THE LOTTERY: A DREAM COME TRUE OR A TAX ON PEOPLE WHO ARE BAD AT MATH?

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ABSTRACT: We present a lottery project for lower level mathematics courses that demonstrates financial decision making in a context familiar to students. Students use elementary principals of probability and financial mathematics to compare the expected values of buying Powerball lottery tickets to some basic savings plans. Then, they compare the expected values of the current lottery game with its previous versions. We also provide suggestions for extending the project for more advanced courses and an annotated resource list.

KEYWORDS: Mathematics of finance, finite mathematics, lottery, expected value, annuity, probability, group project, practical applications.

INTRODUCTION

We have all dreamed of winning the big prize, one of the astronomical jackpots offered by some multi-state lotteries. We revel in rags-to-riches stories of big lottery jackpot winners, such as the fifty-five year old construction worker who, on Christmas day of 2002, won the $314.9 million Powerball jackpot, believed to be the largest prize ever won by a single ticket. However, shattering these glamorous fantasies is one of our favorite bumper-stickers, which reads "The lottery: A tax on people who are bad at math". We use these contrasting points of view as discussion starters in our classes, along with such questions as: Who has ever bought a lottery ticket? What do you think your chances of winning are? How big does the jackpot have to be before it is worth playing? Is the lottery just a big rip-off? What if you didn't play the lottery, but instead invested your money elsewhere? Was the Powerball lottery better before recent rule changes, or after? Why?

To address these questions, we have developed a lottery project for our lower level mathematics courses. Specifically, we discuss the Powerball lottery, which is played in 26 states, including our home state of Vermont. In this lottery, players currently choose five out of 53 numbered white balls, and one of 42 red balls. A player who matches three, four, or five white balls, either with or without matching the red ball, wins a prize, as does a player matching both the red ball and zero, one, or two of the white balls. A player may also choose a "Power Play" option at the time of ticket purchase that will multiply any winnings by a factor of two, three, four, or five. The size of the white and red ball sets have varied over the years, affecting the winning probabilities. Comparing the probabilities and expected values as the Powerball game has evolved provides students with experience in how mathematics can reveal financial consequences not apparent to the casual consumer.

Lotteries provide an excellent opportunity to use elementary financial mathematics, as well as some probability, in a context familiar to students. Most of our students do not have major financial decisions to make, so the principles of financial mathematics may seem far removed from their lives. However, nearly all of them have played the lottery at one time or another, and thus the topic readily engages them.

For the project, students work in groups to address several questions related to the Powerball lottery. First, they consider the lottery on a personal level: Should they buy lottery tickets or use their money otherwise? To do this, they use basic combinatorics to determine the probability of winning any of the first through ninth prizes currently awarded by the lottery. Then, they compute the expected value of buying a ticket, and compare the likely results of buying a ticket daily to a few other simple investments of their money. Students actually change their behavior (or claim to!) based on their analysis, and it is rewarding to watch them integrate course material into their everyday decision making, albeit in a minor way.

Students then begin analyzing more complex issues. They compare expected values of buying a Powerball ticket from before and after the Powerball rule changes in November 1997 and October 2002. This strongly reinforces and develops conceptual and computational skills, while allowing students to discover for themselves how an organization can manipulate its policies in its own best interests with repercussions for the unwary consumer. The students become convinced of the necessity of probabilistic analysis to see who profits from the lottery and by exactly how much. Ideally, they will later apply this conviction to other financial transactions in their lives.

We typically use this project in Finite Mathematics, a lower level mathematics course for non-majors. Our students are understandably far more interested in their chosen fields than in mathematics, and their basic computational skills tend to be weak. We find that this lottery project meets four very important criteria for providing a positive mathematical experience for such students. It analyzes spending money on an attractive, media-sensationalized, topical, and ubiquitous game, so students find it both relevant and engaging. It provides a concrete application for the specific concepts (basic interest formulas, expected value, and combinations) included in the curriculum. It involves computations at a level that stretches our students without prohibitively frustrating them. Finally, it builds critical thinking skills from which they will benefit long after completing the course.

Lotteries also provide rich resources for more advanced investigations. There are several directions for further research, such as examining sociological aspects of the lottery, for students who would like to address related questions with tools from other disciplines. Also, lottery-related topics such as survey design and analysis as well as assessing winning strategies would be suitable for courses with a greater emphasis on statistics or more sophisticated combinatorial techniques. We provide a list at the end of this paper of some project extension ideas and lottery resources available for further investigation.

