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New Results for the Martin Polynomial

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Algebraic techniques are used to find several new combinatorial interpretations for valuations of the Martin polynomial, $M(G; s)$, for unoriented graphs. The Martin polynomial of a graph, introduced by Martin in his 1977 thesis, encodes information about the families of closed paths in Eulerian graphs. The new results here are found by showing that the Martin polynomial is a translation of a universal skein-type graph polynomial $P(G)$ which is a Hopf map, and then using the recursion and induction which naturally arise from the Hopf algebra structure to extend known properties. Specifically, when $P(G)$ is evaluated by substituting s for all cycles and 0 for all tails, then $P(G)$ equals $sM(G; s+2)$ for all Eulerian graphs G. The Hopf-algebraic properties of $P(G)$ are then used to extract new properties of the Martin polynomial, including an immediate proof for the formula for $M(G; s)$ on disjoint unions of graphs, combinatorial interpretations for $M(G; 2+2^k)$ and $M(G; 2-2^k)$ with $k \in \mathbf{Z}^{\geqslant 0}$, and a new formula for the number of Eulerian orientations of a graph in terms of the vertex degrees of its Eulerian subgraphs. © 1998 Academic Press

Key Words: Martin polynomial; graph invariants; Eulerian graphs; Eulerian orientations; Hopf algebras; graph polynomials; skein decomposition; invariants given by linear recursion; algebraic combinatorics.

1. INTRODUCTION

The Martin polynomial of a graph, introduced by Martin in his 1977 thesis [9], encodes information about the families of closed paths in Eulerian graphs. This polynomial and its properties were further refined and developed by Las Vergnas in *Le Polynome de Martin d'un Graphe Eulerien* in 1983 [8]. Here in this paper, the Martin polynomial is shown to be a translation of a skein-type graph polynomial defined on a Hopf algebra of graphs. As such, the induction and recursion inherent in the skein decomposition and Hopf algebra structure render it relatively easy to determine several hitherto unknown combinatorial interpretations for various evaluations of the Martin polynomial. A brief description of the Hopf algebraic structures needed is given, as is the construction of Γ, a Hopf-algebra of graphs. This

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is followed by outlines of the main theorems concerning the unique universal skein-type graph polynomial P given in [2]. The results of these universality theorems are used to show that the Martin polynomial is essentially an evaluation of P, and so from the Hopf algebra structure a recursion formula is derived. This formula is then easily manipulated to get a series of new interpretations for the Martin polynomial on unoriented graphs. The final section contains computations and examples.

First, the graph theory terminology used in the following needs to be established. Unless otherwise indicated, the graphs in this paper are allowed to have loops and multiple edges. A graph is called Eulerian if all its vertices have even degrees, but connectedness is not required. A closed path in a graph is a sequence, $\{v_0, e_1, v_1, e_2, v_2, ..., e_n, v_n = v_0\}$, of distinct edges, e_i, and not necessary distinct vertices v_i, so that the vertices v_{i-1} and v_i are the endpoints of the edge e_i. Closed paths are referred to as cycles by some authors, but this terminology is avoided here to preclude confusion with the specific definition of the term cycle given below. An Eulerian k-partition of a graph G is an unordered partition of the edge set of G into $k \geqslant 1$ closed paths. An oriented graph is one in which each edge is assigned a direction, and an Eulerian orientation of a graph G is an orientation in which the in-degree equals the out-degree at each vertex. A circuit is a closed path in an oriented graph so that the edges are oriented in a consistent direction. An anticircuit is a closed path in an oriented graph so that the edges are oriented in directions which alternate as the closed path is traversed. A tail of length i is a graph isomorphic to the graph G defined by $v(G) = \{a_0, ..., a_n\}$, and $e(G) = \{(a_0, a_1), (a_1, a_2), ..., (a_{n-1}, a_n)\}$. A cycle of length i is a graph isomorphic to the graph G defined by $v(G) = \{a_1, ..., a_n\}$, and $e(G) = \{(a_n, a_1), (a_1, a_2), ..., (a_{n-1}, a_n)\}$. Write t_i for a tail of length i, and c_i for a cycle of length i, so for example, t_6 looks like

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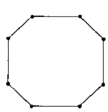
and c_8 looks like



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2. HOPF ALGEBRA STRUCTURE

Below is a brief outline of the Hopf algebra structure which provides the algebraic machinery to manipulate the graph polynomials. Although this paper is fairly self-contained, some familiarity with the essentials of Hopf algebras is helpful. These can be reviewed in [1, 4, 12]. For incidence Hopf algebras, of which Γ below is an example, see [10, 11].

