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$$J(G;x)=\frac{\prod_{i=0}^{p-1}(x+2i)}{\prod_{i=0}^{m-1}(x+2i)\prod_{i=0}^{n-1}(x+2i)}J(H;x)J(I;x).$$

Proof: By lemma 2.9 $J(G;x)=\prod_{i=0}^{p-1}(x+2i)\sum_{\substack{\text{choices of}\\ \text{flower of } G}} J\left(G\big|_{F^c};x\right)$. But $G\big|_{F^c}$ can be partitioned into the disjoint union of $H\big|_{H\cap F^c}$ and $I\big|_{I\cap F^c}$, so the right-hand side of this becomes $\prod_{i=0}^{p-1}(x+2i)\sum_{\substack{\text{choices of}\\ \text{flower of } G}} J\left(H\big|_{H\cap F^c};x\right)J\left(I\big|_{I\cap F^c};x\right)$. Notice that a choice of flower F_p in G consists of the disjoint union of a flower F_m centered at x in H and a flower F_n centered at x in I. Writing the complement of the flower in H as K, and the complement of the flower in I as M, this becomes

$$\prod_{i=0}^{p-1}(x+2i)\sum_{\substack{\text{choices of}\\ \text{flower of } H \text{ and } I}} J\left(H\big|_{K^c};x\right)J\left(I\big|_{M^c};x\right)$$

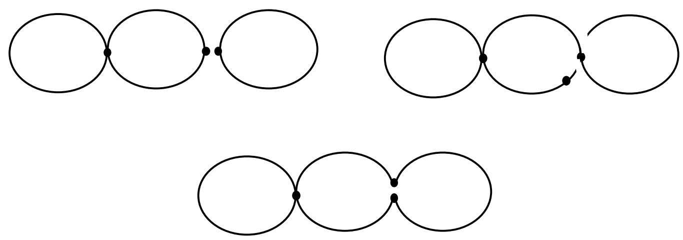
$$=\prod_{i=0}^{p-1}(x+2i)\sum_{\substack{\text{choices of}\\ \text{flower of } H}} J\left(H\big|_{K^c};x\right)\sum_{\substack{\text{choices of}\\ \text{flower of } I}} J\left(I\big|_{M^c};x\right),$$ and the result follows from lemma 2.9. ///

Note that by lemma 2.9 this result does not contradict that $J(G;x)$ is a polynomial.

3. CONCLUSION

The results given here only tidy up a few loose ends. There is certainly much, much more yet to be discovered about the Martin polynomial and its applications. In particular, the Martin polynomial has applications broader than the restriction to Eulerian graphs considered here. For example, Martin gave an analogous polynomial for oriented graphs [M] which was in turn studied by Jaeger and Las Vergnas, and Bouchet has extended it applicability to isotropic systems [B].

then, with v as the center of the flowers F_2, the possible configurations are:



These correspond to the three different vertex states at *w*. Note that the last two are considered different choices of flowers, since there is a different state at the vertex *w* in the two of them, even though they have the same sizes if petals.

With this, the proof follows from definition 2.3 and writing $J(G;x)=J(D^{\bar{v}}(G);x)$ where $\bar{v}$ is the set of all vertices except *v*. As noted above, such as state consists of some flower with *n* petals and some state of the edges not in the flower. Thus, since J is multiplicative, $J(R;x)=J(F_n;x)J(R|_{F^c};x)$, were R is a summand of $D^{\bar{v}}(G)$ and $R|_{F^c}$ is the state of the edges not in F_n.. Thus by factoring out each choice of flower from all the different states in its complement and then summing over all the different flowers that appear, $J(D^{\bar{v}}(G);x)=\sum J(F_n)\sum J(R|_{F^c};x)$ where the outer sum is over all choices of flowers, and the inner sum is over all states of the complement of the flower. But the inner sum then is just $J(G|_{F^c};x)$. Thus since by lemma 2.8, $J(F_n;x)=\prod_{i=0}^{n-1}(x+2i)$ the result follows. ///

COROLLARY 2.10 *If* maxdeg(G) = 2n, *then –2i is a root of* $J(G;x)$ *for all* $i\in\{0,\ldots n-1\}$.

