Decycling of Fibonacci Cubes

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Abstract

The decycling number $\nabla(G)$ of a graph G is the smallest number of vertices that can be deleted from G so that the resultant graph contains no cycle. Here we study the decycling number for the Fibonacci cubes, which are a family of graphs with applications in interconnection topologies. We present lower and upper bounds of the decycling number for the Fibonacci cubes.

1 Introduction

In 1986, Erdős, Saks and Sós presented the problem of finding, for a given graph G, the size $t(G)$ of a maximum subset T of $V(G)$ that would induce a tree [5]. They also studied the problem of finding the path number, i.e. the maximum size of a vertex subset that would induce a path. Meanwhile, the more general problem of finding the size of maximum subset F of $V(G)$ that induces a forest was beginning to receive attention for various types of graphs, such as cubic graphs [4, 12] and planar graphs [1, 2].

Finding a maximum subset F of $V(G)$ that induces a forest can be expressed as the problem of determining a minimum subset S of $V(G)$ for which $G - S$ is acyclic. Any vertex subset S for which $G - S$ contains no cycles is a *decycling set*. The size of a minimum decycling set S in a graph is the *decycling number* of G and will be denoted by $\nabla(G)$. A decycling set of size $\nabla(G)$ is called a ∇-*set*. Finding a minimum decycling set of a graph is quite difficult even for some simple graphs, and has been proved to be NP-complete in general [8].

In [3], various introductory decycling results were presented, followed by investigation into hypercube and 2-dimensional grid graphs. The hypercube of dimension n, denoted by Q_n, consists of 2^n vertices labelled with distinct n-bit (0,1) strings (or binary strings); two vertices are adjacent if and only if their labels differ in exactly one bit. Further results concerning the decycling number of the n-th order hypercube were presented in [6, 11] .

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The hypercube is a popular interconnection topology for parallel processing since it has many appealing properties. However, the number of vertices in the hypercube must be a power of 2, which restricts the permissible size of the hypercube. In [7], Hsu introduced a new interconnection topology — Fibonacci cubes, which have a slower growth rate than hypercubes and can be embedded in hypercubes.

Recall that the Fibonacci numbers form a sequence of positive integers $\{f_n\}_{n=0}^{\infty}$ where $f_0 = 1$, $f_1 = 2$ and $f_n = f_{n-1} + f_{n-2}$. Given that i is a non-negative integer such that $i \leqslant f_{n-1} - 1$, then i can be uniquely represented as a sum of distinct non-consecutive Fibonacci numbers (*Zeckendorf's Theorem* [14]) in the form $i = \sum_{j=0}^{n-1} b_j f_j$ where b_j is either 0 or 1, for $0 \leqslant j \leqslant n-1$ with the condition $b_j b_{j+1} = 0$ for $0 \leqslant j < n-1$. The sequence $[b_{n-1}, \cdots, b_1, b_0]$ is called the order-n Fibonacci code of i, and uniquely determines i. For example, $i = 12 = f_5 - 1$ has Fibonacci code 10101.

The Fibonacci cube Γ_n of order n is the graph (V_n, E_n) where $V_n = \{0, 1, \cdots, f_n - 1\}$ and two vertices i and j are adjacent if and only if their Fibonacci codes differ in exactly one bit. The Fibonacci cubes for the first few values of n are depicted in Figure 1. A number of properties of Fibonacci cubes are described in [7, 9, 10, 13]. In this paper, we study the decycling number of the Fibonacci cubes. To simplify notation, we write ∇_n for $\nabla(\Gamma_n)$.

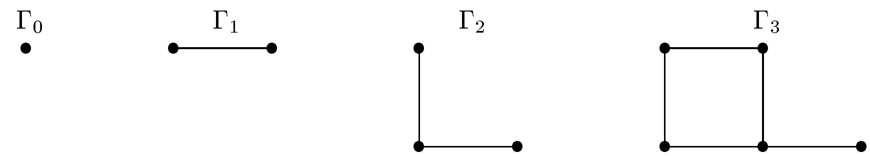

Figure 1. The n-dimensional Fibonacci cubes Γ_n for $n = 0, 1, 2, 3$.