The Hopf algebra of graphs, Γ, is constructed in the following way. Let **G** be the set of finite abstract graphs i.e., graphs defined combinatorially by their vertex and edge sets, independent of any embedding. Loops and multiple edges are allowed (such graphs are sometimes referred to as "multigraphs"). Graphs G and H are identified if $G|_{e(G)}$ is graph isomorphic to $H|_{e(H)}$. Here $e(G)$ and $e(H)$ are the edge sets of G and H respectively. The vertical bar indicates the restriction of a graph to the edges noted in the subscript and their incident vertices. Colloquially, this identification is "discard isolated vertices and identify isomorphic graphs." Note that a graph consisting only of isolated vertices is then identified with the empty graph, denoted E.

Let Γ be the infinite dimensional vector space over the complex numbers **C** with basis **G**.

PROPOSITION 2.1. *Γ has a Hopf algebra structure given as follows: Multiplication m and unit u on $\Gamma \otimes \Gamma$ are given by*

$$m: \Gamma \otimes \Gamma \to \Gamma \qquad \text{by} \quad m(G \otimes H) = G \dot\cup H$$
$$u: \mathbf{C} \to \Gamma \qquad \text{by} \quad u(z) = zE,$$

where $\dot\cup$ indicates disjoint union.

Comultiplication Δ and counit ε are given by

$$\Delta: \Gamma \to \Gamma \otimes \Gamma \qquad \text{by} \quad \Delta(G) = \sum_{F \subseteq e(G)} G|_F \otimes G|_{F^c},$$

where F^c is the set complement of F in $e(G)$.

$$\varepsilon: \Gamma \to \mathbf{C} \qquad \text{by} \quad \varepsilon(G) = \begin{cases} 1 & \text{if } \quad G = E, \text{ the class of the empty graph} \\ 0 & \text{otherwise.} \end{cases}$$

The antipode is given recursively by $s(E) = 1$, and $s(G) = -\sum G|_F \cdot s(G|_{F^c})$.

Proof. Associativity, coassociativity, the unitary and counitary properties, the commutativity of m and the cocommutativity of Δ are all easy to verify, as is the antipode. Showing that Γ is a bialgebra is also straightforward. These proofs appear in [2, propositions 1.1–1.5]. An example of the comultiplication is given in Section 6 of this paper.

PROPOSITION 2.2. *Hereditary sets of graphs, i.e. those which are closed under the formation of subgraphs and disjoint unions, form bases for sub-Hopf algebras of Γ.*

Of importance is the sub-Hopf algebra **X** whose basis is the hereditary set of disjoint unions of tails and cycles.

LEMMA 2.3. *The coalgebra structure of* **X**, *induced by* Γ, *is given by the following explicit formulas*:

Let

$$\bar{a}=(a_1, ..., a_n), \qquad \bar{b}=(b_1, ..., b_n),$$

$$|\bar{a}|=\sum_{i=1}^{n} a_i, \qquad |\bar{b}|=\sum_{j=1}^{n} b_j, \qquad \textit{and} \qquad \|\bar{a}\|=\sum_{i=1}^{n} a_i i, \qquad \|\bar{b}\|=\sum_{j=1}^{n} b_j j,$$