THEOREM 2.11 *Let x be a cut vertex of G so that there are at least two subgraphs H and I which intersect only at x and which when identified at x yield G. Suppose* $\deg_x H=2m$ *and* $\deg_x I=2n$ *so that* $\deg_x G=2p=2(n+m)$. *Then*

Proof: The proof is by induction on n. Since $J(c_i;x)=x$ for all i, the result is trivial for n = 1. Now suppose the result holds for F_k whenever $k \le n-1$, and consider $J(F_n;x)$. Select one of the petals at random and call it P. Then $J(F_n;x) = \sum J(S;x)$ where the sum is over all Eulerian vertex states S at the center vertex v. (Note that since v is the only vertex with degree greater than 2, this is the same as all graph states of F_n.) This is the same as $\sum_{P \text{ intact}} J(S;x) + \sum_{P \text{ spliced}} J(S;x)$, where the sum over "$P$ intact" means those states where P comprises one of the components of the state, and "P spliced" means those states where P is part of some larger component. Note that a state S of F_n with P intact is equal to the disjoint union of P and a state of F_{n-1}, the remaining n – 1 petals. Also a state S of F_n with P spliced in is some state of F_{n-1} with P spliced in one of two ways at one of the n – 1 new vertices in the state. Splicing in P does not change the value of J on the states, since splicing in P only lengthens some component without changing the number of components. Thus the sum becomes

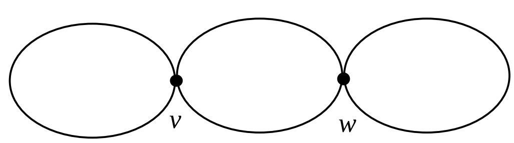
$$J(P;x) \sum_{\substack{S \text{ as state} \\ \text{of } F_n}} J(S;x) + 2(n-1) \sum_{\substack{S \text{ as state} \\ \text{of } F_n}} J(S;x) = (x+2(n-1))J(F_{n-1};x)$$

$$= (x+2(n-1)) \prod_{i=0}^{n-2} (x+2i) = \prod_{i=0}^{n-1} (x+2i) \text{ as claimed. ///}$$

LEMMA 2.9 *Let G be a graph with a vertex v of degree $2n$. Then* $J(G;x) = \prod_{i=0}^{n-1} (x+2i) \sum_{F \in e(G)} J(G|_{F^c};x)$ *where the sum is over all subsets F of the edge set that result from all possible ways of choosing a flower in G with n petals and center v.*

Proof: First a word of clarification about what it means to "choose a flower in G". This means to choose a state at every vertex except v. The result will be the disjoint union of a flower with n petals centered at v and some (possibly empty) set of cycles.

For example, if G is the following graph,



The proof consists of showing that $J(G;2x)=\sum_{A\in U(G)} J(G|_{A};x)J(G|_{A};x)$

and then using lemma 2.4 to translate this to a result for the Martin polynomial.

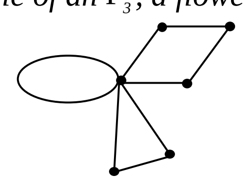
By rewriting in one of the forms given in lemma 2.4, $J(G;2x)=\sum_{A\in U(G)} J(G|_{A};x)J(G|_{A};x)$ becomes

$$\sum_{k=1} \hat{f}_k(G)(2x)^k = \sum_{A\in U(G)} \sum_{i=1} \hat{f}_i(G|_A)x^i \sum_{j=1} \hat{f}_j(G|_{A^c})x^j ,$$

which is true if and only if $2^k \hat{f}_k(G)=\sum_{A\in U(G)} \sum_{i+j=k} \hat{f}_i(G|_A)\hat{f}_j(G|_{A^c})$ for all k. But in any state of G with k components, it is possible to choose any i of the components to make up A, namely $\sum_{i=0}^{k}\binom{k}{i}=2^k$ ways. Thus it is true that $J(G;2x)=\sum_{A\in U(G)} J(G|_{A};x)J(G|_{A};x)$, and so the result follows from lemma 2.4 ///

The following theorem gives a reduction formula for $J(G;x)$ when G has a cut vertex. An analogous result is given in [LV] for a version of the Martin polynomial on oriented graphs, and indicated in the non-oriented case. However, because the computation of $J(G;x)$ retains vertices of degree two, the details of the proof differ slightly.

