Medial Graphs and the Penrose Polynomial

Joanna A. Ellis-Monaghan
Department of Mathematics and Statistics
University of Vermont
Burlington, VT 05405

Irasema Sarmiento
Departamento de Matemáticas
CINVESTAV
Av. IPN 2508
CP 07000, México D. F.

Abstract

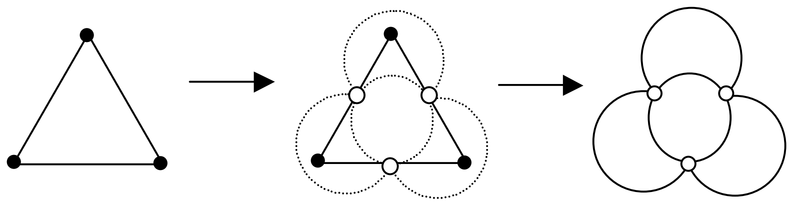
We construct a one variable graph polynomial, $N(G,W;x)$, on the space of planar drawings of Eulerian graphs with *maxdeg* = 4 and weight system *W*. When G_m is the medial graph of a planar graph *G*, with an appropriate weight system *W*, then $N(G_m,W;x) = P(G;x)$, where $P(G;x)$ is the Penrose polynomial of *G*. We give a simple combinatorial proof of an identity for $N(G,W;x)$, which has as a special case an identity very similar to one derived using Hopf algebraic techniques in [Sar]. This identity then gives relations for the Penrose polynomial of a graph in terms of the Eulerian subgraphs of its medial graph.

Introduction

The Penrose polynomial of a graph, defined implicitly by Roger Penrose in *Applications of Negative Dimensional Tensors*, [Pen69], encodes edge coloring information of planar graphs. This polynomial can be computed via weighted skein, or transition, relations in its medial graph. In this paper we construct a new polynomial defined on drawings of graphs in the plane with specified weights assigned to each vertex state. Properties of this polynomial are then used to get an identity for the Penrose polynomial.

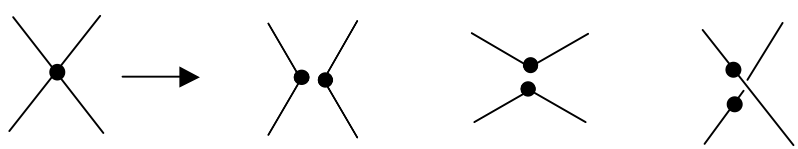
The following conventions are used through this paper. Graphs may have loops and multiple edges. A graph is said to be Eulerian if all its vertices have even degrees, but connectedness is not required. A cycle is a graph isomorphic to a polygon. The notation $k(G)$ will be used to indicate the number of components of G, *not* counting isolated vertices.

Recall that the medial graph of a connected planar graph G is constructed by putting a vertex on each edge of G and drawing edges around the faces of G. Specifically, two vertices of the medial graph of G are joined by an edge if the corresponding edges in G are neighbors in the cyclic order of edges around a vertex. For example:

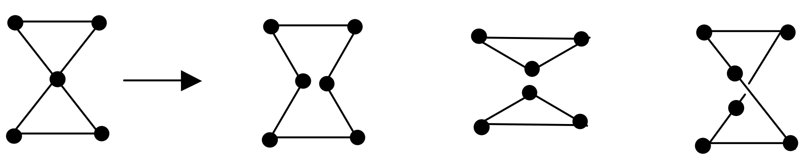


The graph G The medial graph of G

A vertex state is a choice of local reconfiguration of a graph at a vertex by pairing the edges incident with that vertex. In this paper we will only consider those vertex states where all the edges are paired (no singleton edges). For example, the three possible vertex states of a vertex of degree 4 are:



A graph state is the result of choosing an allowed vertex state (different applications may use only specific vertex states) at each vertex of degree greater than 2. For example, the three possible graph states of this graph are:



We will write $St(G)$ for the set of graph states of G.