In Section 2 of this paper, we will discuss lower bounds on ∇_n as well as, for small values of n, the exact value of ∇_n. In Section 3, we present upper bounds on ∇_n. Our results, as they apply to Γ_n for $1 \leqslant n \leqslant 14$, are summarised in Table 1.

2 Lower Bound

To establish our first lower bound, we will take advantage of some properties of Γ_n that we now review.

LEMMA 2.1 [7] (*Decomposition of Fibonacci cube*) *Let* $\Gamma_n = (V_n, E_n)$ *denote the Fibonacci cube of order* n *where* $n \geqslant 2$. *Let* $\mathrm{LOW}(n)$ (*resp.* $\mathrm{HIGH}(n)$) *denote the subgraph induced by the set of vertices* $\{0, 1, \cdots, f_{n-1} - 1\}$ (*resp.* $\{f_{n-1}, f_{n-1}+1, \cdots, f_n\}$). *Then*

1. $\mathrm{LOW}(n) \cong \Gamma_{n-1}$;

2. $\mathrm{HIGH}(n) \cong \Gamma_{n-2}$.

Moreover, if we let $\mathrm{LINK}(n) = \big\{\{i, j\} : |i - j| = f_{n-1}, \{i, j\} \in E_n\big\}$, *then the two disjoint subgraphs* $\mathrm{LOW}(n)$ *and* $\mathrm{HIGH}(n)$ *are connected exactly by the set of edges* $\mathrm{LINK}(n)$.

Table 1. Decycling number of Γ_n.

n	$\lvert V_n \rvert$	$\lvert E_n \rvert$	∇_n
1	2	1	0
2	3	2	0
3	5	5	1
4	8	10	1
5	13	20	3
6	21	38	6
7	34	71	11
8	55	130	19
9	89	235	33
10	144	420	53-55
11	233	744	86-94
12	377	1308	139-158
13	610	2285	225-264
14	987	3970	364-439

By Lemma 2.1, the Fibonacci cube Γ_n of order n can be decomposed into two disjoint subgraphs Γ_{n-1} and Γ_{n-2}. Therefore, to decycle Γ_n, we need to decycle each of the two subgraphs, yielding the following lower bound for the decycling number of Γ_n.

COROLLARY 2.2 $\quad \nabla_n \geqslant \nabla_{n-1} + \nabla_{n-2}$.

In [7], Hsu gave a formula to calculate the number of edges in Γ_n just in terms of the Fibonacci numbers $\left(\lvert E_n \rvert = \dfrac{2(n+1)f_n - (n+2)f_{n-1}}{5}\right)$. In order to establish lower bounds and exact values of ∇_n for small n, we will also want to know about the degree sequence of Γ_n. If we let $d_n(i)$ denote the degree of vertex i in Γ_n, then we have the following lemma:

LEMMA 2.3 *For $n \geqslant 2$,*

$$d_n(i) = \begin{cases} d_{n-1}(i) + 1, & 0 \leqslant i < f_{n-2}, \\ d_{n-1}(i), & f_{n-2} \leqslant i < f_{n-1}, \\ d_{n-2}(i - f_{n-1}) + 1, & f_{n-1} \leqslant i < f_n, \end{cases}$$

where $d_0(0) = 0$, $d_1(0) = 1$, and $d_1(1) = 1$. The maximum degree $\Delta(\Gamma_n) = n$, and vertex 0 is the only vertex of degree n. For $n \geqslant 4$, vertices 1 and f_{n-1} are the only vertices of degree $n - 1$.

Proof. For $d_n(i)$, the proof follows immediately from Lemma 2.1. The remaining statements easily follow by induction, again using Lemma 2.1. □

We next prove a sequence of lemmas that will provide the decycling number of the n-dimensional Fibonacci cubes for $n \leqslant 7$.

LEMMA 2.4 $\nabla_4 = 1$*, and there is only one ∇-set (for which the corresponding maximum induced forest is isomorphic to the path P_7).*

Proof. The value of ∇_4 follows from Corollary 2.2 and Figure 2 in which the vertex surrounded by $\Box$ is in S. It is easy to see $S = \{0\}$ is the unique ∇-set for Γ_4, and the left-over graph $\Gamma_4 - S$ is a path of order 7, so we also find the path number of Γ_4. $\Box$

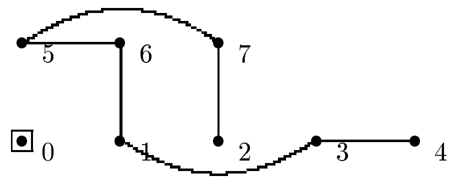

Figure 2. Decycling set of Γ_4.

