6]. (The fact that every such computation involves the same subsets B actually characterizes the basis clutters of matroids [12].) If B is a basis and $e_i \in B$ then there is a unique cocircuit $\text{cut}(B, e_i)$ of M with $\text{cut}(B, e_i) \cap B = \{e_i\}$, and clearly e_i is a coloop at the time of its contraction in the calculation if and only if the rest of $\text{cut}(B, e_i)$ has already been deleted; that is, e_i contributes a factor of X_{e_i} if and only if $\text{cut}(B, e_i)$ contains no e_j with $j < i$. Thus $\{e_i \in B$ which contribute a factor of X_{e_i} to the summand corresponding to $B\} = IA(B)$, the set of *internally active* elements of B. Similarly, if B is a basis and $e_i \notin B$ then there is a unique circuit $\text{cyc}(B, e_i)$ of M with $\text{cyc}(B, e_i) \subseteq B \cup \{e_i\}$, and clearly e_i is a loop at the time of its deletion in the calculation if and only if the rest of $\mathrm{cyc}(B, e_i)$ has already been contracted; that is, e_i contributes a factor of Y_{e_i} if and only if $\mathrm{cyc}(B, e_i)$ contains no e_j with $j < i$. Thus $\{e_i \notin B$ which contribute a factor of Y_{e_i} to the summand corresponding to $B\} = EA(B)$, the set of *externally active* elements of B. The result is the following *activities formula* for $T(M)$.

Proposition 2.11. *Let T be a parametrized Tutte polynomial on $\mathcal{M}$, and let $M \in \mathcal{M}$. Then*

$$T(M) = T(\emptyset) \cdot \sum_B \left(\prod_{e \in IA(B)} X_e \right) \left(\prod_{e \in B - IA(B)} x_e \right) \left(\prod_{e \in EA(B)} Y_e \right) \left(\prod_{e \notin B \cup EA(B)} y_e \right).$$

We have chosen to define parametrized Tutte polynomials as in Definition 2.3, and then deduce Proposition 2.11 as just discussed. This choice can be reversed: Proposition 2.11 may be taken as a definition, along with the requirement that $T(M)$ be independent of the order on $E(M)$ used to define the activities, and then Definition 2.3 follows. (This is the structure of the discussion in [2].) That is, a parametrized Tutte polynomial can be defined using either a deletion-contraction recursion or an activities formula. This will not come as a surprise to the reader who is familiar with the classical Tutte polynomial; what may come as a surprise is the fact that the subset expansion of a parametrized Tutte polynomial is not generally a corank-nullity expansion. See Section 5 for details.

3. Definitions and results for graphs

In this section we discuss the modifications which are appropriate to apply the above discussion to graphs rather than matroids. There is of course a close connection between graphs and matroids: if G is a graph then it has an associated circuit matroid M on the edge-set $E(G)$. The circuit matroid contains a great deal of information about the structure of G; for instance M determines which sets of edges of G are circuits and which are cutsets ("cocircuits" in matroid terminology). However the vertices of G are not present in the circuit matroid and consequently some information about G — for instance its number of connected components — cannot be determined by M.

Considering the three different portions of the recursion of Definition 2.3, we give the following definition.

Definition 3.1. *A* minor-closed class of graphs *is closed under the deletion of loops, the contraction of isthmuses (or "coloops") , and the contraction and deletion of other edges.*

This is a modification of the usual terminology in that we do not require a minor-closed class of graphs to be closed under the deletion of isthmuses. A consequence is the following.

Proposition 3.2. *Let $\mathcal{G}$ be a class of graphs, and for $k \geq 1$ let $\mathcal{G}_k$ consist of those $G \in \mathcal{G}$ with precisely k connected components. Then $\mathcal{G}$ is minor-closed if and only if every $\mathcal{G}_k$ is minor-closed.*

Definition 3.3. *Let U be an R-parametrized class, and let $\mathcal{G}$ be a minor-closed class of graphs with $E(G) \subseteq U$ $\forall G \in \mathcal{G}$. Then* a parametrized Tutte polynomial on $\mathcal{G}$ *is a function $T : \mathcal{G} \to R$ which satisfies the following: $T(G) = X_e T(G/e)$ for any isthmus e of $G \in \mathcal{G}$, $T(G) = Y_e T(G - e)$ for any loop e of $G \in \mathcal{G}$, and $T(G) = y_e T(G - e) + x_e T(G/e)$ for any other edge e of $G \in \mathcal{G}$.*

Before proceeding we make several observations, analogous to the observations in Section 2.

1. When we write *parametrized Tutte polynomial on $\mathcal{G}$* we actually mean *Tutte polynomial on the minor-closed class $\mathcal{G}$ of graphs whose edges are elements of the R-parametrized class U.*

2. $T(G)$ is not generally invariant under graph isomorphisms which are not parameter-preserving.

3. The parameters X_e and Y_e need not coincide with the values of T on the two graphs whose only edge is e.

4. If G has $k = k(G)$ connected components then $T(G)$ is completely determined by x, y, X, Y and the value of T on the edgeless graph E_k with k vertices. We call $T(E_k)$ an *initial value* of T, and denote it α_k.

5. Both the classical Tutte and dichromatic polynomials satisfy Definition 3.3.

Proposition 3.4 follows from the fact that every graph which appears in a calculation of $T(G)$ has $k(G)$ connected components.

Proposition 3.4. *Let U be an R-parametrized class, and let $\mathcal{G}$ be a minor-closed class of graphs with $E(G) \subseteq U$ $\forall G \in \mathcal{G}$. Then a function $T : \mathcal{G} \to R$ is a parametrized Tutte polynomial on $\mathcal{G}$ if and only if for every $k \geq 1$, the restriction of T to $\mathcal{G}_k$ is a parametrized Tutte polynomial on $\mathcal{G}_k$.*

The definition of the circuit matroid of a graph is compatible with the definitions of deletion and contraction for graphs and matroids, and an edge of a graph is a loop or isthmus if and only if it is a loop or coloop (respectively) in the graph's circuit matroid. We deduce the following proposition.

Proposition 3.5. *Let U be an R-parametrized class, let $\mathcal{G}$ be a minor-closed class of graphs with $E(G) \subseteq U$ $\forall G \in \mathcal{G}$, and for each integer $k \geq 1$ let $\mathcal{M}_k$ be the class of circuit matroids of graphs which appear in $\mathcal{G}_k$. Then a function $T : \mathcal{G} \to R$ is a parametrized Tutte polynomial on $\mathcal{G}$ if and only if for every $k \geq 1$, T induces a well-defined function $T_k : \mathcal{M}_k \to R$ which is a parametrized Tutte polynomial on $\mathcal{M}_k$.*

Combining Propositions 3.2, 3.4 and 3.5 we see that the theory of parametrized Tutte polynomials defined on a minor-closed class of graphs $\mathcal{G}$ may be deduced from the theory of parametrized Tutte polynomials defined on minor-closed classes of matroids, by applying the matroidal theory to the various $\mathcal{M}_k$ separately.

Corollary 3.6 (The Generalized Zaslavsky-Bollobás-Riordan Theorem for Graphs). *Let R be a commutative ring, let $\mathcal{G}$ be a minor-closed class of graphs whose edge-sets are contained in an R-parametrized class U, and let $\alpha_1, \alpha_2, ... \in R$. Then there is a parametrized Tutte polynomial T on $\mathcal{G}$ with $T(E_k) = \alpha_k$ $\forall k | E_k \in \mathcal{G}$ if and only if the following identities are satisfied.*

(a) Whenever e_1 and e_2 appear together in a circuit of a k-component graph $G \in \mathcal{G}$,

$$\alpha_k \cdot (x_{e_1} Y_{e_2} + y_{e_1} X_{e_2}) = \alpha_k \cdot (x_{e_2} Y_{e_1} + y_{e_2} X_{e_1}).$$

(b) Whenever e_1, e_2 and e_3 appear together in a circuit of a k-component graph $G \in \mathcal{G}$,

$$\alpha_k \cdot X_{e_3} \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2}) = \alpha_k \cdot X_{e_3} \cdot (Y_{e_1} x_{e_2} + x_{e_1} y_{e_2}).$$

(c) Whenever e_1, e_2 and e_3 are parallel to each other in a k-component graph $G \in \mathcal{G}$,

$$\alpha_k \cdot Y_{e_3} \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2}) = \alpha_k \cdot Y_{e_3} \cdot (Y_{e_1} x_{e_2} + x_{e_1} y_{e_2}).$$

If $k \neq k'$ there need not be any connection at all between the behavior of a parametrized Tutte polynomial on $\mathcal{G}_k$ and its behavior on $\mathcal{G}_{k'}$. In particular, α_k and $\alpha_{k'}$ may appear in different instances of the identities of the generalized Zaslavsky-Bollobás-Riordan theorem for graphs, if different combinations of edges appear in digons, triangles and triads in $\mathcal{G}_k$ and $\mathcal{G}_{k'}$. We leave it to the reader to formulate an appropriate version of Corollary 2.10, generalizing Theorem 6 of [2], with attention to the fact that the generators of $I^{\mathcal{G}}$ involving α_k arise only from the digons, triangles and triads in $\mathcal{G}_k$.