A skein-relation for graphs is a formal sum of possibly weighted vertex states, together with an evaluation of the terminal forms (the graph states, typically a disjoint union of graphs with maxdeg=2). See [E-M98] for a detailed discussion of these concepts, which are appropriated from knot theory, in their most general form.

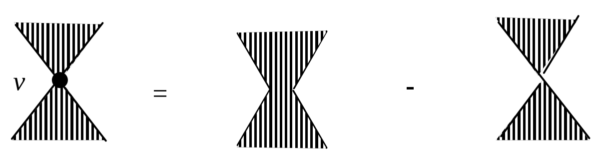
A skein-type, or transition, polynomial is one which is computed by repeated applications of skein relations. See [Jae90] for a comprehensive treatment of these in the case of 4-regular graphs.

Definition 1: A *weight system W* on a planar drawing of an Eulerian graph with maxdeg=4 is an association of a value (a vertex state weight) to each state at each vertex of degree 4 in *G*. These values may in general be in any ring, but here they will typically be integers. Thus there are three values associated to each vertex of degree 4 in *G*.

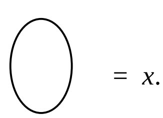
Definition 2: The *state weight* of a graph state *S* of a graph *G* with weight system *W* is $\omega(S)=\prod\omega(S,v)$, where $\omega(S,v)$ is the vertex state weight of the vertex state at *v* in the graph state *S*, and where the product is over all vertices of deg 4 in *G*.

The Penrose Polynomial $P(G;x)$

The Penrose polynomial of a planar graph was defined implicitly in [Pen69], and an excellent exposition can be found in [Aig97]. It can be computed via skein relations applied to its medial graph (see [Jae90] for example), and we will use this approach here. Let *G* be a planar graph, and let G_m be its medial graph, two-colored with the unbounded face colored white. Then $P(G;x)$ can be computed by applying the following skein-relation to the vertices of degree 4 in G_m:



This reduces the graph to a formal sum of unions of circles (cycles with one edge and no vertices), which are then each evaluated to *x*:

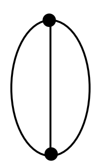


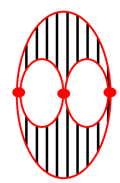
The Penrose polynomial also has the following state model formulation:

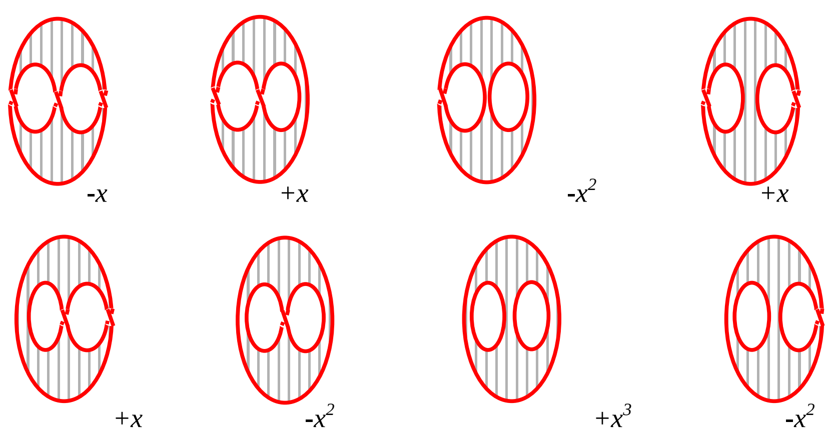
$$P(G;x)=\sum_{St(G_m)}\left(\left(\prod_{v\in G_m}\omega(S,v)\right)x^{k(S)}\right)=\sum_{St(G_m)}\left((-1)^{cr(S)}\,x^{k(S)}\right),$$

where $\omega(S,v)$ is $+1$ for the white non-crossing state at v, 0 for the black non-crossing state, and -1 for the crossing state, $k(S)$ is the number of components in the graph state S, and $cr(S)=$ the number of crossing vertex states chosen in the state S, so that $\left(\prod_{v\in G_m}\omega(S,v)\right)=(-1)^{cr(S)}$.