LEMMA 2.5 $\nabla_5 = 3$.

Proof. By Lemma 2.2, $\nabla_5 \geqslant 2$. To decycle Γ_5, we need to decycle the two disjoint subgraphs LOW(5) and HIGH(5). If $\nabla_5 = 2$, then we need to remove the unique ∇-set in LOW(5), and one more vertex in HIGH(5) to decycle HIGH(5), i.e. vertex 8, 9, 11 or 12. In each case it is a simple exercise to see that Γ_5 contains a cycle even after deleting both vertex 0 and vertex i, for $i \in \{8, 9, 11, 12\}$. Hence $\nabla_5 \geqslant 3$. Figure 3 shows that there exists a decycling set of size 3 in Γ_5. Note in Figure 3, the maximum induced forest is a path of order 10, so we also get the path number for Γ_5. $\Box$

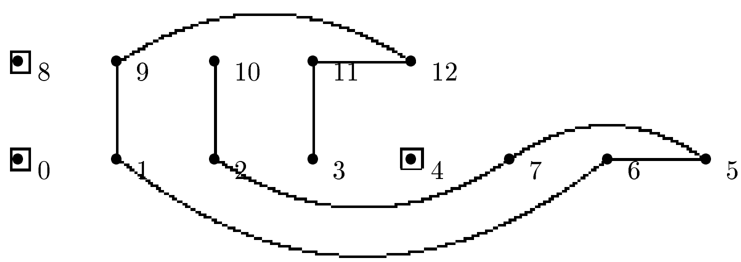

Figure 3. A decycling set of Γ_5.

By an exhaustive search in Γ_5, we found 6 decycling sets of size 3. Two of them $\big(\{0,4,9\}$ and $\{0,9,11\}\big)$ are independent; the other four are $\{0,1,11\}$, $\{0,1,12\}$, $\{0,4,8\}$, and $\{0,8,12\}$.

LEMMA 2.6 $\nabla_6 = 6$.

Proof. By Lemma 2.2, $\nabla_6 \geqslant 4$. If $\nabla_6 = 4$, then by Lemma 2.3 at most $\Delta(\Gamma_6)+2(\Delta(\Gamma_6)-1)+(\Delta(\Gamma_6)-2) = 20$ edges are removed, and at least 18 edges remain, which is too many for a forest on 17 vertices. Hence $\nabla_6 \geqslant 5$.

If $\nabla_6 = 5$, we count the number of vertices and edges in the left-over graph. Let S be a putative decycling set of size 5 in Γ_6. If the degree sequence of S is $6, 5, 5, 4, 4$, then observe that all the vertices of degree 5 have a neighbour of degree 6. Also it is easy to verify that each vertex of degree 4 in Γ_6 has at least one neighbour of degree 5 or 6. So $\varepsilon(S) \geqslant 4$ (where $\varepsilon(S)$ denotes the number of edges in the induced subgraph $\Gamma_n[S]$), and at least 18 edges are left in $\Gamma_6 - S$, which is too many for a forest on 16 vertices. If the degree sequence is $6, 5, 4, 4, 4$, we have $\varepsilon(S) \geqslant 1$, and at least 16 edges remaining, which is still too many for a forest on 16 vertices. Any other degree sequence will also result in $\Gamma_6 - S$ having at least 16 edges. So we have $\nabla_6 \geqslant 6$. $\{0, 4, 7, 8, 14, 18\}$ is a decycling set of size 6 for Γ_6. □

By computer search, we found 19 decycling sets of size 6 for Γ_6 (5 are independent decycling sets). More details are available online at `www.math.mun.ca/~yubo/research/fib_cube/`.

LEMMA 2.7 $\nabla_7 = 11$.

Proof. By Lemma 2.2, we have $\nabla_7 \geqslant 3 + 6 = 9$. By using techniques similar to those described in Lemma 2.6, we can easily improve the lower bound to get $\nabla_7 \geqslant 10$.