On the other hand, it may happen that identical matroids appear in $\mathcal{M}_k$ and $\mathcal{M}_{k'}$. For instance, it may happen that $\mathcal{G}$ contains both G and a graph G' obtained from G by adjoining some isolated vertices, or by taking one-point unions of some components of G. In such a circumstance the calculations of $T(G)$ and $T(G')$ are identical except for the fact that a computation of $T(G)$ involves $\alpha_{k(G)}$ and a computation of $T(G')$ involves $\alpha_{k(G')}$. We deduce the following proposition.

Proposition 3.7. *Let U be an R-parametrized class, let $\mathcal{G}$ be a minor-closed class of graphs with $E(G) \subseteq U$ $\forall G \in \mathcal{G}$, and suppose $G, G' \in \mathcal{G}$ have the same circuit matroid. Then there is an $r \in R$ with $T(G) = \alpha_{k(G)} \cdot r$ and $T(G') = \alpha_{k(G')} \cdot r$.*

The following version of Proposition 2.4 holds.

Proposition 3.8. *Suppose a parametrized Tutte polynomial is defined on $\mathcal{G}$. If e is an edge of $G \in \mathcal{G}$ then there are $r, s \in R$ with $T(G) = (rX_e + sY_e) \cdot \alpha_{k(G)}$.*

We use the notation $G = G_1 \amalg G_2$ to indicate that G is the disjoint union of two subgraphs G_1 and G_2.

Proposition 3.9. *Suppose a parametrized Tutte polynomial is defined on $\mathcal{G}$. If $G_1, G_2, G_1 \amalg G_2 \in \mathcal{G}$ then there is an $r \in R$ with $r \cdot \alpha_{k(G_1)}\alpha_{k(G_2)} = T(G_1)T(G_2)$ and $r \cdot \alpha_{k(G_1)+k(G_2)} = T(G_1 \amalg G_2)$.*

Proof. Suppose we calculate $T(G_1 \amalg G_2)$ by first removing all the edges of G_1 through contraction and deletion, and then removing all the edges of G_2. If we compare this to a calculation of $T(G_1)$, we see that at the moment when the edges of G_1 have all been removed we have obtained an element $s \in R$ with $s\alpha_{k(G_1)} = T(G_1)$. There is a $t \in R$ with $T(G_2) = \alpha_{k(G_2)}t$, and a calculation of such a t may be replicated repeatedly within the calculation of $T(G_1 \amalg G_2)$ to conclude that it is also true that $T(G_1 \amalg G_2) = st\alpha_{k(G_1)+k(G_2)}$. Then $T(G_1)T(G_2) = s\alpha_{k(G_1)} \cdot \alpha_{k(G_2)}t$, so the proposition is satisfied by $r = st$. ■

We say a parametrized Tutte polynomial on a class $\mathcal{G}$ of graphs is *multiplicative with respect to disjoint unions* if $T(G_1 \amalg G_2) = T(G_1)T(G_2)$ whenever $G_1, G_2, G_1 \amalg G_2 \in \mathcal{G}$.

Corollary 3.10. *A parametrized Tutte polynomial is multiplicative with respect to disjoint unions if and only if $\alpha_{k_1}\alpha_{k_2} = \alpha_{k_1+k_2}$ for all k_1, k_2 such that $\mathcal{G}_{k_1}, \mathcal{G}_{k_2}$ and $\mathcal{G}_{k_1+k_2}$ are nonempty.*

Proof. Observe that because $\mathcal{G}$ is closed under minors, $\mathcal{G}_k$ is nonempty if and only if $E_k \in \mathcal{G}_k$.

Certainly if T is multiplicative with respect to disjoint unions and $E_{k_1}, E_{k_2}, E_{k_1+k_2} \in \mathcal{G}$ then $\alpha_{k_1+k_2} = T(E_{k_1+k_2}) = T(E_{k_1} \amalg E_{k_2}) = T(E_{k_1})T(E_{k_2}) = \alpha_{k_1}\alpha_{k_2}$. Conversely, if $\alpha_{k_1}\alpha_{k_2} = \alpha_{k_1+k_2}$ for all k_1, k_2 such that $\mathcal{G}_{k_1}, \mathcal{G}_{k_2}$ and $\mathcal{G}_{k_1+k_2}$ are nonempty then Proposition 3.9 implies that $T(G_1 \amalg G_2) = T(G_1)T(G_2)$ for every disjoint union in $\mathcal{G}$. ■

Corollary 3.11. *Suppose $\mathcal{G}_k$ is nonempty for all $k \geq 1$. Then a parametrized Tutte polynomial on $\mathcal{G}$ is multiplicative with respect to disjoint unions if and only if $\alpha_k = \alpha_1^k$ for all $k \geq 1$.*

If G is the union of subgraphs G_1 and G_2 which share precisely one vertex and no edges then G is the *one-point union* of G_1 and G_2. The proofs of the next three results are very similar to those of Proposition 3.9 and Corollaries 3.10 and 3.11.

Proposition 3.12. *Suppose a parametrized Tutte polynomial is defined on $\mathcal{G}$. If $G_1, G_2, G \in \mathcal{G}$ and G is the one-point union of G_1 and G_2 then there is an $r \in R$ with $r \cdot \alpha_{k(G_1)}\alpha_{k(G_2)} = T(G_1)T(G_2)$ and $r \cdot \alpha_{k(G_1)+k(G_2)-1} = T(G)$.*

Corollary 3.13. *A parametrized Tutte polynomial is multiplicative with respect to one-point unions if and only if $\alpha_{k_1}\alpha_{k_2} = \alpha_{k_1+k_2-1}$ for all k_1, k_2 such that $\mathcal{G}_{k_1}, \mathcal{G}_{k_2}$ and $\mathcal{G}_{k_1+k_2-1}$ are nonempty.*

Corollary 3.14. *Suppose $\mathcal{G}_k$ is nonempty for all $k \geq 1$. Then a parametrized Tutte polynomial on $\mathcal{G}$ is multiplicative with respect to one-point unions if and only if $\alpha_1 = \alpha_1^2$, $\alpha_1\alpha_2 = \alpha_2$, and $\alpha_k = \alpha_2^{k-1}$ for all $k \geq 2$.*

A parametrized Tutte polynomial T on $\mathcal{G}$ is a *strong Tutte function* if $T(G_1)T(G_2) = T(G)$ whenever $G, G_1, G_2 \in \mathcal{G}$ are graphs whose associated circuit matroids satisfy $M = M_1 \oplus M_2$. Such a parametrized Tutte polynomial must be multiplicative with respect to both disjoint unions and one-point unions. In general the combination of these two multiplicative properties is not enough to guarantee that T is a strong Tutte function, both because $\mathcal{G}$ need not contain disjoint or one-point unions of its elements and because there are many ways for different graphs to have the same circuit matroid. (For instance, $\mathcal{G}$ might contain G_1, G_2 and $G_1 \amalg G_2 \amalg E_2$ but not contain $G_1 \amalg G_2$ itself.) However we do have the following.

Corollary 3.15. *Suppose T is a parametrized Tutte polynomial on a minor-closed class $\mathcal{G}$ of graphs. Suppose further that $\mathcal{G}_k$ is nonempty for every $k \geq 1$ and that $\mathcal{G}$ is closed under one-point unions and the removal of isolated vertices. Then the following are equivalent.*

(a) T is a strong Tutte function.

(b) T is multiplicative with respect to both disjoint unions and one-point unions.

(c) The initial value $\alpha_1 = T(E_1)$ is idempotent, and $\alpha_k = \alpha_1$ $\forall k \geq 1$.

