Generalized Transition Polynomials

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Abstract

We construct a one variable graph polynomial, $q(G,W;x)$, on the space of Eulerian graphs G with weight system W. This polynomial generalizes the transition polynomial of Jaeger from 4-regular Eulerian graphs to all Eulerian graphs. Furthermore, we provide a Hopf algebraic structure for the space of weighted Eulerian graphs and show that $q(G,W;x)$ is a Hopf algebra map. Many polynomials, such as the Tutte, Penrose, and Martin polynomials, as well as the Kaufman bracket of knot theory and Bollobas and Riordan's Tutte polynomial for coloured graphs, can be formulated, at least partially, as evaluations of $q(G,W;x)$. Thus, the comultiplication and antipode of the Hopf algebra give new tools for analyzing these polynomials as well.

Introduction

In "On Transition Polynomials of 4-Regular Graphs", [Jae90], Jaeger develops a graph polynomial, $Q(G,A,\tau)$, for 4-regular graphs which unifies common characteristics of the Martin, Penrose and (via the medial graph of a planar graph) Tutte polynomial. In the current paper, we extend this polynomial, with slight modification, to $q(G,W;x)$, a generalized transition polynomial for all Eulerian graphs. Furthermore, and this is the central result of this paper, we give a Hopf algebra structure for the space of Eulerian graphs with weight systems and show that $q(G,W;x)$ is a Hopf algebra map from this space to the binomial bialgebra $R[x]$, where R is a commutative ring with unit.

The comultiplication and antipode in particular give tools that can be used to analyze this polynomial.

Jaeger [Jae90] shows that, if the weight systems are chosen appropriately, $Q(G,A,\tau)$ gives the Martin polynomials of oriented and unoriented 4-regular graphs ([Las83], [Mar77]), the Penrose polynomial ([Aig97], [Pen69]) of a planar graph via the medial graph, the Tutte polynomial ([Tut67]) of a planar graph when $x=y$ (again via the medial graph), and the Kauffman bracket ([Kau87]) of knot theory. Since $q(G,W;x)$ generalizes $Q(G,A,\tau)$, it also gives these polynomials. Furthermore, since $q(G,W;x)$ is defined for all Eulerian graphs, it also gives the Martin polynomials for all oriented and unoriented Eulerian graphs, and, for planar graphs, it can be shown to agree with the Tutte polynomial for coloured graphs of Bollobas and Riordan [BR99] via the medial graph.

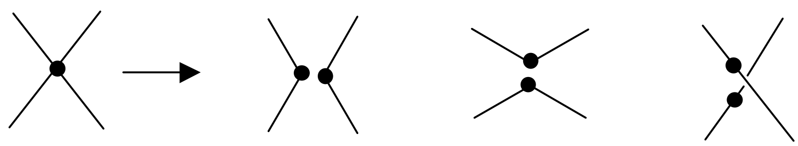
Identities from Hopf algebra structures have already been used, either directly or implicitly, to get new combinatorial interpretations for several of these polynomials (see [Bol], [ES01], [E-M99], [E-Ma], [E-Mb], [Sar01]), and it is hoped that the foundations laid here will lead to more such results.

Preliminaries

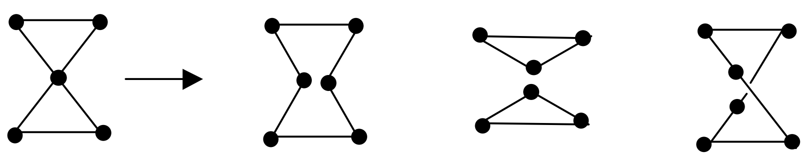
The following conventions are used throughout this paper. Graphs may have loops and multiple edges. A graph is said to be Eulerian if all its vertices have even degrees, but connectedness is not required. A cycle is a graph isomorphic to a polygon. We permit *free loops*, which consist of an edge in the shape of a circle, but with no vertex. The set of free loops of a graph *G* is denoted $fl(G)$. The notation $k(G)$ will be used to indicate the number of components of *G*, *not* counting isolated vertices. Two sets of terminologies appear in the literature for the concepts we discuss here, one using *transition systems* (see [Fle90] for example), and one using *state models* (see [Kau87] for example). We try to indicate corresponding terms in the definitions.