For example:

$G =$ 

$G_m =$ 



Thus, $P(G;x)=x^3-3x^2+2x$.

The Penrose polynomial has some surprising properties which make its study so enticing, particularly with respect to the famous Four Color Theorem. The Four Color Theorem, which states that the regions of every planar bridgeless graph can be properly colored with four colors, can be shown to be equivalent to showing that every planar, cubic, connected graph can be properly edge-colored with three colors. The Penrose polynomial, when applied to planar, cubic, connected graphs, encodes exactly this information (see [Pen69]):

$$P(G;3) = \left(\frac{-1}{4}\right)^{\frac{|V|}{2}} P(G;-2) = \#\text{ edge-3-colorings of } G.$$

The polynomial $N(G;x)$

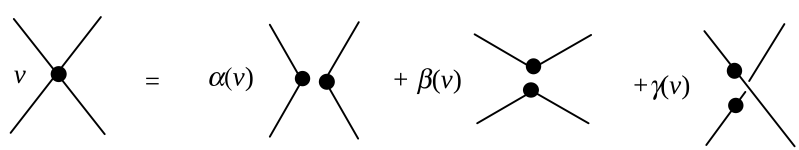
We now define a new graph polynomial which will have the property that $N(G_m, W; x) = P(G;x)$ whenever G is a planar graph and an appropriate weight system W is assigned to G_m. Thus studying this new polynomial will lead to insights into the Penrose polynomial.

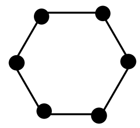
Let G be an Eulerian multigraph (loops and multiple edges allowed) with $\max\deg = 4$, drawn in the plane simply (only two edge crossing at any given point), and with the regions two-colored with the unbounded region colored white. Identify two graphs which are ambient isotopic to each other, or which can be transformed one into the other by switching which of two edges crosses over the other, or which are the same if isolated vertices are ignored. Write G,W for such a graph drawing with an associated weight system W.

Let $\Gamma = span_{\mathbf{C}}\{\text{all such } G,W \text{ under this equivalence}\}$.

Recursive Definition:

Define $N:\Gamma \to \mathbf{C}[x]$ recursively by repeatedly applying the following skein relation at any vertex of degree 4, and then evaluating the terminal forms (the graph states) by identifying each cycle with the variable x. Here, $\alpha(v)$, $\beta(v)$ and $\gamma(v)$ are the vertex state weights assigned to each state at v by the weight system W.





$= x.$

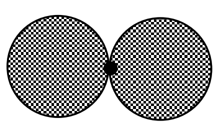
State Model Definition:

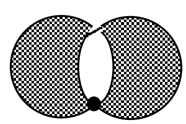
$$N(G,W;x) = \sum_{St(G)} \left(\left(\prod \omega(S,v) \right) x^{k(S)} \right) = \sum_{St(G)} \omega(S) x^{k(S)} .$$

The polynomial $N(G,W;x)$ is essentially the same as the transition polynomial $Q(G,W,x)$ in [Jae90]. They differ by a factor of *x*, and here we retain vertices of deg 2 in the recursion while they are elided in [Jae90].

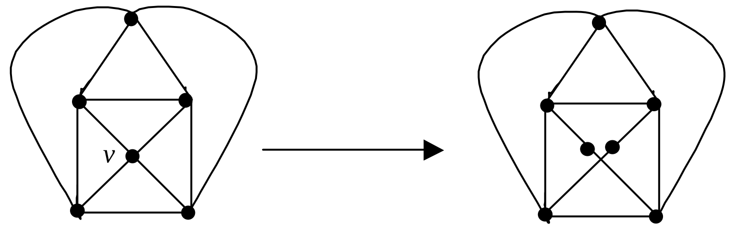
Note that since this state model definition is clearly equivalent to the recursive definition, and is independent of the order of the vertices to which the recursion is applied, $N(G,W;x)$ is well-defined. Also, since the states, both vertex and graph, are in terms of local reconfigurations, $N(G,W;x)$ is well-defined with respect to the equivalence relations within the definition of Γ.