Assume that $\nabla_7 = 10$ and S is a decycling set of size 10 in Γ_7. Since $|V_7| = 34$, $|E_7| = 71$, and $\Gamma_7 - S$ has 24 vertices, that means at most 23 edges are left in $\Gamma_7 - S$. Define $E' = E_7 - E(\Gamma_7 - S)$. In order for $\Gamma_7 - S$ to be a forest, it is necessary that $|E'| \geqslant 71 - 23 = 48$. By observation, we also have the following useful information in Γ_7: all the neighbours of vertex 0 have degree 6 or degree 5; there are 8 vertices of degree 5; each of vertices 1 and 21 has 2 neighbours of degree 5; $\{3, 4\}, \{8, 29\} \in E_7$. There are the following 5 cases:

1. All the vertices of degree 7 and 6 are in S. There are two subcases:

   (a) There exists no degree 4 vertex in S. Notice that each vertex of degree 5 has a neighbour of degree 6 or 7, then $\varepsilon(S) = |E(\Gamma_7[S])| > 6$. Hence $|E'| \leqslant 47$.

   (b) There exists at least one degree 4 vertex in S. We have $\varepsilon(S) \geqslant 6$. Hence $|E'| \leqslant 47$.

2. Vertex 0 is in S, exactly one of vertices 1 or 21 is not in S.

   (a) All vertices of degree 5 are in S. Then $\varepsilon(S) \geqslant 6$ and $|E'| \leqslant 47$.

   (b) At least one of degree 5 vertex is not in S. Then $|E'| \leqslant 47$.

3. Vertex 0 and exactly one of vertices 1 or 21 are not in S.

   (a) All vertices of degree 5 are in S.

   (b) At least one vertex of degree 5 is not in S.

   For above two subcases, it is easy to verify that $|E'| \leqslant 47$.

4. Vertices 1 and 21 are in S, vertex 0 is not in S. There are several subcases:

   (a) All vertices of degree 5 are in S, which means that $\Gamma_7 - S$ has at least 2 components, and $\varepsilon(S) \geqslant 4$. Hence $|E'| \leqslant 48$, and at least 23 edges remain in $\Gamma_7 - S$, which is too many for a forest on 24 vertices with 2 components.

   (b) Exactly one vertex of degree 5 is not in S. Then $\varepsilon(S) \geqslant 4$. Hence $|E'| \leqslant 47$.

   (c) Exactly two vertices of degree 5 are not in S. Then either $\Gamma_7 - S$ has at least two components and $|E'| \leqslant 48$, or $\varepsilon(S) \geqslant 4$ which means $|E'| \leqslant 46$.

(d) Exactly three vertices of degree 5 are not in S. Either $\varepsilon(S) \geqslant 2$ which means $|E'| \leqslant 47$, or $\Gamma_7 - S$ has at least 3 components and $|E'| \leqslant 49$. Hence 22 edges remain which is too many for a forest with 3 components and 24 vertices.

(e) Exactly four vertices of degree 5 are not in S. Either $\Gamma_7[S]$ has an edge, so $|E'| \leqslant 47$, or this forces using vertices 12, 25, 30, and 32 (all of degree 4) to avoid any edges in $\Gamma_7[S]$, but then this isolates, for example, vertex 9 or 24, and hence increases the number of components.

(f) More than 5 vertices of degree 5 are not in S. In this case, $|E'| \leqslant \sum_{v \in S} d_7(v) \leqslant 47$.

5. Vertices 0, 1 and 21 are not in S. The only possibility is all the vertices of degree 5 are in S. Hence $\varepsilon(S) = 2$ and $|E'| \leqslant 46$.

In all of the above cases, there are not enough edges removed for decycling, and therefore $\nabla_7 \geq 11$. By a computer search, we find 192 decycling sets of size 11 in Γ_7 (18 are independent). These decycling sets are listed online at `www.math.mun.ca/~yubo/research/fib_cube/`. □

Computer Search Heuristics. For $n \leqslant 7$, an exhaustive computer search to find all the decycling sets of size ∇_n in Γ_n is possible. For $n \geqslant 8$, Lemma 2.1 suggests the following heuristic to find $\nabla(\Gamma_n)$

Step 1. Find all the decycling sets of size $\nabla_{n-2} + i$ in Γ_{n-2} without any sub-decycling set, where $i = 0, 1, \cdots$. Do the same thing in Γ_{n-1}, obtaining all the decycling sets of size $\nabla_{n-1} + j$ $(j = 0, 1, \cdots)$ without any sub-decycling set.