Definition 2.7 *A* flower graph F_n *consists of* n *disjoint cycles of any lengths identified at a common center vertex (which then has degree 2*n*). An example of an* F_3*, a flower with three petals is:*



LEMMA 2.8 $J(F_n;x)=\prod_{i=0}^{n-1}(x+2i)$ *for any flower F_n.*

As shown below, $J(G;x)$ is just a translation of the Martin polynomial, but the algebra structure motivates computing with $J(G;x)$ rather than $M(G;x)$ in some situations.

LEMMA 2.4 $J(G;x) = xM(G;x+2)$ when $G \neq E$.

Proof: $J(G;x) = \sum_{S \in C(G)} J(S;x) = \sum_{S \in C(G)} J\left(\prod_{i=1} c_i^{a_i(S)};x\right)$ since each Eulerian state *S* of *G* is a disjoint union of cycles of various lengths, i.e. *S* has the form $\prod_{i=1} c_i^{a_i(S)}$. But each cycle evaluates to just *x*, so $\sum_{S \in C(G)} J\left(\prod_{i=1} c_i^{a_i(S)};x\right) = \sum_{S \in C(G)} x^{\sum a_i(S)}$. However, $\sum_{i=1} a_i(S)$ is just the number of components of *S*, so by collecting powers of *x*, $\sum_{S \in C(G)} x^{\sum a_i(S)} = \sum_{k=1} \hat{f}_k(G)x^k = x\sum_{k=0} \hat{f}_{k+1}(G)x^k = xM(G;x+2)$. ///

COROLLARY 2.5 $\frac{1}{x-2} J(G;x-2) = M(G;x)$.

If the Martin polynomial of an edgeless graph *E* is defined to be $\frac{1}{x-2}$, then lemma 2.4 and corollary 2.5 hold when *G* = *E* as well. Note that when considering the possibility that *G* = *E* the sum should start at zero when $J(G;x)$ is written in the form $\sum_{k=0} \hat{f}_k(G)x^k$. Since $\hat{f}_0(G) = 1$ if *G* = *E* and zero otherwise, the form given in the proof of lemma 2.4 is consistent.

THEOREM 2.6 $M(G;x) = \frac{x-2}{4} \sum_{A \in U(G)} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}; \frac{x+2}{2}\right)$, *where* $U(G) = \{A \subseteq e(G) \text{ such that both } G|_A \text{ and } G|_{A^c} \text{ are Eulerian}\}$.

Proof: What follows is a short combinatorial proof of the same result in [E-M]. The considerably longer, algebraic method in [E-M] however has the advantage of showing where $J(G;x)$ comes from, and the interested reader is referred there.

graph separated by v. The last equation tells how to evaluate the Martin polynomial on a "free loop", i.e. a graph consisting of one edge and no vertices.

In [LV], the generalization of this approach to vertices of any even degree is given, and the distinction between cut and non-cut vertices is removed.

This approach exploits the fact that the Martin polynomial is invariant under the addition or removal of vertices of degree 2 since such vertices do not affect the numbers of Eulerian k-partitions. Notice that if vertices of degree 2 are replaced, the first equation has the same form as $D^v(G)$ when v is a vertex of degree 4. The approach used in the present paper, that of retaining the degree 2 vertices, follows [E-M, d]. There the more general polynomial developed encodes information about the *lengths* of the various components as well as just the number of them.