For an edge *e*, we denote the half edge of *e* that is incident to a vertex *v* by e_v. We distinguish the two half edges of a loop. The set of half edges of a graph *G* is denoted by $h(G)$. A *pairing* at a vertex *v* is a choice of two half edges incident to *v*.

A *vertex state*, or *transition*, is a choice of local reconfiguration of a graph at a vertex by pairing the half edges incident with that vertex. A vertex state typically retains the new vertices of degree 2, while a transition elides them. We use the former configuration here, although in many applications, this distinction is irrelevant. In this paper, we will only consider Eulerian graphs and those vertex states where all the half edges are paired (no singleton edges). For example, the three possible vertex states of a vertex of degree 4 are:



A *graph state*, or *transition system*, $S(G)$, is the result of choosing a vertex state at each vertex of degree greater than 2. For example, the three possible graph states of this graph are:



We will write $St(G)$ for the set of graph states of *G*, and when the graph is clear from context write *S* for $S(G)$.

A *skein-relation* for graphs is a formal sum of weighted vertex states, together with an evaluation of the terminal forms (the graph states, which here are a disjoint union of 2-regular graphs). See [E-M98] for a detailed discussion of these concepts, which are appropriated from knot theory, in their most general form, and [Yet90] for a general theory of invariants given by linear recursion relations.

A *skein-type*, (or *state model*, or *transition*) *polynomial* is one which is computed by repeated applications of skein relations. See [Jae90] for a comprehensive treatment of these in the case of 4-regular graphs.

Definition 1: A *pair weight* is an association of a value $p(e_v, e_v')$ to a pair of half edges incident with a vertex *v* in *G*. In general, we assume these values are in *R*, a commutative ring with unit.

Definition 2: A *weight system*, $W(G)$, of an Eulerian graph *G* is an assignment of a pair weight to every possible pair of adjacent half edges of *G*. We will simply write *W* for $W(G)$ when the graph is clear from context.

Definition 3: The *vertex state weight* of a vertex state is $\prod p(e_v, e'_v)$ where the product is over the pairs of half edges comprising the vertex state. Thus, associated to each vertex of degree 2*n* in *G*, there are $\prod_{i=0}^{n-1}(2n-(2i+1))$ not necessarily distinct values corresponding to each possible vertex state.

Definition 4: The *state weight* of a graph state *S* of a graph *G* with weight system *W* is $\omega(S)=\prod \omega(v,S)$, where $\omega(v,S)$ is the vertex state weight of the vertex state at *v* in the graph state *S*, and where the product is over all vertices of *G*.

Definition 5: Let *G* and *H* be Eulerian graphs with weight systems *W* and *Y*, respectively. A map $f:(G,W)\to(H,Y)$ is an *isomorphism of graphs with weight systems* if *f* is a bijection between $h(G)\cup fl(G)$ and $h(H)\cup fl(H)$ such that

1. e_v, $e'_v \in h(G)$ are incident to the same vertex *v* in *G* if and only if $f(e_v), f(e'_v)\in h(H)$ are incident to the same vertex in *H*,
2. e_v, $e_w \in h(G)$ are the two different half edges of the same edge *e* in *G* if and only if $f(e_v), f(e_w)\in h(H)$ are the two different half edges of the same edge in *H*,
3. $p(e_v,e'_v)=p(f(e_v), f(e'_v))$ for every pair of adjacent half edges in $h(G)$,
4. $e\in fl(G)\Leftrightarrow f(e)\in fl(H)$.

This map of half edges and free loops is used rather than simple graph isomorphism for two reasons. One is to distinguish the two half edges in the case of a loop so that requirement 3 means that *f* also preserves vertex and state weights. The other is to identify graphs that are isomorphic except for possibly some isolated vertices.

Note that "isomorphism of graphs with weight systems" gives an equivalence relation for graphs with weight systems. We write $(\mathcal{E},W)$ for the class of graphs with no edges (and empty weight system).

Hereafter, we identify graphs with weight systems which are isomorphic as in Definition 5, and let $\Gamma = span_R\{\ (G,W)\ \}$.