Step 2. By the Decomposition Lemma(2.1), if S is a decycling set in Γ_n, then $S \cap V(\text{LOW}(n))$ must be a decycling set in Γ_{n-1}, and $S \cap V(\text{HIGH}(n))$ must be a decycling set in Γ_{n-2}. Therefore, to find a decycling set of size $\nabla_{n-1} + \nabla_{n-2} + k$ in Γ_n, we test all the combinations of a decycling set of size $\nabla_{n-2} + i$ in Γ_{n-2} and a decycling set of size $\nabla_{n-1} + j$ in Γ_{n-1} together with x more vertices in the left-over graph of Γ_n, where $i + j + x = k$, and i, j, x are non-negative integers.

Such a computer search shows that there is no decycling set of size less than 19 and 33 for Γ_8 and Γ_9 respectively, so $\nabla_8 = 19$ and $\nabla_9 = 33$. The details of the results are listed in Table 2, for $n \leqslant 9$. Blank entries in the table correspond to computational tasks that are prohibitively time consuming.

Table 2. Computer search results of decycling sets for Γ_n

Γ_n	$\lvert V_n \rvert$	∇_n	Number of decycling sets, without sub-decycling sets, of size $\nabla_n + i$								
			$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
5	13	3	6	58	36	4	0	0	0	0	0
6	21	6	19	704	1933	639	65	0	0	0	0
7	34	11	192	8528	113175	454916	453985	112185	7996	96	0
8	55	19	33	10649	778540	21836552					
9	89	33	58								

Based on the computer search, we found that $\nabla_n = \nabla_{n-1} + \nabla_{n-2} + \mu(n)$ where $\mu(n)$ is small integer value (for $n = 4, 5, 6, 7, 8, 9$ it would be $0, 1, 2, 2, 2, 3$ respectively). This observation leads to the following conjecture:

CONJECTURE 2.8 $\nabla_n = \nabla_{n-1} + \nabla_{n-2} + \mu(n)$, *where* $\mu(n)$ *is a non-decreasing function of* n.

From Table 2, we found that there are still too many choices to test to feasibly find the decycling number of Γ_n if $n \geqslant 10$. In order to decrease the number of choices, we then consider the independent decycling sets, denoting the independent decycling number of Γ_n by ∇_n^0. We have already noted that $\nabla_5^0 = 3$, $\nabla_6^0 = 6$ and $\nabla_7^0 = 11$. By Lemma 2.2, we know that $\nabla_8^0 \geqslant 6 + 11 = 17$, and from the results summarized in Table 2, $\nabla_8^0 \geq 19$. However, it can also be argued theoretically that $\nabla_8^0 = 19$ by considering a line of reasoning similar to that presented in the proof of Lemma 2.7.

As with ∇_n, determining exact values of ∇_n^0 becomes increasingly difficult as n increases. We can, however, make some progress by using the same method described earlier, and noting that if S is an independent decycling set of Γ_n, then $S \cap \mathrm{LOW}(n)$ $\big($resp. $S \cap \mathrm{HIGH}(n)\big)$ must be an independent decycling set for Γ_{n-1} (resp. Γ_{n-2}). By such an exhaustive search, there is no independent decycling set of size less than 55 and 94 for Γ_{10} and Γ_{11} respectively, so $\nabla_{10}^0 = 55$ and $\nabla_{11}^0 = 94$. The details of the results for the independent decycling sets for Γ_n are presented in Table 3 and the corresponding decycling sets are listed online at `www.math.mun.ca/~yubo/research/fib_cube/`.