Definition 2.2 Following the discussion in the introduction which established that the Eulerian states of a graph G with k components correspond to the Eulerian k-partitions of G counted with appropriate loop multiplicities, the Martin polynomial can be given as $M(G;x)=\sum_{k=0} \hat{f}_{k+1}(G)(x-2)^k$, where $\hat{f}_k(G)$ is the number of states of G with k components.

Definition 2.3 The polynomial $J(G;x)$ is given by the following recursion relations:

$J(G;x)=J\left(D^v(G;x)\right)$, $J(c_n;x)=x \;\; \forall i \geq 1$, and $J(E;x)=1$, where E is a graph with no edges.

In this definition, a single loop evaluates to x instead of $\frac{x}{2}$ as in [E-M]. This corrects a miscounting of loop multiplicity in that paper, but does not significantly alter any results.

Note that $J(G;x)$ can equivalently be defined by $J(G;x)=\sum_{S \in C(G)} J(S;x) = J\left(D^{\bar{v}}(G;x)\right)$. In other words, J can be computed either by looking at all vertex states at one vertex at a time, or by summing over all possible graph states, or at any stage in-between. A formal proof of this, applicable to general graphs, can be found in [E-M, d]. That $J(G;x)$ is an algebra map (so that if G and H are disjoint graphs then $J(GH;x)=J(G;x)J(H;x)$) and in fact a Hopf-map, can also be found in [E-M, d and E-M].

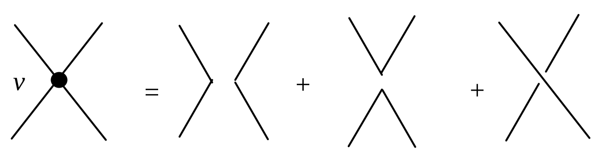
2. The Martin polynomial and $J(G;x)$

Definition 2.1 Following [LV], the Martin polynomial of a non-empty graph *G* is given by

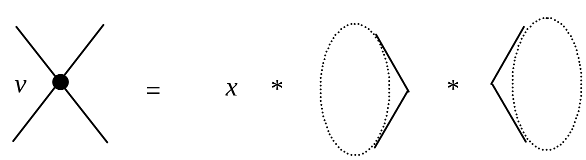
$$M(G;x) = \sum_{k=0} f_k(G)(x-2)^k \ ,$$

where *x* is the independent variable, and $f_k(G)$ is the number of Eulerian *k*-partitions of *G*, *each weighted by the multiple ways of traversing any loops that appear.*

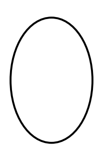
In [J], considering the special case of 4-regular graphs, the multiplicity of loops is addressed by defining loops to consist of two half-edges. The Martin polynomial is then modeled as a transition system, or skein-type polynomial, by the following recursion relations:



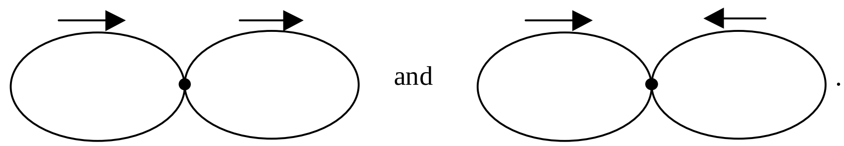
if *v* is not a cut vertex of *G*, and



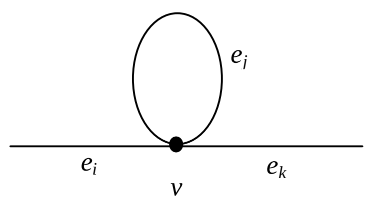
if *v* is a cut vertex, and


$= 1.$

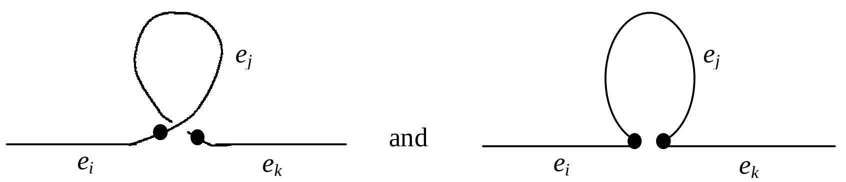
In this definition, the diagrams represent a local reconfiguration at the vertex *v*, with the rest of the graph remaining unchanged. The top equation means that the Martin polynomial of the graph indicated on the left-hand side is equal to the sum of Martin polynomial of the graphs on the right-hand side. The middle equation means that the Martin polynomial of a graph with a cut vertex *v* is the product of *x* and the Martin polynomial of the two components of the