Because the assignment of vertex state weights may depend on the drawing, it is certainly possible that *N* may depend on the particular drawing of a graph. For example, consider the following two drawings of the graph with one vertex and two edges. Determine the weight systems by two-coloring the regions with the outer region colored white, and apply the same weights as for the Penrose polynomial (1 for white non-crossing, 0 for black non-crossing, and –1 for crossing). Let *L* be the drawing on the left, and *R* be the drawing on the right. Then $N(L,W,x) = 0$, but $N(R,W,x) = x - x^2$.





However, extending the class of graphs on which *N* is defined from planar graphs to graph drawings is necessitated by the fact that a choice of vertex states in the recursion relation may transform a planar graph into a non-planar graph drawn in the plane. This can be seen in the following example, where for the crossing state at *v*, the resulting graph is homeomorphic to K_5.



Generating Function Definition:

Collecting like terms in the preceding state model definition leads to the following generating function form of $N(G,W;x)$.

$$N(G,W;x)=\sum f_n(G)x^n ,$$

where $f_n(G)=\sum \omega(S)$, where the sum is over all states of G with n components.

LEMMA 3 *Let G_m be the medial graph of a planar graph G. Give G_m a weight system W by two-face-coloring G_m (unbounded region colored white), and assigning a value of +1 for the white non-crossing state at each vertex v, 0 for the black non-crossing state, and –1 for the crossing state. Then* $N(G_m,W;x)=P(G;x)$**.**

Proof: With this weight system, $N(G_m,W;x)$ is exactly the same as the state model definition for the Penrose polynomial, except that the graph states here retain information about the vertices of degree 2. However, since cycles of all lengths all evaluate to *x*, the end result is the same. Furthermore, since the medial graphs of any two ambient isotopies of a connected planar graph are ambient isotopic, it follows that $N(G_m,W;x)=P(G;x)$ is consistent as well. ///

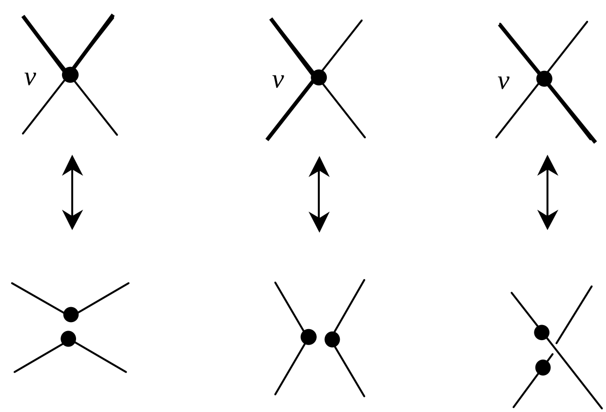
The structural properties of $N(G;x)$.

We now turn our attention to the structural properties of $N(G,W;x)$. Let *GH* be the disjoint union of two graphs *G* and *H* drawn in the plane with weight systems *W* and *W'*, respectively. Since the skein relation can be applied first to all the degree 4 vertices in *G*, and then to those in *H*, it follows that $N(G,W;x)$

is multiplicative on disjoint unions of graphs in Γ, so that $N(GH,W'';x) = N(G,W;x)N(H,W';x)$, where W" is the weight system which assigns to $v \in V(GH)$ the weights it originally received from W (or W') as a vertex of G (or H). Thus $N:\Gamma \to \mathbf{C}[x]$ is an algebra map. Theorem 7 below will have as a corollary that $N:\Gamma \to \mathbf{C}[x]$ is also a coalgebra map.