Then the comultiplication is

$$\Delta t_n=\sum (2-\big||\bar{a}|-|\bar{b}|\big|)\frac{|\bar{a}|!\ |\bar{b}|!}{\prod_{i=1}^{n} a_i!\ \prod_{j=1}^{n} b_j!}\prod_{i=1}^{n} t_i^{a_i}\otimes\prod_{j=1}^{n} t_j^{b_j},$$

where the sum is over pairs of n-tuples $(\bar{a}, \bar{b})$ *such that* $\|\bar{a}\|+\|\bar{b}\|=n$, *and* $\big||\bar{a}|-|\bar{b}|\big|\leqslant 1$, *and by*

$$\Delta c_n=E\otimes c_n+c_n\otimes E+\sum\sum_{k=1}^{n} k\left(\frac{a_k}{|\bar{a}|}+\frac{b_k}{|\bar{b}|}\right)$$
$$\times\frac{|\bar{a}|!\ |\bar{b}|!}{\prod_{i=1}^{n} a_i!\ \prod_{j=1}^{n} b_j!}\prod_{i=1}^{n} t_i^{a_i}\otimes\prod_{i=1}^{n} t_j^{b_j},$$

where the sum is over pairs of n-tuples $(\bar{a}, \bar{b})$ *such that* $\|\bar{a}\|+\|\bar{b}\|=n$, *and* $|\bar{a}|-|\bar{b}|=0$.

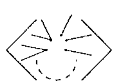
Proof. Essentially, computing Δt_n means, for each pair of elements $\prod_{i=0}^{n} t_i^{a_i}$ and $\prod_{j=0}^{n} t_j^{b_j}$, counting how many subsets $F\subseteq e(t_n)$ there are such that $t_n|_F=\prod_{i=0}^{n} t_i^{a_i}$ and $t_n|_{F^c}=\prod_{j=0}^{n} t_j^{b_j}$. Another way of describing this counting (cf. [6, p. 96]) is that it is computing the "section coefficients" $(t_n \mid \prod_{i=0}^{n} t_i^{a_i}, \prod_{j=0}^{n} t_j^{b_j})$. Similarly, to compute the section coefficients in Δc_n, first notice that $E\otimes c_n+c_n\otimes E$ comes from $F=\varnothing$ or $e(c_n)$, so for the rest of the terms, neither F nor F^c is empty. Notice that if neither F nor F^c is empty, then $c_n|_F=\prod_{i=0}^{n} t_i^{a_i}$ and $c_n|_{F^c}=\prod_{j=0}^{n} t_j^{b_j}$ for some $\bar{a}$ and $\bar{b}$, so the middle terms are very similar to those that arise in computing Δt_n. The counting is straightforward combinatorics and given in full detail in [2, Lemma 1.11].

3. A UNIQUE UNIVERSAL GRAPH POLYNOMIAL COMPATIBLE WITH Γ

The graph polynomial $P(G)$ developed in this section was motivated by knot polynomials computed via skein relations, or more generally, linear recursion relations. (See [13] for a discussion of linear recursion relations with many examples both in knot and graph theory). One example is the Kauffman bracket given in [7] which is computed using the following skein relation:

The angle brackets here indicate a link diagram of which only one crossing (on the left hand side of the equation) or choice of state or “splitting” (on the right hand side of the equation) is depicted. The diagrams away from the depicted sites are identical for each of the summands. A specific choice of configuration at a crossing site is called a choice of state at that crossing. The diagram which results from choosing a state at each crossing is called a state of the link, and consists of a disjoint (unlinked, unknotted) union of circles. Repeated applications of the skein relation (this is the linear recurrence) eventually decompose the link diagram into a formal sum of link states with appropriate coefficients. To then give the invariant, each set of disjoint circles in the resulting summands is replaced by an appropriate choice of variables. Thus a graph polynomial which appropriates this knot theory notion of choosing states at crossing sites would “decompose” the graph at each vertex, the vertices of a graph playing a role similar to the crossing sites of a knot diagram.