Definition 6: Let A be an Eulerian subgraph of an Eulerian graph G, and let $g: h(A) \to h(G)$ be the canonical immersion. We say that A has a weight system $W(A)$ that is *inherited* from $W(G)$ if $p(e_v, e'_v) = p(g(e_v), g(e'_v))$ for every pair of adjacent half edges in $h(A)$.

Note that if A is a subgraph of G, and B is a subgraph of A, then the weight system that B inherits as a subgraph of A is exactly the same as the weight system that it inherits as a subgraph of G.

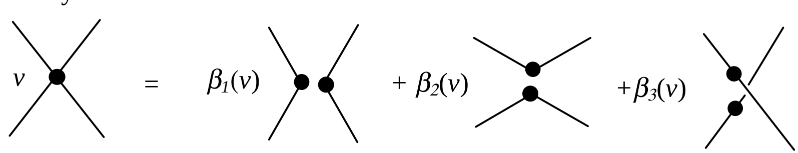
The generalized transition polynomial $q(G,W;x)$

Definition 7 (Recursive Definition):

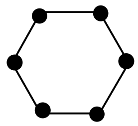
Define $q: \Gamma \to R[x]$ recursively by repeatedly applying the skein relation $q(G,W) = \sum \beta_i q(G_i, W(G_i))$ at any vertex v of degree > 2. Here the G_i's are the graphs that result from locally replacing a vertex v of degree $2n$ in G by one of its $\prod_{i=0}^{n-1}(2n-(2i+1))$ vertex states. The β_i's are the vertex state weights. Furthermore, each G_i has a weight system, $W(G_i)$, based on the canonical identification between the half edges of G_i and G as follows. For any pair of adjacent half edges in G_i which are not both part of the new vertex state (i.e. which are not both incident to v in G), the pair weight in G_i is the same as it was in G. All the pairs of half edges adjacent to the newly formed vertices of degree 2 in G_i have pair weight equal to 1, the unit in R.

Repeated application of this relation reduces G to a weighted sum of disjoint unions of cycles, (the graph states). These terminal forms are evaluated by identifying each cycle (including any free loops) with the variable x, weighted by the product of the pair weights over all pairs in the cycle.

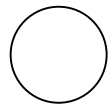
For example, if v is a vertex of degree 4, this definition can be represented pictorially as:



where each of the new vertices of degree two have pair weight 1 for the unique pair of edges incident to them, and where the terminal evaluation is then



= Cx, where C is the product of all the pair weights for each pair of adjacent edges, and



= x for free loops,

Definition 8 (State Model Definition):

$$q(G,W;x)=\sum_{St(G)}\left(\left(\prod\omega(v,S)\right)x^{k(S)}\right)=\sum_{St(G)}\omega(S)\,x^{k(S)}\,.$$

Note that since this state model definition is clearly equivalent to the recursive definition, and is independent of the order of the vertices to which the recursion is applied, $q(G,W;x)$ is well defined. In addition, since the states, both vertex and graph, are in terms of local reconfigurations, $q(G,W;x)$ is well-defined with respect to the equivalence relations within the definition of Γ.

Definition 9 (Generating Function Definition):

Collecting like terms in the preceding state model definition leads to the generating function form, $q(G,W;x)=\sum f_n(G)x^n$. Here, $f_n(G)=\sum\omega(S)$, where the sum is over all states of G with n components.

In the special case that G is 4-regular, the polynomial $q(G,W;x)$ is essentially the same as the transition polynomial $Q(G,A,\tau)$ in [Jae90]. Simply choose pair weights to give the desired vertex state weights determined by A. Specifically, if the vertex state weight in (G,A) is w, then the pair weights for each of the pairs of edges determined by the state would be $\sqrt{w}$ in (G,W). The two polynomials differ by a factor of x, and here we retain vertices of deg 2 in the recursion while they are elided in [Jae90]. Thus $q(G,W;x)$ gives a generalization of Jaeger's transition polynomials to all Eulerian graphs.