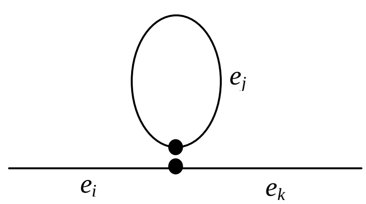
However, when considering vertex states at a vertex incident to one or more loops, there is no ambiguity: each way of traversing the loop appears as a separate state. For example, since a vertex state is a choice of *local* configuration, the vertex v in the following configuration



has the following two vertex states:



Thus, when loops are counted with the appropriate multiplicity, an Eulerian k-partition is exactly an Eulerian graph state with k components. Note that in the above example, the third choice of vertex states at v is:

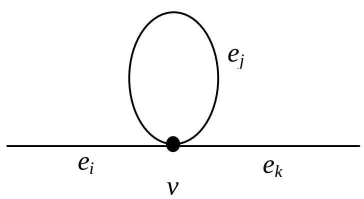


This state has one more component and would correspond to an Eulerian k+1-partition which includes a closed path of length one, {v, e_j, v), and another closed path with the sequence $(\ldots e_i, v, e_k \ldots)$.

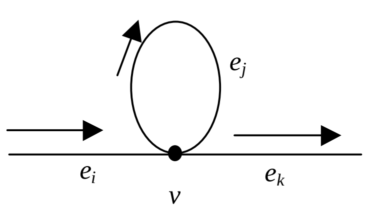
A closed path in a graph is a sequence, $\{v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n = v_0\}$, of distinct edges, e_i, and not necessary distinct vertices v_i, so that the vertices v_{i-1} and v_i are the endpoints of the edge e_i. Two sequences which are simply reversals or cyclic permutations of each other are considered to be the same closed path. Closed paths are referred to as cycles by some authors, but this terminology is avoided here to preclude confusion with the specific definition of the term cycle given above. An Eulerian *k*-partition of a graph *G* is an unordered partition of the edge set of *G* into $k \geq 1$ closed paths.

The definition of a closed path given above, and the relationship between an Eulerian *k*-partition of a graph *G* and an Eulerian graph state of *G* need to be treated with care in the case of a graph with one or more loops. The following discussion clarifies this.

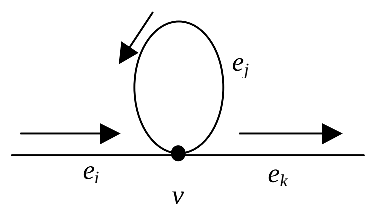
The definition of a closed path above does not distinguish two ways of traversing a loop. Consider the following configuration for example:



The closed path sequence $(\ldots e_i, v, e_j, v, e_k \ldots)$ does not take into account the two different possibilities:

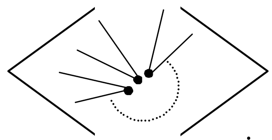


and



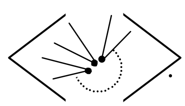
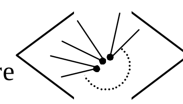
Thus, an Eulerian *k*-partition of a graph *G* is an unordered partition of the edge set of *G* into $k \geq 1$ closed paths, *with loops counted with appropriate multiplicity*. However, it is not simply a matter of counting loops with multiplicity two. For example, whenever an isolated two-loop configuration appears, there are only two ways of traversing the configuration, not four:

Definition 1.1 An *Eulerian vertex state* is a choice of reconfiguration at a vertex of a graph *G*. The reconfiguration consists of replacing a 2*n*-valent vertex *v* with *n* 2-valent vertices joining pairs of edges originally adjacent to *v*. A vertex state may be represented pictorially as



Here the angle brackets indicate a graph which is identical to *G* except in a small neighborhood of *v*. The small neighborhood of *v* is replaced by the chosen vertex state in the new graph. Note that this definition is just a special restriction to Eulerian graphs of the definition of a general vertex state given in [E-M].