DEFINITION 3.1. A *vertex state* is a choice of reconfiguration at a vertex of a graph G. The reconfiguration consists of replacing an n-valent vertex v with j 2-valent vertices joining pairs of edges originally adjacent to v, and $n-2j$ 1-valent vertices. A vertex state may be represented pictorially as



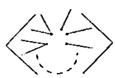
Here the angle brackets indicate a graph which is idertical to G except in a small neighborhood of v. The small neighborhood of v is replaced by the chosen vertex state in the new graph. Note that this definition may be formalized by defining the vertices and edges of the new graph strictly in terms of the vertices and edges of the original graph, but the bulky notation needed would obscure this very simple concept.

DEFINITION 3.2. A *graph state* of a graph G is the result of choosing a vertex state at each vertex of G. Note that a graph state is a disjoint union of cycles and tails. Let $S(G)$ denote the set of graph states of a graph G.

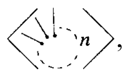
DEFINITION 3.3.

$$D_{2j}^{v,n}(G) = \sum$$  .

Here



is a choice of states at the vertex v which has degree n. The sum is taken over all the vertex states at v which have $2j$ paired edges. For example, if G has an n-valent vertex v, then

$$D_0^{v,n}(G) =$$ 

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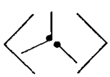
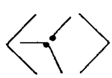
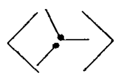
and if G has a 3-valent vertex v, then

$$D_2^{v,3}(G) =$$  +  +  .

Note that in each summand, the graph away from the reconfigured vertex remains unchanged. $D_{2j}^{v,n}(G)$ will be denoted $\sum_{r=1}^{\beta(n,j)} D_{2j,r}^{v,n}(G)$ when it is necessary to differentiate among the summands. Each $D_{2j,r}^{v,n}(G)$ is a choice of vertex state at v, and the vertex states are arbitrarily indexed by $r = 1 \ldots \beta(n, j)$. Here $\beta(n, j)$ is simply the number of ways of choosing j pairs of edges from the n edges adjacent to the vertex, so

$$\beta(n, j) = \frac{(2j)!}{j!\,2^j} \binom{n}{2j}.$$

Notice that the order of the summands indexed by r in $\sum_{r=1}^{\beta(n,j)} D_{2j,r}^{v,n}(G)$ is (necessarily!) arbitrary since there is no consistent way to distinguish among the summands, at least not for the abstract graphs considered here.

For graphs with particular embeddings (e.g. in the plane) it often is possible to distinguish among the summands and so give an explicit indexing by r as well. In fact, letting coefficients depend on r leads to invariants such as the Penrose polynomial and other such transition polynomials. The interested reader is referred to [5].

DEFINITION 3.4. For any complex-valued function $s(n, j)$ defined for pairs of integers (n, j) with $0 \leqslant n$ and $0 \leqslant j \leqslant [n/2]$, let $V^{v,n}(G) = \sum_{j=0}^{[n/2]} s(n, j)\, D_{2j}^{v,n}(G)$.

DEFINITION 3.5. Let $m = |v(G)|$, and let $v_1, ..., v_m$ be an arbitrary ordering of the vertices of G. Let $n_i = \deg(v_i)$. Define

$$D_{2(j_{x_1}, ..., j_{x_s}), (r_{x_1}, ..., r_{x_s})}^{(v_{x_1}, ..., v_{x_s}), (n_{x_1}, ..., n_{x_s})}(G)$$

(or more briefly $D_{2\bar{j}, \bar{r}}^{\bar{v}, \bar{n}}(G, \overline{x_s})$) to be the graph G with vertex states chosen at $v_{x_1}, ..., v_{x_s}$, where the vertex state replacing vertex v_k has j_k 2-valent vertices and is choice r_k (in some fixed, arbitrary, indexing of the states of v_k).

The expression $D_{2\bar{j}, \bar{r}}^{\bar{v}, \bar{n}}(G, \overline{x_s})$ can equivalently be defined as the composition

$$D_{2j_{x_s}, r_{x_s}}^{v_{x_s}, n_{x_s}} \circ \cdots \circ D_{2j_{x_1}, r_{x_1}}^{v_{x_1}, n_{x_1}}(G).$$

These compositions are performed with the understanding that there is a canonical correspondence between the vertices of a graph G, excluding v, and the vertices in each summand of $D_{2j}^{v,n}(G)$. Observe that the $D_{2j, r}^{v,n}(G)$ commute with one another. Also note that $D_{2\bar{j}, \bar{r}}^{\bar{v}, \bar{n}}(G, \overline{x_m})$, where $m = |v(G)|$, is simply a graph state of G, since the m in the subscript of x indicates that vertex states are chosen at all m vertices.