Definition 4: Let A be an Eulerian subgraph of a graph G in Γ. Let $V_2(A/G)$ be the vertices of degree 4 in G that have degree 2 when restricted to A, and let $V_4(A/G)$ be those that have degree 4. When there is no danger of confusion about the underlying graph G, we will write $V_i(A)$ for $V_i(A/G)$.

Definition 5: Let A be an Eulerian subgraph of a graph G in Γ, and let $v \in V_2(A/G)$. Then there are three different possible configurations for A in G at v, and they correspond to the three possible vertex states at v. (The heavy edges are in A.) These are the three different *transitions* of A in G.



Definition 6: $\sigma(A/G) = \prod \omega(A,v)$, where the product is over $V_2(A/G)$.

Here $\omega(A,v)$ is the weight of the vertex state at v which corresponds to the transition of A in G. When there is no danger of confusion, $\sigma(A)$ will be written for $\sigma(A/G)$.

THEOREM 7 $N(G,W;x+y) = \sum \sigma(A) N(A,W_A;x) N(A^c,W_{A^c};y)$, *where the sum is over all Eulerian subgraphs A of G.*

Here A is written for $G|_A$, the restriction of G to A (as a drawing). The weight system for A, denoted W_A, is inherited from W in such a way that if $v \in V_4(A)$, then the state weights at v as a vertex of A are the same as they were as for v as a vertex of G. This holds similarly for A^c.

Proof: By the state model definition ,

$$N(G,W;x+y)=\sum_{St(G)}\omega(S)(x+y)^{k(S)}=\sum_{St(G)}\omega(S)\sum_{r=0}^{k(S)}\binom{k(s)}{r}x^r y^{k(S)-r}.$$

Thus, the coefficient of $x^a y^b$ on the left-hand-side is:

$$\sum\omega(S)\binom{a+b}{a},$$

where the sum is over $S \in St(G)$ such that $k(S)=a+b$.

By the same state model definition, the right-hand-side becomes

$$\sum\sigma(A)\sum\omega(S_A)x^{k(S_A)}\sum\omega(S_{A^c})y^{k(S_{A^c})},$$

where the outer sum is over all Eulerian subgraphs A of G, the first inner sum is over all graph states S_A of A, and the second inner sum is over all graph states S_{A^c} of A^c.

Thus, the coefficient of $x^a y^b$ on the right-hand-side is:

$$\sum\sum\sum\sigma(A)\omega(S_A)\omega(S_{A^c}),$$

where the outer sum is over all Eulerian subgraphs A of G, the first inner sum is over all graph states S_A of A with a components, and the second inner sum is over all graph states S_{A^c} of A^c with b components.

Now consider a state S of G with $a+b$ components. Choose a of those components to comprise S_A. Thus A consists of the edges in those a components and hence is Eulerian. The state S_A is determined by the state S, as is consequently the state S_{A^c} which has b components. Note that there are

$\binom{a+b}{a}$ ways to do this. Thus the coefficients of $x^a y^b$ are equal on both sides of the equation if and only if $\omega(S) = \sigma(A)\omega(S_A)\omega(S_{A^c})$.

However, notice that $\omega(S) = \prod \omega(S,v) =$

$$\prod_{v \in V_2(A)} \omega(A,v) \prod_{v \in V_4(A)} \omega(S_A) \prod_{v \in V_4(A^c)} \omega(S_{A^c}) = \sigma(A)\omega(S_A)\omega(S_{A^c}).$$

Thus this is true, which completes the proof. ///

COROLLARY 8:

$$N(G,W;1 \otimes x + x \otimes 1) = \sum \sigma(A) N(A, W_A; x) \otimes N(A^c, W_{A^c}; x).$$

Proof: This follows immediately from letting $x = 1 \otimes x$ and $y = x \otimes 1$. ///

This now means that $N : \Gamma \to \mathbf{C}[x]$ is a coalgebra map, where $\mathbf{C}[x]$ is given the structure of the binomial bialgebra, and Γ has a comultiplication given by $\Delta(G,W) = \sum \sigma(A)(A, W_A) \otimes (A^c, W_{A^c})$, where the sum is over all Eulerian subgraphs *A* of *G* with weight systems inherited from *W*.