Proof. We have already observed that (a) implies (b), and Corollaries 3.11 and 3.14 indicate that (b) implies (c).

Suppose now that (c) holds. Observe that if C is a connected component of $G \in \mathcal{G}$ then contracting all the edges of G not in C results in a graph $C' = C \amalg E_{k(G)-1} \in \mathcal{G}$; by hypothesis we may deduce that $C \in \mathcal{G}_1$ by removing $k(G) - 1$ isolated vertices from C'. It follows that every $G \in \mathcal{G}$ has the same circuit matroid as some $G' \in \mathcal{G}_1$, obtained by taking a one-point union of the connected components of G. Proposition 3.7 shows that (c) implies $T(G) = T(G')$ under such circumstances.

It follows that T is a strong Tutte function on $\mathcal{G}$ if and only if the restriction $T|\mathcal{G}_1$ is a strong Tutte function on $\mathcal{G}_1$. To show that (c) implies that $T|\mathcal{G}_1$ is a strong Tutte function it remains only to observe that according to Proposition 3.5, $T|\mathcal{G}_1$ must induce a well-defined parametrized Tutte polynomial on the class $\mathcal{M}_1$ of circuit matroids of graphs in $\mathcal{G}_1$, and then Corollary 2.6 implies that this induced parametrized Tutte polynomial is a strong Tutte function because E_1 is the only graph in $\mathcal{G}_1$ with $\emptyset$ as its circuit matroid. ■

4. Proof of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids

The generalized Zaslavsky-Bollobás-Riordan theorem for matroids could be proven using [2] as a model if the activities expansion of Proposition 2.11 were used as a definition instead of Definition 2.3. We choose to give a proof modeled on [19] instead, because this proof seems to be shorter. We present the argument in detail so that the reader can conveniently verify that it is valid in our more general context.

Let R be a commutative ring, let $\mathcal{M}$ be a minor-closed class of matroids defined on subsets of an R-parametrized class U, and suppose we are given a parametrized Tutte polynomial T on $\mathcal{M}$. We claim that the identities given in the generalized Zaslavsky-Bollobás-Riordan theorem for matroids must hold, with $\alpha = T(\emptyset)$.

If e_1 and e_2 are digonal in $\mathcal{M}$ then there is a matroid $M \in \mathcal{M}$ whose only elements are e_1 and e_2, in which $\{e_1, e_2\}$ is the only circuit. Calculating $T(M)$ by removing e_1 through contraction and deletion before removing e_2 leads to the conclusion that $T(M) = T(\emptyset) \cdot (x_{e_1}Y_{e_2} + y_{e_1}X_{e_2})$, while calculating $T(M)$ by removing e_2 first leads to the conclusion that $T(M) = T(\emptyset) \cdot (x_{e_2}Y_{e_1} + y_{e_2}X_{e_1})$. The identity of part (a) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids follows.

If e_1, e_2 and e_3 are triangular in $\mathcal{M}$ then there is a matroid $M \in \mathcal{M}$ whose only elements are e_1, e_2 and e_3 and whose only circuit is $\{e_1, e_2, e_3\}$. Calculating $T(M)$ by removing e_1 and then e_2 leads to the conclusion that

$$T(M) = T(\emptyset) \cdot (x_{e_1}x_{e_2}Y_{e_3} + x_{e_1}y_{e_2}X_{e3} + y_{e_1}X_{e_2}X_{e_3}),$$

while calculating $T(M)$ by removing e_2 and then e_1 leads to the conclusion that

$$T(M) = T(\emptyset) \cdot (x_{e_2}x_{e_1}Y_{e_3} + x_{e_2}y_{e_1}X_{e3} + y_{e_2}X_{e_1}X_{e_3});$$

comparing these we see that

$$T(\emptyset) \cdot X_{e_3} \cdot (x_{e_1}y_{e_2} + y_{e_1}X_{e_2}) = T(\emptyset) \cdot X_{e_3} \cdot (x_{e_2}y_{e_1} + y_{e_2}X_{e_1}).$$

The identity of part (a) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids holds, because e_1 and e_2 are parallel in M/e_3; hence

$$\begin{aligned} & T(\emptyset) \cdot X_{e_3} \cdot (x_{e_1}Y_{e_2} + y_{e_1}X_{e_2}) - T(\emptyset) \cdot X_{e_3} \cdot (x_{e_1}y_{e_2} + y_{e_1}X_{e_2}) \\ = \quad & T(\emptyset) \cdot X_{e_3} \cdot (x_{e_2}Y_{e_1} + y_{e_2}X_{e_1}) - T(\emptyset) \cdot X_{e_3} \cdot (x_{e_2}y_{e_1} + y_{e_2}X_{e_1}). \end{aligned}$$

The identity of part (b) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids follows.

If e_1, e_2 and e_3 are triadic in $\mathcal{M}$ then there is a matroid $M \in \mathcal{M}$ whose only elements are e_1, e_2 and e_3 and whose circuits are $\{e_1, e_2\}$, $\{e_1, e_3\}$, and $\{e_2, e_3\}$. Calculating $T(M)$ by removing e_1 and then e_2 leads to the conclusion that

$$T(M) = T(\emptyset) \cdot (x_{e_1}Y_{e_2}Y_{e_3} + y_{e_1}x_{e_2}Y_{e3} + y_{e_1}y_{e_2}X_{e_3}),$$

while calculating $T(M)$ by removing e_2 and then e_1 leads to the conclusion that

$$T(M) = T(\emptyset) \cdot (x_{e_2}Y_{e_1}Y_{e_3} + y_{e_2}x_{e_1}Y_{e3} + y_{e_2}y_{e_1}X_{e_3}).$$

Comparing these we see that

$$T(\emptyset) \cdot Y_{e_3} \cdot (x_{e_1}Y_{e_2} + y_{e_1}x_{e_2}) = T(\emptyset) \cdot Y_{e_3} \cdot (x_{e_2}Y_{e_1} + y_{e_2}x_{e_1});$$

this is the identity of part (c) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids.

Now let R be a commutative ring and let $\mathcal{M}$ be a minor-closed class of matroids defined on subsets of an R-parametrized class U. Suppose $\alpha \in R$ and the identities of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids all hold in R. We claim that there is a $T : \mathcal{M} \to R$ which has $T(\emptyset) = \alpha$ and satisfies Definition 2.3.

Suppose instead that there is no such function T, and let $M \in \mathcal{M}$ have as few elements as possible in a matroid for which $T(M)$ is not uniquely defined by Definition 2.3. We may then refer unambiguously to $T(N)$ for any proper minor N of M. If e is a loop of M then e is also a loop of every minor of M which contains it, so every possible computation using Definition 2.3 has the same result $Y_e \cdot T(M - e)$, a contradiction. Similarly, if any element of M is a coloop then $T(M)$ is uniquely defined by Definition 2.3.

Consequently we may presume that no element of M is a loop or coloop. Certainly M must have at least two elements, because Definition 2.3 is unambiguous for matroids with fewer than two elements. If M has exactly two elements then since neither is a loop or coloop they must be parallel, and the identity of part (a) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids guarantees that $T(M)$ is unambiguously defined. If M has exactly three elements then since none is a loop or coloop, they are either triangular or triadic and the identities of parts (a), (b) and (c) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids guarantee that $T(M)$ is unambiguously defined.

Hence M has at least four elements. Choose any element $e_0 \in E(M)$, and let $D = \{e \in E(M)$ such that $y_e T(M - e) + x_e T(M/e) = y_{e_0} T(M - e_0) + x_{e_0} T(M/e_0)\}$. As $T(M)$ is not uniquely defined by Definition 2.3, M must have at least one element $f \notin D$.

We recall that according to Lemma 3.3 of [19], the fact that M has no loops or coloops implies that if e and f are any two elements of M then e cannot be a loop in $M - f$, and e is a coloop in $M - f$ if and only if f is a coloop in $M - e$; if this occurs then e and f are in series. Dually, if e and f are any two elements of M then e cannot be a coloop in M/f, and e is a loop in M/f if and only if f is a loop in M/e; if this occurs then e and f are parallel.