Definition 1.2 An *Eulerian graph state* of a graph *G* is the result of choosing one vertex state at each vertex of *G*. Note that a graph state is a disjoint union of cycles. Let *C*(*G*) denote the set of Eulerian graph states of a graph *G*. However, *C*(*G*) is not "up to isomorphism", so that each individual state is listed.

Definition 1.3 $D^{v}(G)=\sum$ . Here  is a choice of states at the vertex *v* which has degree 2*n*. The sum is taken over all choices of Eulerian vertex states at *v*.

Definition 1.4 Let $m=|v(G)|$, and let $v_1,\ldots,v_m$ be an arbitrary ordering of the vertices of *G*. Define $D^{(v_{x_1},\ldots,v_{x_s})}(G)$, or more briefly $D^{\bar{v}}(G)$, to be the graph *G* with vertex states chosen at $v_{x_1},\ldots v_{x_s}$. Note that $D^{(v_{x_1},\ldots,v_{x_s})}(G)$ can equivalently be defined as the composition $D^{v_{x_s}}\circ\ldots\circ D^{v_{x_1}}(G)$. These compositions are performed with the understanding that there is a canonical correspondence between the vertices of a graph *G*, (excluding *v*) and the vertices in each summand of $D^{v}(G)$. Observe that the $D^{v}(G)$ commute with one another. Also note that when $s=m$ then $D^{\bar{v}}(G)$ is simply the formal sum of all Eulerian graph states of *G*, since $s=m$ means that vertex states are chosen at all *m* vertices.

Abstract

The Martin polynomial of a graph, introduced by Martin in his 1977 thesis [M], encodes information about the families of closed paths in Eulerian graphs. A translation of the Martin polynomial, $J(G;x)$, is used in this paper to give a short combinatorial proof of a reduction identity derived algebraically in [E-M]. A reduction formula for $J(G;x)$ involving cut-vertices, analogous to one for the Martin polynomial in [LV], is also derived.

Key words and phrases: Martin polynomial, graph invariants, Eulerian graphs, Eulerian orientations, graph polynomials.

Mathematics subject classification: primary--05C38, secondary--16W30.

1. Introduction

The Martin polynomial of a graph, introduced by Martin in his 1977 thesis [M], encodes information about the families of closed paths in Eulerian graphs. This polynomial and its properties were further refined and developed by Las Vergnas in *Le Polynome de Martin d'un Graphe Eulerien* in 1983 [LV]. A reduction identity for the Martin polynomial was derived in [E-M] using Hopf-algebraic techniques and the fact that the Martin polynomial is essentially an evaluation of a unique universal graph polynomial. The current paper gives a simple, albeit unmotivated, combinatorial proof of the same reduction identity. The evaluation used, $J(G;x)$, is a translation of the Martin polynomial. Without the algebraic context, the form of $J(G;x)$ given in this paper may seem unmotivated, but the interested reader can see [E-M] for details. However, $J(G;x)$ is in some ways easily manipulated algebraically. These manipulations are then translated back to give results for the Martin polynomial. This is the basis for the combinatorial proof of the reduction identity and other formulas given here.

The following conventions are used through this paper. Unless otherwise indicated, graphs may have loops and multiple edges. A graph is called Eulerian if all its vertices have even degrees, but connectedness is not required. Unless otherwise indicated, all graphs in this paper are assumed to be Eulerian. A cycle is a graph isomorphic to a polygon, and is denoted c_n when it has n vertices.

MARTIN POLYNOMIAL MISCELLANEA

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