The structural properties of $q(G,W;x)$

We now turn our attention to the structural properties of $q(G,W;x)$. We first show that Γ is a Hopf algebra over R. We give $R[x]$ the structure of the binomial Hopf algebra by taking x to be primitive. We then show that $q(G,W;x)$ is a Hopf algebra homomorphism from Γ to $R[x]$. As such, both the comultiplication and antipode provide useful tools for determining properties of graph polynomials that are evaluations of $q(G,W;x)$.

PROPOSITION 10. Γ *is a Hopf algebra over R with multiplication, unit, comultiplication, counit and antipode defined (respectively) as follows:*

Multiplication: $m:\Gamma\otimes\Gamma\to\Gamma$ *by*

$m\left(\left(G,W(G)\right)\otimes\left(H,W(H)\right)\right)=\left(GH,W(GH)\right)$, *where GH is the disjoint union of G and H, and* $W(GH)$ *is the union weight system which assigns to every pair of adjacent half edges in* $h(GH)$ *the weight it originally received from* $W(G)$ *(or* $W(H)$*) as a pair of half edges of* $h(G)$ *(or* $h(H)$*).*

Unit: $\mu:R\to\Gamma$ *by* $\mu(r)=r(\mathcal{E},W)$.

Comultiplication: $\Delta:\Gamma\to\Gamma\otimes\Gamma$ *by*

$\Delta(G,W)=\sum\left(A_1,W(A_1)\right)\otimes\left(A_2,W(A_2)\right)$, *where the sum is over ordered partitions* (A_1,A_2) *of G into edge disjoint Eulerian subgraphs, and where the weight systems* $W(A_i)$ *are inherited.*

Counit: $\varepsilon:\Gamma\to R$ *by* $\varepsilon(G,W)=1$ *if* $G=\mathcal{E}$, *and 0 else.*

Antipode: For $G\neq\mathcal{E}$, $\zeta(G,W)=-\sum\left(A_1,W(A_1)\right)\zeta\left(A_2,W(A_2)\right)$, *where the sum is over ordered partitions* (A_1,A_2) *of G into edge disjoint Eulerian subgraphs such that* $A_1\neq\mathcal{E}$, *where the weight systems* $W(A_i)$ *are inherited. When* $G=\mathcal{E}$, *then* $\zeta(\mathcal{E},W)=1$.

Alternatively, $\zeta(G,W)=\sum(-1)^{|P|}\left(A_1\ldots A_{|P|},W\left(A_1\ldots A_{|P|}\right)\right)$, *where the sum is over all ordered partitions P of G into* $|P|$ *edge-disjoint Eulerian subgraphs. Here* $A_1\ldots A_{|P|}$ *is the disjoint union of the subgraphs, and* $W\left(A_1\ldots A_{|P|}\right)$ *is the union weight system as in the definition of the multiplication.*

Proof: Most of the criteria for Γ being a Hopf Algebra (see [Swe69] or [Sch94] for example—in fact Γ may be interpreted as an incidence Hopf algebra as in [Sch94]) are straightforward, so we will only prove coassociativity and that Γ is a bialgebra here.

Coassociativity is verified by showing that $(I \otimes \Delta) \circ \Delta(G,W) = (\Delta \otimes I) \circ \Delta(G,W)$. The left hand side is equal to $\sum \left((A_1, W(A_1)) \otimes \sum (B_1, W(B_1)) \otimes (B_2, W(B_2)) \right)$, where the outer sum is over all ordered partitions (A_1, A_2) of G into edge disjoint Eulerian subgraphs, and where the inner sum is over all ordered partitions (B_1, B_2) of A_2 into edge disjoint Eulerian subgraphs.

Note that if B is an Eulerian subgraph of A, and A is an Eulerian subgraph of G, then the weight system that B inherits from A is the same as the weight system that B would inherit from G as a subgraph of G. Thus, by writing C_1, C_2, C_3 for A_1, B_1, B_2, respectively, the sum can be written as $\sum (C_1, W(C_1)) \otimes (C_2, W(C_2)) \otimes (C_3, W(C_3))$, where the sum is over all ordered partitions (C_1, C_2, C_3) of G into edge disjoint Eulerian subgraphs with inherited weight systems.

A similar argument rewrites $(\Delta \otimes I) \circ \Delta(G,W)$ in the same form, which completes the verification of coassociativity.