Let $e \in D$ and $f \notin D$. If f is neither a loop nor a coloop in M/e or $M - e$ then

$$\begin{aligned}
& y_e T(M-e) + x_e T(M/e) \\
= \;& y_e y_f T(M-e-f) + y_e x_f T((M-e)/f) + x_e y_f T((M/e)-f) + x_e x_f T((M/e)/f) \\
= \;& y_f \cdot (y_e T(M-f-e) + x_e T((M-f)/e)) + x_f \cdot (y_e T((M/f)-e) + x_e T((M/f)/e)) \\
= \;& y_f T(M-f) + x_f T(M/f),
\end{aligned}$$

a contradiction.

Clearly it is not possible for a non-parallel series pair and a parallel pair to share exactly one element: the minor obtained by deleting the rest of M would have rank 2 and contain a parallel pair, so the third element would be a coloop; however an element in series with a coloop must be a coloop, an impossibility for an element of a parallel pair. It follows that either every $e \in D$ is parallel to every $f \notin D$ (in which case the elements of M are all parallel to each other) or every $e \in D$ is in series with every $f \notin D$ (in which case M is a single circuit).

Suppose the elements of M are all parallel. Let $f \notin D$. Then

$$\begin{aligned}
& y_f T(M-f) + x_f T(M/f) \\
= \;& y_f \cdot (y_{e_0} T(M-f-e_0) + x_{e_0} T((M-f)/e_0)) + x_f \cdot T(\emptyset) \cdot \left(\prod_{e \neq f} Y_e \right) \\
= \;& y_f y_{e_0} T(M-f-e_0) + y_f x_{e_0} \cdot T(\emptyset) \cdot \left(\prod_{e_0 \neq e \neq f} Y_e \right) + x_f Y_{e_0} \cdot T(\emptyset) \cdot \left(\prod_{e_0 \neq e \neq f} Y_e \right) \\
= \;& y_{e_0} y_f T(M-e_0-f) + (y_f x_{e_0} + x_f Y_{e_0}) \cdot T(\emptyset) \cdot \left(\prod_{e_0 \neq e \neq f} Y_e \right).
\end{aligned}$$

The identity of part (c) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids then implies that

$$\begin{aligned}
& y_f T(M-f) + x_f T(M/f) \\
= \;& y_{e_0} y_f T(M-e_0-f) + (y_{e_0} x_f + x_{e_0} Y_f) \cdot T(\emptyset) \cdot \left(\prod_{e_0 \neq e \neq f} Y_e \right) \\
= \;& y_{e_0} y_f T(M-e_0-f) + y_{e_0} x_f \cdot T(\emptyset) \cdot \left(\prod_{e_0 \neq e \neq f} Y_e \right) + x_{e_0} \cdot T(\emptyset) \cdot \left(\prod_{e \neq e_0} Y_e \right) \\
= \;& y_{e_0} T(M-e_0) + x_{e_0} T(M/e_0),
\end{aligned}$$

contradicting the assumption that $f \notin D$.

A similar contradiction is obtained from the identity of part (b) of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids if M is a single circuit. ■

5. Generalized activities

If we adopt the convention that $M - e = M/e$ whenever e is a loop or coloop of M, then a parametrized Tutte polynomial may be calculated using the following modification of Definition 2.3: $T(M) = (X_e - a)T(M/e) + aT(M - e)$ for any coloop e of $M \in \mathcal{M}$ and any $a \in R$, $T(M) = aT(M - e) + (Y_e - a)T(M/e)$ for any loop e of $M \in \mathcal{M}$ and any $a \in R$, and $T(M) = y_e T(M - e) + x_e T(M/e)$ for any other element e of $M \in \mathcal{M}$. (N.b. The use of the single letter a is not meant to imply that a single value of a must be used in all instances of the definition; rather each individual equation involving a must be true for all values of a in R.) Different sets of choices of a in this definition result in different *generalized activities expansions* of $T(M)$. For instance, taking $a = 0$ for every coloop e and $a = Y_e$ for every loop e leads to the activities formula of Proposition 2.11. Several such expansions of the ordinary Tutte polynomial were introduced in [6], and the notion was extended to the coloured Tutte polynomial of [2] in [11].

In particular, suppose we calculate $T(M)$ using the modified version of Definition 2.3, with $a = X_e - x_e$ for every coloop e and $a = y_e$ for every loop e. The calculation may follow a linear order of $E = E(M)$, like the calculation in the discussion of Proposition 2.11, but it need not. We associate each term of the resulting sum with the set S of elements of M which were contracted in obtaining that term, and we let $EI(S) = \{e \in E - S$ which are deleted as isthmuses in obtaining this term$\}$ and $IL(S) = \{e \in S$ which are contracted as loops in obtaining this term$\}$. The calculation results in the *subset expansion* of $T(M)$:

$$T(M) = \alpha \cdot \sum_{S \subseteq E} \left(\prod_{e \in EI(S)} (X_e - x_e) \right) \left(\prod_{e \in E-S-EI(S)} y_e \right) \left(\prod_{e \in IL(S)} (Y_e - y_e) \right) \left(\prod_{e \in S-IL(S)} x_e \right).$$

In general this formula cannot be simplified. However, if it happens that the summands can be rewritten to allow the multiplication of like factors in the products over $EI(S)$ and $IL(S)$, rather than the given factors (which vary with e), then the subset expansion can be considerably simplified using the fact that $|IL(S)| = |S| - r(S)$ and $|EI(S)| = r(M) - r(S)$ are the nullity and corank of S, respectively. Such a rewriting is sometimes possible for an appropriate multiple of $T(M)$.

Proposition 5.1. *Suppose $e_0 \in U$, $M \in \mathcal{M}$ and for every $e \in E = E(M)$ there is a triangle or a triad of $\mathcal{M}$ which contains both e and e_0. Then there is an element $Z \in R$, which is a product of some parameters X_f and Y_f, such that*

$$
\begin{aligned}
& Zx_{e_0}^{|E|}y_{e_0}^{|E|} \cdot T(M) \\
= \quad & Z\alpha \cdot \sum_{S \subseteq E} x_{e_0}^{|E|-|S|+r(S)} \left(\prod_{e \in S} x_e \right) (X_{e_0} - x_{e_0})^{r(M)-r(S)} \cdot y_{e_0}^{|E|-r(M)+r(S)} \left(\prod_{e \in E-S} y_e \right) (Y_{e_0} - y_{e_0})^{|S|-r(S)}.
\end{aligned}
$$

Proof. If $e \in E$ then e appears in a triangle or triad of $\mathcal{M}$ with e_0 and some other $f \in U$. According to the generalized Zaslavsky-Bollobás-Riordan theorem for matroids,

$$\alpha \cdot (x_e Y_{e_0} + y_e X_{e_0}) = \alpha \cdot (x_{e_0} Y_e + y_{e_0} X_e)$$

and

$$\alpha Z_f \cdot (x_e Y_{e_0} + y_e x_{e_0}) = \alpha Z_f \cdot (Y_e x_{e_0} + x_e y_{e_0})$$

for at least one $Z_f \in \{X_f, Y_f\}$. These two equations imply that

$$\alpha Z_f x_{e_0} \cdot (Y_e - y_e) = \alpha Z_f x_e \cdot (Y_{e_0} - y_{e_0})$$

and

$$\alpha Z_f y_{e_0} \cdot (X_e - x_e) = \alpha Z_f y_e \cdot (X_{e_0} - x_{e_0}).$$

Let Z be a product which includes at least one Z_f for each $e \in E$. Then using the subset expansion of