NECESSARY BACKGROUND

Students need only elementary concepts of combinations, probability, expected value, and interest formulas to complete this project. We briefly outline them here.

1. *Combinatorics*: The multiplication principle for independent choices, and ${}_nC_r = \frac{n!}{(n-r)!r!}$.

2. *Probability*: $p(B)$, the probability of B occurring, is the number of ways B can occur divided by the total number of outcomes in the sample space. The probability of B not occurring, in other words, the probability that its complement B^c does occur, is $p(B^c) = 1 - p(B)$.

3. *Expected value*: Expected Value $= \sum p(B)\, v(B)$, where the sum is taken over all possible outcomes B and where $v(B)$ is the value of the outcome B. To make this formal definition more accessible to our

students, we also simply make a table with the probabilities in one column and the values in the other, take the product in each row, and then sum the results.

4. *Financial Formula*: Future Value Ordinary Annuity= pymt$\frac{((1+i)^n-1)}{i}$, where pymt is the amount of deposit, i is the periodic interest rate (so i is the given rate divided by the number of compoundings per year), and n is the total number of payments made.

THE PROJECT

The Multi-State Lottery Association publishes the following table listing the chance of winning each prize under the current Powerball game matrix of 5/53 and 1/42, where a player selects 5 of 53 white balls, and 1 of 42 red balls [7]. In the first part of the lottery project, students simply verify the odds for each prize. Since odds are often defined as the ratio of success to failure, we clarify for our students that the numbers in the table actually represent each prize probability, i.e. the ratio of success to total number of outcomes.

Match	Prize	Odds
5 white and 1 red	Grand Prize	1 in 120,526,770.00
5 white	$100,000	1 in 2,939,677.32
4 white and 1 red	$5,000	1 in 502,194.88
4 white	$100	1 in 12,248.66
3 white and 1 red	$100	1 in 10,685.00
3 white	$7	1 in 260.61
2 white and 1 red	$7	1 in 696.85
1 white and 1 red	$4	1 in 123.88
1 red	$3	1 in 70.39

Students then use this information to compute the expected value of a one-dollar lottery ticket assuming a 10 million dollar grand prize. From this, they estimate their expected losses if they played the lottery daily for ten years. They compare that loss to the benefit gained from investing the same amount of money into an ordinary annuity for ten years, or into a matched retirement account. Finally, they compare the expected value of the currnet game matrix to those of two previous version s of the Powerball lottery.

Questions to guide the analysis:

1. Determine the number of possible tickets in a given Powerball prize drawing. Then, verify the odds of winning each prize given in the table.

2. Determine the probabilities of matching exactly two white balls and exactly one white ball in Powerball without a matching red ball. In these cases, why do you think that prizes are not given?

3. How accurate is the claim on the website that 1 in 36.06 tickets is a winner? What percentage of tickets are winners? What percentage are losers? What does this mean? For example, if everyone in a class of 35 people bought a ticket, about how many winners would you expect?

4. Compute the expected value of a $1 ticket under the current Powerball rules. Remember that you have to buy the ticket, so that the value of, for example, a $5,000 prize, is actually $4,999.

5. Assuming a constant 10 million dollar jackpot, if you buy one lottery ticket every day for the next ten years, how much money would you expect to win or lose?

6. How much money would you have if you put $1 into a jar in your room every day for the next ten years?

7. How much money would you have after ten years if you put $1 per day into an ordinary annuity that earns 3.25% compounded daily?

8. Suppose that, instead of buying lottery tickets, you put $30 per month extra into your retirement account (an ordinary annuity). Further suppose that your company matches your contributions—i.e. they put in an additional $30 whenever you do. Assume that the retirement fund earns 7% compounded monthly. How much extra money will be in your account after ten years?