Table 3. Computer search results of independent decycling set for Γ_n

Γ_n	$\lvert V_n \rvert$	∇_n^0	Number of independent decycling sets, without sub-decycling sets, of size $\nabla_n^0 + i$								
			$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
5	13	3	2	11	7	1	0	0	0	0	0
6	21	6	5	16	7	0	1	0	0	0	0
7	34	11	18	53	34	4	1	0	0	0	0
8	55	19	6	162	586	392	92	4	0	0	0
9	89	33	6	818	6325	10437	7358	2455	394	13	0
10	144	55	1	135	9100	102334	444544	899721	273817		
11	233	94	166								

Based on Table 3, we found that the decycling number and the independent decycling number of Γ_n are equal for $n \leqslant 9$, which suggests this conjecture:

CONJECTURE 2.9 *For the n-dimensional Fibonacci cube* Γ_n,$\nabla_n = \nabla_n^0$.

3 Upper Bound

In Section 2, we studied the lower bound of the decycling number of Fibonacci cubes, and got the exact decycling number of Γ_n when n is small. In this section, we will discuss the upper bound of the decycling number of Γ_n.

By the Decomposition Lemma, we have the following upper bound:

LEMMA 3.1 $\nabla_n \leqslant \nabla_{n-1} + \left\lfloor \dfrac{f_{n-2}}{2} \right\rfloor$.

Proof. Γ_n can be decomposed into two disjoint subgraphs which are isomorphic to Γ_{n-1} and Γ_{n-2} respectively. And these two subgraphs are connected by $\mathrm{LINK}(n)$ (Lemma 2.1). Consider the subgraph induced by a maximum induced forest in LOW(n) and a maximum independent set in HIGH(n); such a graph is clearly acyclic. Since Γ_n is a bipartite graph, it contains a maximum independent set of size at least $\left\lceil \frac{f_n}{2} \right\rceil$, and in [10] it was proved that Γ_n has independence number $\alpha(\Gamma_n) = \left\lceil \frac{f_n}{2} \right\rceil$. So $\nabla_n \leqslant \nabla_{n-1} + f_{n-2} - \left\lceil \frac{f_{n-2}}{2} \right\rceil$, and the upper bound follows. □

Furthermore, we found several independent decycling sets of size 94 in Γ_{11}. After analyzing these 166 decycling sets, we observed that there always exist 19 components in the left-over graph $\Gamma_{11} - S$. This may be used to decrease the upper bound of ∇_n when $n \geqslant 12$. Here is the idea:

Consider the induced forest F formed by the method described in Lemma 3.1. The number of components in F cannot be less than the number of components in the induced forest in LOW(n). Let (X, Y) be the bipartition of HIGH(n), in which one of X or Y consists of n-bit strings (each beginning with 10) having an odd number of 1's (i.e. having odd Hamming weight), and the other set consists of n-bit strings having an even number of 1's. Let $Z \in \{X, Y\}$, such that each vertex of Z is in F and let $Z' = V\big(\mathrm{HIGH}(n)\big) - Z$ be the other set in $\{X, Y\}$. For each vertex $v \in Z'$, if no two of the undeleted neighbours of v in Γ_n are located in the same component of F, then we can undelete v, and the resultant graph clearly remains acyclic. This decreases the upper bound of ∇_n by one. The vertices in Z' whose neighbours are in different components of F form a vertex subset we denote by A. Let C denote the components of F and then form a bipartite graph B with bipartition (A, C), where each component in F and each vertex in A will be a vertex in B, and $E(B) = \{(i, C_j) \,:\, i \in A, C_j \in C, \left|N_{\Gamma_n}(i) \cap V(C_j)\right| = 1\}$. Here $N_G(v)$ denotes the neighbours of vertex v in G. We then find a minimum decycling set S_A of B with the property that $S_A \subseteq A$. Then in Γ_n, remove the vertices in $A - S_A$ from S; the resultant graph is acyclic.

Since B has fewer vertices than Γ_n, we can easily find a minimum decycling set of B by choosing vertices in A. Also we noticed that the size of A is relatively smaller than the size of C in $B = (A, C)$.

By pursuing this idea we were able to reduce the upper bound on ∇_{12} to 158 from the value of 166 obtained from Lemma 3.1; this new upper bound is reflected in Table 1. Examples of decycling sets matching the presented upper bound for Γ_{12}, Γ_{13} and Γ_{14} are archived online at `www.math.mun.ca/~yubo/research/fib_cube/`.

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