$T(M)$, we see that

$$
\begin{aligned}
& Zx_{e_0}^{|E|}y_{e_0}^{|E|}\cdot T(M) \\
= \; & \alpha\cdot\sum_{S\subseteq E} Zy_{e_0}^{|E|}\cdot\left(\prod_{e\in EI(S)}(X_e-x_e)\right)\left(\prod_{e\in E-S-EI(S)} y_e\right)x_{e_0}^{|E|}\left(\prod_{e\in IL(S)}(Y_e-y_e)\right)\left(\prod_{e\in S-IL(S)} x_e\right) \\
= \; & \alpha\cdot\sum_{S\subseteq E} Z\cdot\left(\prod_{e\in EI(S)}(X_{e_0}-x_{e_0})\right)\left(\prod_{e\in E-S} y_e\right)y_{e_0}^{|E|-|EI(S)|}\cdot x_{e_0}^{|E|}\left(\prod_{e\in IL(S)}(Y_e-y_e)\right)\left(\prod_{e\in S-IL(S)} x_e\right) \\
= \; & \alpha\cdot\sum_{S\subseteq E}(X_{e_0}-x_{e_0})^{|EI(S)|}\cdot y_{e_0}^{|E|-|EI(S)|}\left(\prod_{e\in E-S} y_e\right)\cdot Zx_{e_0}^{|E|}\left(\prod_{e\in IL(S)}(Y_e-y_e)\right)\left(\prod_{e\in S-IL(S)} x_e\right) \\
= \; & \alpha\cdot\sum_{S\subseteq E}(X_{e_0}-x_{e_0})^{|EI(S)|}\cdot y_{e_0}^{|E|-|EI(S)|}\left(\prod_{e\in E-S} y_e\right)\cdot Z\cdot\left(\prod_{e\in IL(S)}(Y_{e_0}-y_{e_0})\right)x_{e_0}^{|E|-|IL(S)|}\left(\prod_{e\in S} x_e\right) \\
= \; & \alpha\cdot\sum_{S\subseteq E}(X_{e_0}-x_{e_0})^{|EI(S)|}\cdot y_{e_0}^{|E|-|EI(S)|}\left(\prod_{e\in E-S} y_e\right)\cdot Z\cdot(Y_{e_0}-y_{e_0})^{|IL(S)|}\cdot x_{e_0}^{|E|-|IL(S)|}\left(\prod_{e\in S} x_e\right).
\end{aligned}
$$

The result follows from the equalities $|IL(S)|=|S|-r(S)$ and $|EI(S)|=r(M)-r(S)$. ■

If x_{e_0}, y_{e_0} or Z is a zero divisor in R then a corank-nullity formula for $T(M)$ itself cannot be derived from Proposition 5.1. However if it happens that there is a Z as described in the proof which is not a zero divisor in R, then of course Z may be cancelled. Moreover, if x_{e_0} and y_{e_0} are not zero divisors in R then R may be enlarged by adjoining multiplicative inverses for them through localization, so we may as well assume that these multiplicative inverses are present in R. Under these circumstances Proposition 5.1 yields a corank-nullity formula for $T(M)$ which generalizes the corresponding formula for the classical Tutte polynomial.

Corollary 5.2. *Suppose $e_0\in U$ and both x_{e_0} and y_{e_0} have multiplicative inverses in R; let $u=(X_{e_0}-x_{e_0})y_{e_0}^{-1}$ and $v=(Y_{e_0}-y_{e_0})x_{e_0}^{-1}$. Suppose further that $M\in\mathcal{M}$ and for every $e\in E=E(M)$ there is either a triangle $\{e,e_0,f\}$ of $\mathcal{M}$ such that X_f is not a zero divisor in R or a triad $\{e,e_0,f\}$ such that Y_f is not a zero divisor in R. Then*

$$T(M)=\alpha\cdot\sum_{S\subseteq E}(\prod_{e\in S}x_e)(\prod_{e\in E-S}y_e)u^{r(M)-r(S)}v^{|S|-r(S)}.$$

Proof. The equality is obtained by cancelling Z from the equality of Proposition 5.1 and replacing $(X_{e_0}-x_{e_0})$ by uy_{e_0} and $(Y_{e_0}-y_{e_0})$ by vx_{e_0}. ■

6. When R is a field

In [19] Zaslavsky classifies the field-valued strong Tutte functions which are defined on domains in which all triples of elements are triadic and triangular, and all pairs of elements are digonal; there are seven possible types. If the domain is coloured – that is, for every $e_1 \in U$ there are $e_2, e_3 \in U$ with $e_1 \neq e_2 \neq e_3 \neq e_1$ such that x_{e_i}, y_{e_i}, X_{e_i}, and Y_{e_i} do not vary with i – then there are only four types. In this section we briefly describe the derivation of these classifications from the generalized Zaslavsky-Bollobás-Riordan theorem. The subset expansion of Section 5 gives formulas for the various types.

Suppose then that $\mathcal{M}$ is a minor-closed class of matroids in which all triples are triangular and triadic, and all pairs of elements are digonal. Suppose we are given a parametrized Tutte polynomial on $\mathcal{M}$ with values in a field R. As in Corollary 5.2 these hypotheses could be weakened to assumptions that $\mathcal{M}$ has sufficiently many triangles and triads (rather than all possible triangles and triads) and that sufficiently many elements of R are not zero divisors, but to keep the discussion simple we leave such sharpening to the interested reader. Moreover we could assume that T is a strong Tutte function with no loss of generality, for if T is not identically 0 then $\alpha^{-1} \cdot T$ is strong, according to Proposition 2.5 and Corollary 2.6; however this would not simplify our discussion, so we do not assume T is strong.

If $T(\emptyset) = 0$ then of course T is identically 0. If $X_e = 0 = Y_e$ for all $e \in U$ then $T(M) = 0$ for all $M \neq \emptyset$ by Proposition 2.4. These are the *nil* parametrized Tutte polynomials.

Assume that T is not nil and there are $e_1, e_2 \in U$ such that $X_e = Y_e = 0$ for all $e \notin \{e_1, e_2\}$. (This is impossible if $\mathcal{M}$ is coloured.) The generalized Zaslavsky-Bollobás-Riordan theorem for matroids implies that for every $e \notin \{e_1, e_2\}$,

$$x_{e_1} Y_e + y_{e_1} X_e = 0 = x_e Y_{e_1} + y_e X_{e_1};$$

consequently the vectors (x_e, y_e) and (Y_{e_1}, X_{e_1}) are perpendicular. Similarly, the vectors (x_e, y_e) and (Y_{e_2}, X_{e_2}) are perpendicular for every $e \notin \{e_1, e_2\}$. These parametrized Tutte polynomials are *pointlike* if $e_1 = e_2$ and *pairlike* if $e_1 \neq e_2$; there are two types of the latter, according to whether or not (Y_{e_1}, X_{e_1}) and (Y_{e_2}, X_{e_2}) are collinear.

Assume that T is not nil and there are at least three different $e \in U$ such that at least one of X_e, Y_e is nonzero. If $x_e = 0 = y_e$ for all $e \in U$ then clearly $T(M) \neq 0$ only if every element of M is either a loop or a coloop. These parametrized Tutte polynomials are *paranil*.

Assume that T is not nil or paranil, there are at least three different $e \in U$ such that at least one of X_e, Y_e is nonzero, and it is not true that $x_e = 0 = y_e$ for all $e \in U$. The generalized Zaslavsky-Bollobás-Riordan theorem for matroids then implies that for all $e_1 \neq e_2 \in U$

$$\begin{aligned} x_{e_1}Y_{e_2} + y_{e_1}X_{e_2} &= x_{e_2}Y_{e_1} + y_{e_2}X_{e_1} \\ \text{and } x_{e_1}Y_{e_2} + y_{e_1}x_{e_2} &= Y_{e_1}x_{e_2} + x_{e_1}y_{e_2}, \end{aligned}$$

and consequently

$$y_{e_1}(X_{e_2} - x_{e_2}) = y_{e_2}(X_{e_1} - x_{e_1}).$$

If $x_e = 0$ for all $e \in U$ then it follows that $y_{e_1}X_{e_2} = y_{e_2}X_{e_1}$ for all $e_1 \neq e_2 \in U$, and hence there is a single element $u \in R$ such that $X_e = uy_e$ for every $e \in U$. Similarly, if $y_e = 0$ for all $e \in U$ then there is a single element $v \in R$ such that $Y_e = vx_e$ for every $e \in U$. These parametrized Tutte polynomials are *elementary*.