9. Compare, reflect, and comment upon your answers to questions 5 through 8.

10. After November of 1997 and previous to October 2002, the Powerball game matrix was 5/49 and 1/42 with prizes awarded according to the following table:

Outcome	Prize	Odds
5 white and 1 red	Grand Prize	1 in 80,089,128
5 white	$100,000	1 in 1,953,393
4 white and 1 red	$5,000	1 in 364,042
4 white	$100	1 in 8,879
3 white and 1 red	$100	1 in 8,466
3 white	$7	1 in 207
2 white and 1 red	$7	1 in 605
1 white and 1 red	$4	1 in 118
1 red	$3	1 in 74

Compute the expected value of a $1 ticket under this game format, assuming a 10 million dollar grand prize.

11. Previous to November of 1997, the Powerball game matrix was 5/45 and 1/45 with prizes awarded according to the following table:

Outcome	Prize	Odds
5 white and 1 red	Grand Prize	1 in 54,979,155
5 white	$100,000	1 in 1,249,526
4 white and 1 red	$5,000	1 in 274,896
4 white	$100	1 in 6,248
3 white and 1 red	$100	1 in 7,049
3 white	$5	1 in 160
2 white and 1 red	$5	1 in 556
1 white and 1 red	$2	1 in 120
1 red	$1	1 in 84

Compute the expected value of a $1 ticket under this game format, assuming a 10 million dollar grand prize.

12. Compare, reflect, and comment upon your answers to questions 5, 10, and 11.

Partial Solutions:

(Students appreciate being able to check their answers to questions 1 and 3 against information posted on the Powerball website.)

1. The total number of ways of choosing five of 53 white balls and one of 42 red balls is ${}_{53}C_5 \times {}_{42}C_1 = 120{,}526{,}770$. As an example of one of the winning outcomes, the number of ways of matching 4 white balls

and 1 red balls is: ${}_5C_4 \times {}_{53-5}C_1 \times 1 = 240$ (choose 4 of the 5 correct white balls, 1 of the remaining 48 incorrect white balls, and choose the single correct red ball). Then, the probability of winning a $5,000 prize is 240/(120,526,770), which is almost exactly 1/(502,194.88), so the claim is verified.

2. The probability of matching exactly two white balls is 0.05884, and the probability of matching exactly one white ball is 0.33095. If prizes were awarded for these outcomes, such high probabilities of occurring would diminish profits for the lottery commission.
3. The claim that 1/36.06, or about 2.77% of the tickets are winners is accurate, and can be verified by adding the probabilities of winning each of the prizes. This means that about 97.23% of the tickets are losers, so most of the time there would be only one winner (of any value prize) if a class of 35 people bought tickets.
4. The expected value of $−0.74372 for a $1 ticket under the current rules is computed by multiplying the values (each reduced by $1) and probabilities from the given table, including the losing outcome having value -$1 with probability 0.972271, then summing the result.
5. $10 \times 365 \times -0.57778167 = -2{,}109$, so a loss of $2,109 over ten years.
6. $3,650.
7. Using the Future Value Ordinary Annuity formula, with 3,650 compounding periods, a periodic rate of 0.0325/365, and a payment of $1, gives $4,312.73 after ten years.
8. Using the Future Value Ordinary Annuity formula, with 120 compounding periods, a periodic rate of 0.07/12, and a payment of $60, gives $10,385.09 additional retirement money after ten years.
9. The guaranteed return of any of the savings plans is clearly more advantageous than the expected loss from playing the lottery habitually.
10. The expected value of $−0.66708 for a $1 ticket during the period from November 1997 to October 2002 is computed as in question 3, except with a losing probability of 0.971234.
11. The expected value of $−0.62091 for a $1 ticket during the period previous to November 1997 is computed as in question 3, except with a losing probability of 0.971375.
12. Notice that as the rules changed, the expected values have decreased from $−0.62091 to $−0.66708 to $−0.74372, indicating a trend of diminishing rewards for lottery players and increasing profits for the lottery commission.

Note: In questions 4, 10, and 11, there may be small discrepancies in expected values depending upon whether students use the approximate odds provided in the Powerball tables or the actual probabilities of each outcome they computed themselves. Also, prior to October 2002, Powerball only posted integer values in the odds table.

DIRECTIONS FOR FURTHER INVESTIGATION

A wealth of material is available for customizing or extending this lottery project, for either an entire course or the individual student. This enhancement could take the form of interdisciplinary work with the fields of political science, sociology, or economics, for example, or by using more advanced tools from either statistics or combinatorics to do further analysis.