To verify that Γ is a bialgebra, it suffices to show that $\Delta m(G_1 \otimes G_2) = m \otimes m (I \otimes T \otimes I) \Delta \otimes \Delta (G_1 \otimes G_2)$, where T is the "twist map", $T(G \otimes H) = H \otimes G$. We supress the weight systems here for ease of notation.

Now, $\Delta m(G_1 \otimes G_2) = \sum A_1 \otimes A_2$, where the sum is over all ordered partitions (A_1, A_2) of $G_1 G_2$ into edge disjoint Eulerian subgraphs. Furthermore, $m \otimes m (I \otimes T \otimes I) \Delta \otimes \Delta (G_1 \otimes G_2) = m \otimes m (I \otimes T \otimes I) \sum B_1 \otimes B_2 \otimes C_1 \otimes C_2 =$

$\sum B_1 C_1 \otimes B_2 C_2$, where the sums are over all ordered partitions (B_1, B_2) of G_1, and over all ordered partitions (C_1, C_2) of G_2. Since each A_i consists of two subgraphs, one from G_1 (call it B_i), and one from G_2 (call it C_i), each A_i can be written as B_iC_i and vice versa, so it follows that $\sum A_1 \otimes A_2 = \sum B_1C_1 \otimes B_2C_2$. ///

THEOREM 11. *$q(G,W;x)$ is a Hopf algebra map from Γ to the binomial bialgebra $R[x]$.*

Proof. Let *GH* be the disjoint union of two graphs *G* and *H* with weight systems $W(G)$ and $W(H)$, respectively. Since the skein relation can be applied first to all vertices in *G*, and then to those in *H*, it follows that $q(G,W;x)$ is multiplicative on disjoint unions of graphs in Γ, so that $q(GH,W(GH);x) = q(G,W(G);x)q(H,W(H);x)$, where $W(GH)$ is the union weight system which assigns to every pair of adjacent half edges in $h(GH)$ the weight it originally received from $W(G)$ (or $W(H)$) as a pair of half edges of $h(G)$ (or $h(H)$). (See [Jae90], prop. 5 for the degree 4 case). Thus $q:\Gamma \to R[x]$ is an algebra map. Lemma 12 below will have as a corollary that $q:\Gamma \to R[x]$ is also a coalgebra map. A bialgebra map between Hopf algebras is automatically a Hopf algebra map (see [Swe, lemma 4.04]). ///

LEMMA 12. *$q(G,W;x+y) = \sum q(A_1,W(A_1);x)q(A_2,W(A_2);y)$, where the sum is over all ordered partitions (A_1,A_2) of G into edge disjoint Eulerian subgraphs, and where the weight systems $W(A_i)$ are inherited.*

Proof: By the state model definition ,

$$q(G,W;x+y) = \sum_{St(G)} \omega(S)(x+y)^{k(S)} = \sum_{St(G)} \omega(S) \sum_{r=0}^{k(S)} \binom{k(s)}{r} x^r y^{k(S)-r} .$$

Thus, the coefficient of $x^a y^b$ in $q(G,W;x+y)$ is $\sum \omega(S)\binom{a+b}{a}$, where the sum is over $S \in St(G)$ such that $k(S) = a+b$.

By the same state model definition, $\sum q(A_1,W(A_1);x)q(A_2,W(A_2);y) =$ $\sum \left(\sum \omega(S(A_1)) x^{k(S(A_1))} \sum \omega(S(A_2)) y^{k(S(A_2))} \right)$, where the outer sum is over all ordered partitions (A_1, A_2) of G into edge disjoint Eulerian subgraphs, the first inner sum is over $St(A_1)$, and the second inner sum is over $St(A_2)$.

Thus, the coefficient of $x^a y^b$ in $\sum q(A_1, W(A_1); x) q(A_2, W(A_2); y)$ is $\sum\sum\sum \omega(S(A_1))\omega(S(A_2))$, where the outer sum is over all ordered partitions (A_1, A_2) of G into edge disjoint Eulerian subgraphs, the middle sum is over all graph states $S(A_1)$ with *a* components, and the innermost sum is over all graph states $S(A_2)$ with *b* components.