It is routine to check that Γ is in fact a Hopf algebra with antipode given recursively by $\zeta(G,W) = -\sum \sigma(A) A \zeta(A^c)$ where the sum is over all non-empty Eulerian subgraphs *A*, and where $\zeta(\mathcal{E}) = 1$ for $\mathcal{E}$, any graph with no edges. Thus, since *N* is a bialgebra map, it is automatically a Hopf map, i.e. respects the antipode (see [Swe, lemma 4.04]). In particular, we get the following lemma:

LEMMA 9 $N(G,W;-x) = N(\zeta(G,W);x)$.

Proof: Since the antipode of $\mathbf{C}[x]$ is given by $\zeta(p(x)) = p(-x)$, and *N* is a Hopf map, this follows immediately. ///

As an important special case, consider the hereditary set given by graphs in Γ where the weights in the weight systems are restricted to {-1, 0, 1}. This is a hereditary set since the substructures (Eulerian subgraphs with inherited weight systems) remain in the set. Because it is a hereditary set, it forms a sub-Hopf algebra of Γ (see [Sch94, Sch95] for more information about hereditary sets and Hopf algebras). By considering the comultipication and antipode of this sub-Hopf algebra together with Lemma 3, we get the follow two results, which are equivalent to those in [Sar].

THEOREM 10 [SAR, THEOREM 4.3] *Let G_m be the medial graph of a planar graph G. Give G_m a weight system W by two-face-coloring G_m (unbounded region colored white), and assigning a value of +1 for the white non-crossing state at each vertex v, 0 for the black non-crossing state, and –1 for the crossing state. Then* $P(G, x+y) = \sum \sigma(A) N(A, W_A; x) N(A^c, W_{A^c}; y)$ *where the sum is over all Eulerian subgraphs A of G with weight systems inherited from W.*

Proof: This follows immediately from Lemma 3 and theorem 7.

THEOREM 11 [SAR, THEOREM 5.2] *Let G_m be the medial graph of a planar graph G. Give G_m a weight system W by two-face-coloring G_m (unbounded region colored white), and assigning a value of +1 for the white non-crossing state at each vertex v, 0 for the black non-crossing state, and –1 for the crossing state. Then* $P(G, -x) = N(S(G_m, W); x)$.

Proof: This follows immediately from lemmas 3 and 9. ///

This theorem is especially useful because it can be used to give interpretations for the Penrose polynomial at negative values (see [Sar]).

Bibliography

[Aig97] M. AIGNER, The Penrose polynomial of a plane graph, *Math. Ann.,* **307** (1997) 173-189.

[E-M99] J. A. ELLIS-MONAGHAN, New Results for the Martin Polynomial, *Journal of Combinatorial Theory,* Series B, **74** (1998) 326-352.

[Jae90] F. JAEGER, On Transition Polynomials of 4-regular Graphs, *Cycles and Rays (Montreal, PQ, 1987)* 123-150, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 301, *Kluwer Acad. Publ., Dordrecht,* 1990.

[Pen69] R. PENROSE, Applications of Negative Dimensional Tensors, *Combinatorial Mathematics and its Applications; Proceedings of a conference held at* *the Mathematical Institue, Oxford, 1969*, London, New York, Academic Press, (1971), 221-244.

[Sar] I. SARMIENTO, "Hopf Algebras and the Penrose Polynomial", preprint.

[Sch94] W. SCHMITT, Incidence Hopf algebras, *Journal of Pure and Applied Algebra*, **96** (1994) 299-330.

[Sch95] W. SCHMITT, Hopf Algebra Methods in Graph Theory, *Journal of Pure and Applied Algebra*, **101** (1995) 77-90.

[Swe69] M. E. SWEEDLER, *Hopf Algebras,* New York: W. A. Benjamin, Inc., 1969.