More advanced students may consider additional constraints when determining expected values. For example, they can determine the effects of the Power Play option, taxes, larger jackpots, and multiple winners on the expected values of buying a lottery ticket. Snell and Peterson have written a "Chance Profile" [8] which contains helpful information for these enrichment ideas, including how Poisson distributions can be used to model diminished jackpot size due to multiple winners. Furthermore, to predict the length of runs between jackpot winners, Bernoulli trials or Markov chain models could be used, as is detailed in R. Iltis' "Runs with No Winner in a Lottery" [2].

Of perennial interest are various schemes to maximize chances of winning lottery jackpots. For example, in P. Shenkin and A. Wieschenberg's article "The Sure Thing" [9], the authors describe a New York State Lotto scenario in which a bettor purchases all possible tickets, and they analyze the results when multiwinners and taxes are considered. A *Chance Newsletter* written by T. Tarpey [10] describes an actual lottery strategy considered by residents of a small Ohio town to maximize their returns in the Ohio Super Lotto game in hopes of being able to generate sufficient funds to protect a local green area. "Winning the lottery" by C. Colbourn [1] provides a good, brief survey of some sophisticated mathematical techniques applied to lottery schemes and lottery wheels, which are systems for improving odds when playing the lottery by buying as few tickets as possible to guarantee some kind of win—although, as he points out, these systems do "... not in any way improve your expected return, just your minimum return." The techniques used, such as codewords and designs, could be suitable for an upper level combinatorics class.

As a single Internet search for "lottery" immediately reveals, there is a vast amount of lottery-related material on the web, ranging from quite helpful to mathematically questionable or even illegal. The various regional lottery commissions are generally reliable. They are good initial sources of information for questions such as: How is the money raised by this particular lottery used? and Who plays the lottery? The lottery commissions publish annual reports that include financial statements, player demographics, and commission activities. They also generally provide links to a variety of regional lotteries, and comparing these to determine which games in the area have the greatest expected value can be an entertaining exercise. The lottery commission in our own state [11] provides information about the lottery games and their proceeds in our own state, and similar websites can be found for other states. Using this information, we sometimes focus our lottery project on some of the smaller state and regional games, such as Tri-State Megabucks (with its 6/42 game matrix) and Tri-State Cash Lotto (with its 4/33 and 1/33 game matrix). These Tri-State lottery games can be played in Maine, New Hampshire, and Vermont, and similar state and regional games can be found across the country. Focusing our lottery project on such games provides variety for our courses from year to year with an activity of local interest.

Detailed information can be gathered about other multi-state lottery games, such as the "Mega Millions" game [4], which is available in eleven states. In fact, like Powerball, Mega Millions has seen changes in its structure over the years. Specifically, since 2002, its game matrix has been 5/52 and 1/52. Previously, under the title "The Big Game", its pre-1999 game matrix was 5/50 and 1/25, and between 1999 and 2002 it was 5/50 and 1/36. In some semesters, we have changed our lottery project focus from Powerball to Mega Millions, which provides more variety to our offerings, ensures that students in successive semesters encounter different projects, and relates to topical events in our region (since a neighboring Mega Millions state often advertises its "mega jackpots" in our area).

An excellent source for generic lottery data is the North American Association of State and Provincial Lotteries [5]. This comprehensive site includes information, resources, and links for lottery history, gambling studies and research, and problem gambling. The National Gambling Impact Study Commission Final Report [6] studied all types of gambling, not just lotteries, and is concerned with the governance and regulation of gambling as well as evaluation of its impact on the population. Not surprisingly, the commission has its detractors, among them in fact the NASPL, which responded to the report in 1999 [5].

Because lotteries, and gambling in general, generate considerable controversy, sites may exhibit a bias, and this can form the basis for productive classroom discussion with questions such as: How is this site displaying data? How is it interpreting the data? How well are claims supported? The NASPL response to the NGISC, which questions both survey data and statistical analysis in fairly strong terms, could be well integrated into a course with statistics content.