Now consider a state *S* of *G* with $a+b$ components. Choose *a* of those components to comprise $S(A_1)$. Thus, A_1 consists of the edges in those *a* components and hence is Eulerian. The state $S(A_1)$ is determined by the state *S*, as is consequently the complementary state $S(A_2)$, which necessarily has *b* components. Note that there are $\binom{a+b}{a}$ ways to do this. Thus the coefficients of $x^a y^b$ are equal on both sides of the equation if and only if $\omega(S) = \omega(S(A_1))\omega(S(A_2))$.

However, notice that $\omega(S) = \prod \omega(S, v) = \prod\prod p(e_a, e_b)$, where the outer product is over all the vertices of *G*, and the inner products is over all the pair weights of the state at *v*. Note that any two of edges that are paired in *S* appear paired in either $S(A_1)$ or $S(A_2)$, and hence the pair weight (as a pair in *G,W*, since the weight systems are inherited) appears exactly in $\omega(S(A_1))$ or $\omega(S(A_2))$. Thus, since the weight systems are inherited, it follows that for any *v*, the weight of each pair in the state *S* appears in exactly one of the products $\omega(S(A_1))$ or $\omega(S(A_2))$.

Hence $\omega(S) = \omega(S(A_1))\omega(S(A_2))$, which completes the proof. ///

COROLLARY 13.

$$q(G, W; 1 \otimes x + x \otimes 1) = \sum q(A_1, W(A_1); x) \otimes q(A_2, W(A_2); x).$$

Proof: This follows immediately from letting $x = 1 \otimes x$ and $y = x \otimes 1$ in Lemma 12, and completes the proof of Theorem 11 that $q : \Gamma \to R[x]$ is a Hopf algebra map. ///

COROLLARY 14. $q(G,W;y) = \sum \prod q(A_i, W(A_i); x_i)$, *where* $y = \sum_{i=1}^{n} x_i$, *and where the sum is over all ordered partitions* $(A_1, \ldots, A_n)$ *of G into* n *edge-disjoint Eulerian subgraphs.*

Proof: This follows from Lemma 12 by induction. ///

LEMMA 15. $q(G,W;-x) = q(\zeta(G,W);x)$.

Proof: Since the antipode, ζ_R, of $R[x]$ is given by $\zeta_R(p(x)) = p(-x)$, where $p(x) \in R[x]$, and $q(G,W;x)$ is a Hopf algebra map, this follows immediately from Proposition 10 and Theorem 11. ///

Evaluations and applications

As noted previously, when G is a 4-regular graph, and with the correct choices of pair weights, $q(G,W;x) = xQ(G,A;x)$, where $Q(G,A;x)$ is Jaeger's transition polynomial for 4-regular graphs (see [Jae90]). Since, with appropriate weight systems, $Q(G,A;x)$ is equal to the Martin polynomial (either oriented or unoriented, but restricted to 4-regular graphs), the Tutte polynomial of a planar graphs when $y = x$ (via the medial graph), the Penrose polynomial, or the Kauffman bracket from knot theory, it follows that $q(G,W;x)$ also determines these polynomials.

However, since $q(G,W;x)$ is a considerably more general object than $Q(G,A;x)$, it also determines, for example, the Martin polynomials for arbitrary Eulerian (di)graphs. Bollobas and Riordin ([BR99]) give a Tutte polynomial for coloured graphs, which is the most general polynomial satisfying both an activities and a deletion/contraction definition. On a restricted ideal, $q(G,W;x)$ can be shown to agree (via the medial graph) with their Tutte polynomial of a coloured graph in the planar case.

More importantly, and this is the significance of the current work, the structural properties of $q(G,W;x)$ now provide algebraic tools for analyzing all of the polynomials which it determines. In particular, Lemma 12, Corollary 14, and Lemma 15 give identities that facilitate the use of recursion and induction to derive new combinatorial interpretations of these polynomials. This type of Hopf algebraic structure has already been used to considerably extend the known evaluations of the Martin, Penrose, and Tutte polynomials implicitly in [Bol], and explicitly in [ES01], [E-M99], [E-Ma], [E-Mb], [Sar01], and is the foundation for continuing research.

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