If $x_e = 0$ for all $e \in U$ then the subset expansion of Section 5 implies

$$\begin{aligned} T(M) &= \alpha \cdot \sum_{S \subseteq E} \left(\prod_{e \in EI(S)} (X_e - x_e) \right) \left(\prod_{e \in E-S-EI(S)} y_e \right) \left(\prod_{e \in IL(S)} (Y_e - y_e) \right) \left(\prod_{e \in S-IL(S)} x_e \right) \\ &= \alpha \cdot \sum_{S=IL(S) \subseteq E} u^{|EI(S)|} \left(\prod_{e \in E-S} y_e \right) \left(\prod_{e \in S} (Y_e - y_e) \right). \end{aligned}$$

If we use E_0 to denote the set of loops of M, then the subsets $S \subseteq E$ with $S = IL(S)$ are simply the subsets of E_0; such a subset has $r(S) = 0$ and hence $|EI(S)| = r(M) - r(S) = r(M)$. Consequently

$$\begin{aligned} T(M) &= \alpha \cdot \sum_{S \subseteq E_0} u^{|EI(S)|} \left(\prod_{e \in E-S} y_e \right) \left(\prod_{e \in S} (Y_e - y_e) \right) \\ &= \alpha u^{r(M)} \cdot \left(\prod_{e \in E-E_0} y_e \right) \cdot \sum_{S \subseteq E_0} \left(\prod_{e \in E_0 - S} y_e \right) \left(\prod_{e \in S} (Y_e - y_e) \right) \\ &= \alpha u^{r(M)} \cdot \left(\prod_{e \in E-E_0} y_e \right) \cdot \left(\prod_{e \in E_0} (Y_e - y_e + y_e) \right) \\ &= \alpha u^{r(M)} \cdot \left(\prod_{e \in E-E_0} y_e \right) \cdot \left(\prod_{e \in E_0} Y_e \right). \end{aligned}$$

Similarly, if $y_e = 0$ for all $e \in U$ then the subset expansion of Section 5 implies

$$T(M)=\alpha v^{|E|-r(M)}\cdot\left(\prod_{e\in E-E_1}x_e\right)\cdot\left(\prod_{e\in E_1}X_e\right)$$

where E_1 denotes the set of coloops of M.

If it is not true that $x_e=0$ for all $e\in U$ or that $y_e=0$ for all $e\in U$ then it follows that there are $u,v\in R$ with $X_e-x_e=uy_e$ and $Y_e-y_e=vx_e$ for all $e\in U$. These parametrized Tutte polynomials are *normal*; they are given by the corank-nullity formula of Corollary 5.2.

7. Parallel and series connections

Suppose M_1 and M_2 are matroids on sets S_1 and S_2, with $S_1\cap S_2=\{e_0\}$. Suppose further that e_0 is neither a loop nor a coloop in either M_1 or M_2. Then the *parallel connection* of M_1 and M_2 is the matroid M on $E=S_1\cup S_2$ whose circuits are the elements of $C(M)=C(M_1)\cup C(M_2)\cup\{C_1\cup C_2-\{e_0\}\mid e_0\in C_1\in C(M_1)$ and $e_0\in C_2\in C(M_2)\}$; $M-e_0$ is the 2-*sum* of M_1 and M_2. It is a useful fact that the classical Tutte polynomials of M and $M-e_0$ are related to those of M_1 and M_2; for instance in [8] the formula

$$(1-uv)T(M-e_0)=T(M_2/e_0)T(M_1-e_0)+T(M_2-e_0)T(M_1/e_0)-uT(M_1/e_0)T(M_2/e_0)-vT(M_2-e_0)T(M_1-e_0)$$

(derived from [3]) is used to show that the classical Tutte polynomial can be computed in polynomial time on certain classes of matroids of bounded width. In this section we observe that these relationships do not generalize fully to parametrized Tutte polynomials. We consider only parallel connections, leaving the dual case of series connections to the reader.

Proposition 7.1. *Suppose a parametrized Tutte polynomial is defined on $\mathcal{M}$ and $M\in\mathcal{M}$ is the parallel connection of M_1 and M_2. Then there are $r,s\in R$ with*

$$T(M_2)=(rX_{e_0}+sY_{e_0})\cdot T(\emptyset)$$

and

$$T(M)=(rx_{e_0}+sY_{e_0})\cdot T(M_1/e_0)+ry_{e_0}\cdot T(M_1-e_0).$$

Proof. Let $E=\{e_0,...,e_m\}$ with $S_1=\{e_0,...,e_i\}$ and $S_2=\{e_0,e_{i+1},...,e_m\}$. As in the proof of Proposition 2.4 we may calculate $T(M_2)$ using Definition 2.3, first removing e_m through deletion and contraction, then removing e_{m-1}, and so on. We conclude that $T(M_2) = (rX_{e_0} + sY_{e_0}) \cdot T(\emptyset)$, with $rX_{e_0}T(\emptyset)$ resulting from the portions of the calculation in which e_0 is not a loop at the time of its removal and $sY_{e_0}T(\emptyset)$ resulting from the portions of the calculation in which e_0 is a loop at the time of its removal.

If we compare this calculation of $T(M_2)$ to a corresponding partial calculation of $T(M)$ we see that the portions of the calculations in which e_0 is or is not a loop at the time of its removal coincide. The portions of the partial calculation in which e_0 is not a loop at the time of its removal produce $rT(M_1)$ rather than $rX_{e_0}T(\emptyset)$, and the portions of the partial calculation in which e_0 is a loop at the time of its removal produce $sT((M_1/e_0) \oplus L)$ rather than $sY_{e_0}T(\emptyset)$, where L is the matroid on $\{e_0\}$ in which e_0 is a loop. Consequently $T(M) = rT(M_1) + sY_{e_0}T(M_1/e_0)$. ■

Suppose $\mathcal{M}$ contains a matroid M_1' which is defined on $S_1' = \{e_0', e_1, ..., e_i\}$, is isomorphic to M_1 under the obvious bijection between S_1 and S_1', and has $x_{e_0'} = rx_{e_0} + sY_{e_0}$ and $y_{e_0'} = ry_{e_0}$ with r and s as in Proposition 7.1. Then $T(M) = T(M_1')$. As S_1 may be considerably smaller than E, calculating $T(M_1')$ may be considerably easier than calculating $T(M)$ directly. However $\mathcal{M}$ might not contain such an M_1', and the fact that r and s are not uniquely determined makes it difficult to ascertain whether or not $\mathcal{M}$ does contain such an M_1'. This leads us to consider the following extension of the discussion of [13].

Suppose $M \in \mathcal{M}$ is the parallel connection of M_1 and M_2 with respect to an element e_0 which is neither a loop nor a coloop in either M_1 or M_2. Let $e_0'' \notin U$, and let $U'' = (S_1 - \{e_0\}) \cup \{e_0''\}$. For each minor N of M_1 let N'' be an isomorphic copy of N which differs from N only in that if $e_0 \in N$ then e_0 is replaced by e_0'' in N''; let $\mathcal{M}'' = \{N'' \mid N \text{ is a minor of } M_1\}$. Parametrize U'' by restricting the parameter functions of U to $S_1 - \{e_0\}$ and defining

$$
\begin{aligned}
x_{e_0''} &= x_{e_0}(x_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_2/e_0) + x_{e_0}y_{e_0}^2 T(M_2 - e_0), \\
y_{e_0''} &= y_{e_0}^2 x_{e_0} T(M_2/e_0) + y_{e_0}^2 (y_{e_0} - Y_{e_0}) T(M_2 - e_0), \\
X_{e_0''} &= (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_2), \text{ and} \\
Y_{e_0''} &= Y_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_2/e_0).
\end{aligned}
$$

Proposition 7.2. *Suppose that (i) whenever $e_0, e_1, e_2 \in S_1$ are triadic or triangular in $\mathcal{M}$ they are both triadic and triangular in $\mathcal{M}$ and (ii) whenever $e_0, e_1 \in S_1$ are digonal in $\mathcal{M}$ there is an $e_2 \in S_2 - \{e_0\}$ such* *that e_0, e_1 and e_2 are both triangular and triadic in $\mathcal{M}$. Then the identities of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids hold in $\mathcal{M}''$, using $\alpha = T(\emptyset)$. The resulting parametrized Tutte polynomial T'' has $T''(\emptyset) = T(\emptyset)$ and*

$$T''(M_1'') = (X_{e_0} y_{e_0} + x_{e_0} Y_{e_0} - X_{e_0} Y_{e_0}) T(\emptyset) T(M).$$

Proof. The proof follows the same outline as the argument of [13]. The most difficult step is the derivation of the digon identities of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids in $\mathcal{M}''$; the key to this derivation is the fact that hypothesis (ii) and Proposition 2.4 imply that if $e_0, e_1 \in S_1$ are digonal in $\mathcal{M}''$ then $T(M_2 - e_0) \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2}) = T(M_2 - e_0) \cdot (Y_{e_1} x_{e_2} + x_{e_1} y_{e_2})$ and $T(M_2/e_0) \cdot (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2}) = T(M_2/e_0) \cdot (Y_{e_1} x_{e_2} + x_{e_1} y_{e_2})$. We do not include the details here; the interested reader may find them posted on the authors' web pages. ■

[[[Here are details.