CONCLUSION

For a concrete sense of what a 1 in 120 million chance of winning that amazing Powerball jackpot is, we like to compare the probability of winning it with some other unlikely events. Larry Laudan's *The Book of Risks: Fascinating Facts About the Chances We Take Everyday* [3] provides the following probability estimates of other unlikely events:

Event	Probability	Comparison to Powerball Jackpot
Winning Current Powerball Jackpot	1/120,526,770	—
Freezing to Death	1/3,000,000	40 times as likely
Being Killed Falling Out of Bed	1/2,000,000	60 times as likely
Being Killed by an Animal	1/2,000,000	60 times as likely
Being Dealt a Royal Flush in Poker	1/649,740	185.5 times as likely
Being Killed by Lightning	1/600,000	201 times as likely
Being Electrocuted	1/350,000	344 times as likely
Dying in a Plane Crash	1/250,000	482 times as likely
Dying from Poisoning	1/86,000	1401 times as likely
Dying from Surgical Complications	1/80,000	1507 times as likely
Dying in a Car Crash	1/5,000	24,105 times as likely

Our students do seem to conclude from their analysis in this project that the lottery really is a tax on people who are bad at mathematics rather than a shortcut to easy street, and that they would be wiser to find a better use for their money. However, as we tell our students, if you play the lottery anyway and lose, we told you so. On the other hand, if you should happen to win, we would gladly accept a 50% commission for demonstrating with mathematical precision exactly how lucky you were! Please send it to the authors at the address provided.

ANNOTATED REFERENCES

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4. Mega Millions. 2004. [Homepage of Mega Millions]. [Online]. Available: `http://www.theofficialbiggame.com/default.asp` [2004, May 8].

5. NASPL. 2004. [Homepage of NASPL]. [Online]. Available: `http://www.naspl.org/` , with 1999 letter from NASPL President to NGISC at `http://www.naspl.org/ccanalys.html` [2004, May 8].

> The NASPL was founded in 1971, and has now grown into an active association of state and provincial lotteries representing 47 lottery organizations throughout North America. Its basic mission remains the same as when it was founded: "to assemble and disseminate information and benefits of state and provincial lottery organizations through education and communications and where appropriate publicly advocate the positions of the Association on matters of general policy."

6. NGISC Final Report. 1999. [Homepage of NGISC Final Report]. [Online]. Available: `http://govinfo.library.unt.edu/ngisc/reports/fullrpt.html` [2004, May 8].

> In 1997 the National Gambling Impact Study Commission was charged by Congress with "a very broad and very difficult task – to conduct a comprehensive legal and factual study of the social and economic implications of gambling in the United States". This final report contains the principal findings of the Commission, and its recommendations for a coherent framework of action.

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8. Snell, J. L. 1999. [Homepage of the Dartmouth College Chance course]. [Online]. Available: `http://www.dartmouth.edu/~chance/`, with 1998 lottery article at `http://www.dartmouth.edu/~chance/teaching_aids/Profiles/using_lotteries.pdf` [2004, May 8].

This article describes how lotteries can be used in a Chance course. Chance is a "quantitative literacy course developed cooperatively by the Chance Team: J. Laurie Snell and Peter Doyle of Dartmouth College, Joan Garfield of the University of Minnesota, Tom Moore of Grinnell College, Bill Peterson of Middlebury College, and Ngambal Shah of Spelman College. The goal of Chance is to make students more informed, critical readers of current news stories that use probability and statistics."

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Chance News is a newsletter that reviewed articles in the news that teachers of probability and statistics might want to use in their classes. The Archives provide all issues of *Chance News* from 1992 to 2003.

11. Vermont State Lottery Commission. 2004. [Homepage of the Vermont Lottery]. [Online]. Available: `http://www.vtlottery.com/cgi-bin/index.pl` [2004, May 8].

BIOGRAPHICAL SKETCHES

George Ashline received his BS from St. Lawrence University, his MS from the University of Notre Dame, and his PhD from the University of Notre Dame in 1994 in value distribution theory. He has taught at St. Michael's College since 1995. He is a participant in Project NExT, a program created for new or recent PhD's in the mathematical sciences who are interested in improving the teaching and learning of undergraduate mathematics. He is also actively involved in professional development programs in mathematics for elementary and middle school teachers.

Joanna Ellis-Monaghan received her BA from Bennington College, her MS from the University of Vermont, and her PhD from the University of North Carolina at Chapel Hill in 1995 in algebraic combinatorics. She has taught at the University of Vermont, and at St. Michael's College since 1992. She is a committed proponent of active learning and has developed materials, projects, and activities to augment a variety of courses.