To verify that the identities of the generalized Zaslavsky-Bollobás-Riordan theorem for matroids hold in $\mathcal{M}''$ we must verify that if $\{e_0, e_1\}$ is a digon in $\mathcal{M}''$ then $T(\emptyset) \cdot (X_{e_0''} y_{e_1} + Y_{e_0''} x_{e_1}) = T(\emptyset) \cdot (X_{e_1} y_{e_0''} + Y_{e_1} x_{e_0''})$, if $\{e_0, e_1, e_2\}$ is a triangle or triad in $\mathcal{M}''$ then $T(\emptyset) \cdot Z_{e_2} x_{e_1} (Y_{e_0''} - y_{e_0''}) = T(\emptyset) \cdot Z_{e_2} x_{e_0''} (Y_{e_1} - y_{e_1})$, where Z_{e_2} is X_{e_2} or Y_{e_2}, and if $\{e_0, e_1, e_2\}$ is a triangle or triad in $\mathcal{M}''$ then $T(\emptyset) \cdot Z_{e_0''} x_{e_1} (Y_{e_2} - y_{e_2}) = T(\emptyset) \cdot Z_{e_0''} x_{e_2} (Y_{e_1} - y_{e_1})$, where $Z_{e_0''}$ is $X_{e_0''}$ or $Y_{e_0''}$.

By hypothesis (i) the third of these identities follows immediately from the fact that $Z_{e_0''}$ is a sum of a multiple of X_{e_0} and a multiple of Y_{e_0}.

The second identity is

$$\begin{aligned}
& T(\emptyset) \cdot Z_{e_2} x_{e_1} (Y_{e_0''} - y_{e_0''}) \\
= \; & T(\emptyset) \cdot Z_{e_2} x_{e_1} (Y_{e_0} X_{e_0} y_{e_0} + x_{e_0} Y_{e_0}^2 - X_{e_0} Y_{e_0}^2 - y_{e_0}^2 x_{e_0}) T(M_2/e_0) - T(\emptyset) \cdot Z_{e_2} x_{e_1} (y_{e_0} - Y_{e_0}) y_{e_0}^2 T(M_2 - e_0) \\
= \; & T(\emptyset) \cdot Z_{e_2} x_{e_1} (Y_{e_0} X_{e_0} (y_{e_0} - Y_{e_0}) + x_{e_0} (Y_{e_0}^2 - y_{e_0}^2)) T(M_2/e_0) - T(\emptyset) \cdot Z_{e_2} x_{e_0} (y_{e_1} - Y_{e_1}) y_{e_0}^2 T(M_2 - e_0) \\
= \; & T(\emptyset) \cdot Z_{e_2} x_{e_1} (Y_{e_0} - y_{e_0}) (-Y_{e_0} X_{e_0} + x_{e_0} (Y_{e_0} + y_{e_0})) T(M_2/e_0) + T(\emptyset) \cdot Z_{e_2} x_{e_0} (Y_{e_1} - y_{e_1}) y_{e_0}^2 T(M_2 - e_0) \\
= \; & T(\emptyset) \cdot Z_{e_2} x_{e_0} (Y_{e_1} - y_{e_1}) (-Y_{e_0} X_{e_0} + x_{e_0} (Y_{e_0} + y_{e_0})) T(M_2/e_0) + T(\emptyset) \cdot Z_{e_2} x_{e_0} (Y_{e_1} - y_{e_1}) y_{e_0}^2 T(M_2 - e_0) \\
= \; & T(\emptyset) \cdot Z_{e_2} x_{e_0''} (Y_{e_1} - y_{e_1}).
\end{aligned}$$

Here $T(\emptyset)\cdot Z_{e_2}x_{e_1}(y_{e_0}-Y_{e_0}) = T(\emptyset)\cdot Z_{e_2}x_{e_0}(y_{e_1}-Y_{e_1})$ and $T(\emptyset)\cdot Z_{e_2}x_{e_1}(Y_{e_0}-y_{e_0}) = T(\emptyset)\cdot Z_{e_2}x_{e_0}(Y_{e_1}-y_{e_1})$ are simply the corresponding triangle or triad identity in $\mathcal{M}$.

The digon identity is much harder to verify. Observe that

$$\begin{aligned}
&T(\emptyset)\cdot(X_{e_0''}y_{e_1}+Y_{e_0''}x_{e_1})\\
&= T(\emptyset)\cdot(Y_{e_0}x_{e_1}+x_{e_0}y_{e_1})(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})T(M_2/e_0)+T(\emptyset)\cdot y_{e_0}y_{e_1}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})T(M_2-e_0)\\
&= T(\emptyset)\cdot X_{e_0}Y_{e_0}x_{e_1}(y_{e_0}-Y_{e_0})T(M_2/e_0)+T(\emptyset)\cdot Y_{e_0}x_{e_1}x_{e_0}Y_{e_0}T(M_2/e_0)\\
&\quad + T(\emptyset)\cdot x_{e_0}y_{e_1}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})T(M_2/e_0)+T(\emptyset)\cdot y_{e_0}(X_{e_0}y_{e_0}y_{e_1}+Y_{e_0}y_{e_1}(x_{e_0}-X_{e_0}))T(M_2-e_0).
\end{aligned}$$

By hypothesis (ii) and Proposition 2.4, $T(M_2-e_0)$ and $T(M_2/e_0)$ are sums of multiples of $T(\emptyset)\cdot X_{e_2}$ and $T(\emptyset)\cdot Y_{e_2}$ for some $e_2\in S_2-\{e_0\}$ which is triangular and triadic with e_0 and e_1 in $\mathcal{M}$. This implies that $x_{e_1}(y_{e_0}-Y_{e_0})T(M_2/e_0) = x_{e_0}(y_{e_1}-Y_{e_1})T(M_2/e_0)$ and $y_{e_1}(x_{e_0}-X_{e_0})T(M_2-e_0) = y_{e_0}(x_{e_1}-X_{e_1})T(M_2-e_0)$, so

$$\begin{aligned}
&T(\emptyset)\cdot(X_{e_0''}y_{e_1}+Y_{e_0''}x_{e_1})\\
= \;& T(\emptyset)\cdot X_{e_0}Y_{e_0}x_{e_0}(y_{e_1}-Y_{e_1})T(M_2/e_0)+T(\emptyset)\cdot Y_{e_0}x_{e_1}x_{e_0}Y_{e_0}T(M_2/e_0)\\
& +T(\emptyset)\cdot x_{e_0}y_{e_1}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})T(M_2/e_0)+T(\emptyset)\cdot y_{e_0}(X_{e_0}y_{e_0}y_{e_1}+Y_{e_0}y_{e_0}(x_{e_1}-X_{e_1}))T(M_2-e_0)\\
= \;& T(\emptyset)\cdot(-X_{e_0}Y_{e_0}x_{e_0}Y_{e_1}+Y_{e_0}x_{e_1}x_{e_0}Y_{e_0}+x_{e_0}y_{e_1}(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}))T(M_2/e_0)\\
& +T(\emptyset)\cdot y_{e_0}(y_{e_0}(X_{e_0}y_{e_1}+Y_{e_0}x_{e_1})-Y_{e_0}y_{e_0}X_{e_1})T(M_2-e_0).
\end{aligned}$$

The triad, triangle and digon identities in $\mathcal{M}$ imply $(x_{e_0}y_{e_1}+x_{e_1}Y_{e_0})T(M_2/e_0) = (x_{e_1}y_{e_0}+Y_{e_1}x_{e_0})T(M_2/e_0)$, $(X_{e_0}y_{e_1}+Y_{e_0}x_{e_1})T(M_2/e_0) = (X_{e_1}y_{e_0}+Y_{e_1}x_{e_0})T(M_2/e_0)$ and $(X_{e_0}y_{e_1}+Y_{e_0}x_{e_1})T(M_2-e_0) = (X_{e_1}y_{e_0}+$

$Y_{e_1}x_{e_0})T(M_2-e_0)$. Hence

$$\begin{aligned}
& T(\emptyset)\cdot(X_{e_0''}y_{e_1}+Y_{e_0''}x_{e_1}) \\
= \quad & T(\emptyset)\cdot(-X_{e_0}Y_{e_0}x_{e_0}Y_{e_1}+x_{e_0}y_{e_1}X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}(x_{e_1}y_{e_0}+Y_{e_1}x_{e_0}))T(M_2/e_0) \\
& +T(\emptyset)\cdot y_{e_0}(y_{e_0}(X_{e_1}y_{e_0}+Y_{e_1}x_{e_0})-Y_{e_0}y_{e_0}X_{e_1})T(M_2-e_0) \\
= \quad & T(\emptyset)\cdot(-X_{e_0}Y_{e_0}x_{e_0}Y_{e_1}+x_{e_0}y_{e_0}(y_{e_1}X_{e_0}+Y_{e_0}x_{e_1})+x_{e_0}Y_{e_0}Y_{e_1}x_{e_0})T(M_2/e_0) \\
& +T(\emptyset)\cdot y_{e_0}(y_{e_0}Y_{e_1}x_{e_0}+y_{e_0}X_{e_1}(y_{e_0}-Y_{e_0}))T(M_2-e_0) \\
= \quad & T(\emptyset)\cdot(-X_{e_0}Y_{e_0}x_{e_0}Y_{e_1}+x_{e_0}y_{e_0}(X_{e_1}y_{e_0}+Y_{e_1}x_{e_0})+x_{e_0}Y_{e_0}Y_{e_1}x_{e_0})T(M_2/e_0) \\
& +T(\emptyset)\cdot y_{e_0}(y_{e_0}Y_{e_1}x_{e_0}+y_{e_0}X_{e_1}(y_{e_0}-Y_{e_0}))T(M_2-e_0) \\
= \quad & T(\emptyset)\cdot(x_{e_0''}Y_{e_1}+y_{e_0''}X_{e_1}).
\end{aligned}$$

This verifies the identities of the generalized Z-B-R theorem in $\mathcal{M}''$.

It remains to prove $T''(M_1'')=(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})T(\emptyset)T(M)$.

Suppose $|S_1|=2$, and $S_1=\{e_0,e_1\}$. Then e_0 and e_1 are parallel (and in series) in M_1 and M_1'', and hence

$$\begin{aligned}
T''(M_1'') \quad = \quad & (x_{e_1}Y_{e_0''}+y_{e_1}X_{e_0''})T(\emptyset) \\
= \quad & (X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})(x_{e_1}Y_{e_0}T(M_2/e_0)+y_{e_1}T(M_2))T(\emptyset) \\
= \quad & (X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})((x_{e_1}Y_{e_0}+y_{e_1}x_{e_0})T(M_2/e_0)+y_{e_1}y_{e_0}T(M_2-e_0))T(\emptyset) \\
= \quad & (X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})((x_{e_1}Y_{e_0}+y_{e_1}x_{e_0})T((M-e_1)/e_0)+y_{e_1}y_{e_0}T(M-e_1-e_0))T(\emptyset) \\
= \quad & (X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})(y_{e_1}x_{e_0}T((M-e_1)/e_0)+y_{e_1}y_{e_0}T(M-e_1-e_0))T(\emptyset) \\
& +(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})(x_{e_1}Y_{e_0}T((M-e_1)/e_0))T(\emptyset) \\
= \quad & (X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})y_{e_1}T(M-e_1)+(X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})(x_{e_1}Y_{e_0}T((M/e_1)-e_0))T(\emptyset) \\
= \quad & (X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})(y_{e_1}T(M-e_1)+x_{e_1}T(M/e_1))T(\emptyset) \\
= \quad & (X_{e_0}y_{e_0}+x_{e_0}Y_{e_0}-X_{e_0}Y_{e_0})T(M)T(\emptyset).
\end{aligned}$$

Suppose now that $|S_1|>2$. If there is an $e_1\in S_1$ which is neither parallel to nor in series with e_0 then Proposition 7.2 may be assumed to hold for both M/e_1 and $M-e_1$, and its validity for M follows using the deletion-contraction formula.

If there is an $e_1 \in S_1$ which is parallel to e_0 then we may assume inductively that Proposition 7.2 holds for $M - e_1$, i.e., $T''(M_1'' - e_1) = (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M - e_1)T(\emptyset)$. On the other hand,

$$\begin{aligned} & T''(M_1''/e_1) \\ = \quad & Y_{e_0''}T''((M_1''/e_1) - e_0) \\ = \quad & Y_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_2/e_0)T((M_1/e_1) - e_0) \\ = \quad & Y_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_2/e_0)T((M_1/e_1) - e_0) \\ = \quad & (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_2/e_0)Y_{e_0}T((M_1/e_1) - e_0). \end{aligned}$$

As M is the parallel connection of M_1 and M_2, M/e_0 is the direct sum of M_2/e_0 and M_1/e_0. As e_1 is parallel to e_0, M_1/e_0 is the direct sum of a loop e_1 and $(M_1/e_0) - e_1$. Moreover, M/e_1 is isomorphic to M/e_0, so M/e_1 is the direct sum of M_2/e_0, a loop e_0, and $(M_1/e_0) - e_1 = (M_1/e_1) - e_0$. It follows that $T(M/e_1)T(\emptyset) = T(M_2/e_0)Y_{e_0}T((M_1/e_1) - e_0)$ and hence $T''(M_1''/e_1) = (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M/e_1)T(\emptyset)$. The validity of Proposition 7.2 for M follows, using the deletion-contraction formula.

Finally, suppose e_0 is in series with every $e_1 \in S_1$, i.e., M_1 is a circuit. Choose a particular $e_1 \neq e_0 \in S_1$. We may assume inductively that Proposition 7.2 holds for M/e_1. $M - e_1$ is the direct sum of M_2 and $M_1 - e_1 - e_0$, so $T(M - e_1)T(\emptyset) = T(M_1 - e_1 - e_0)T(M_2)$. $M_1'' - e_1$ is the direct sum of $\{e_0\}$ and $M_1'' - e_1 - e_0 = M_1 - e_1 - e_0$, so $T''(M_1'' - e_1) = T''(M_1 - e_1 - e_0)X_{e_0''} = T(M_1 - e_1 - e_0)(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_2)$, and we see that Proposition 7.2 holds for $M - e_1$. The validity of Proposition 7.2 for M follows from the deletion-contraction formula.

End of details.]]]

Corollary 7.3. *In the circumstances of Proposition 7.2, the 2-sum $M - e_0$ has*

$$\begin{aligned} & y_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(\emptyset)T(M - e_0) \\ = \quad & x_{e_0}y_{e_0}^2 T(M_2 - e_0)T(M_1/e_0) + x_{e_0}y_{e_0}^2 T(M_2/e_0)T(M_1 - e_0) \\ & + y_{e_0}^2(y_{e_0} - Y_{e_0})T(M_2 - e_0)T(M_1 - e_0) + x_{e_0}y_{e_0}(x_{e_0} - X_{e_0})T(M_1/e_0)T(M_2/e_0). \end{aligned}$$

Proof. Observe that $M/e_0 = (M_1/e_0) \oplus (M_2/e_0)$ and hence

$$
\begin{aligned}
& y_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(\emptyset)T(M - e_0) \\
= \quad & (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(\emptyset)(T(M) - x_{e_0}T(M/e_0)) \\
= \quad & T''(M_1'') - x_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_1/e_0)T(M_2/e_0) \\
= \quad & x_{e_0''}T''(M_1''/e_0'') + y_{e_0''}T''(M_1'' - e_0'') - x_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_1/e_0)T(M_2/e_0) \\
= \quad & x_{e_0''}T(M_1/e_0) + y_{e_0''}T(M_1 - e_0) - x_{e_0}(X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0})T(M_1/e_0)T(M_2/e_0) \\
= \quad & x_{e_0}y_{e_0}^2T(M_2 - e_0)T(M_1/e_0) + y_{e_0}^2x_{e_0}T(M_2/e_0)T(M_1 - e_0) + y_{e_0}^2(y_{e_0} - Y_{e_0})T(M_2 - e_0)T(M_1 - e_0) \\
& +x_{e_0}((x_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0}) - (X_{e_0}y_{e_0} + x_{e_0}Y_{e_0} - X_{e_0}Y_{e_0}))T(M_1/e_0)T(M_2/e_0). \blacksquare
\end{aligned}
$$

The somewhat unnatural hypotheses of Proposition 7.2 and Corollary 7.3 reflect the fact that triangle and triad identities of $\mathcal{M}$ are used to verify the digon identities of $\mathcal{M}''$. We hope that an ingenious reader will improve these results by finding more convenient re-parametrizations of e_0''.

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