DEFINITION 3.6. Let

$$V^{\bar{v}, \bar{n}}(G, \overline{x_s}) = V^{v_{x_s}, n_{x_s}} \circ \cdots \circ V^{v_{x_1}, n_{x_1}}(G),$$

or equivalently,

$$V^{\bar{v}, \bar{n}}(G, \overline{x_s}) = \sum_{\substack{\bar{0} \leqslant \bar{j} \leqslant \overline{[n/2]} \\ \bar{1} \leqslant \bar{r} \leqslant \overline{\beta(n, j)}}} \prod_{i=1}^{s} s(n_{x_i}, j_{x_i})\, D_{2\bar{j}, \bar{r}}^{\bar{v}, \bar{n}}(G, \overline{x_s}),$$

where $\overline{[n/2]} = ([n_{x_1}/2], ..., [n_{x_s}/2])$, and $\overline{\beta(n, j)} = (\beta(n_{x_1}, j_{x_1}), ..., \beta(n_{x_1}, j_{x_1}))$. Notice that the $V^{v,n}(G)$ also commute. Also, if $s = m$, since $D_{2\bar{j}, \bar{r}}^{\bar{v}, \bar{n}}(G, \overline{x_m})$ is nothing more than a choice of graph state, then

$$V^{\bar{v}, \bar{n}}(G, \overline{x_m}) = \sum_{S \in S(G)} \left\{ \prod_{v \in v(G)} s(\deg(v), j(v, S)) \right\} S,$$

where $j(v, S)$ is the number of 2-valent vertices in the vertex state of v appearing in the graph state S. Note that this last form is clearly independent of the ordering of the vertices of G.

To simplify notation in later applications, let $\alpha(v, S) = s(\deg(v), j(v, S))$, so that

$$V^{\bar{v}, \bar{n}}(G, \overline{x_m}) = \sum_{S \in S(G)} \left\{ \prod_{v \in v(G)} \alpha(v, S) \right\} S.$$

Definition 3.7 (Graph Invariants Given by Skein-Type Linear Recurrence Relations). A linear graph invariant Q on span$\{\mathbf{G}\}$ given by skein-type linear recurrence relations is one which is defined by a family of linear recurrence relations of the form $Q(G) = \sum_{j=0}^{[n/2]} s(n, j)\, Q(D_{2j}^{v, n}(G))$, where $s(n, j) \in \mathbf{C}$. Note that $s(n, j)$ can not depend on any indexing by r of $D_{2j}^{v, n}(G) = \sum_{r=1}^{\beta(n, j)} D_{2j, r}^{v, n}(G)$ since the indexing by r is arbitrary. Furthermore, s cannot depend on v because it is in general impossible to distinguish among the vertices of an abstract graph G.

Henceforth, graph invariants given by skein-type linear recurrence relations will be referred to as *skein-type*.

Using definition 3.4, $Q(G) = \sum_{j=0}^{[n/2]} s(n, j)\, Q(D_{2j, r}^{v, n}(G))$ can be written as $Q(G) = Q(V^{v, n}(G))$.

Lemma 3.8. *If G is a graph with m vertices, then $Q(G) = Q(V^{\bar{v}, \bar{n}}(G, \overline{x_s}))$ for any $s \in \{1, ..., m\}$.*

In other words, applying Q to G is the same as applying Q to the sum (weighted by $s(n, j)$) over all possible combinations of vertex states where vertex states are chosen only at some subset of the vertices of G. This is just the linear recursion process being terminated before the vertex decomposition has been applied to all of the vertices.

Proof. The proof is by induction on s and is given in detail in [2, Lemma 2.1].

Corollary 3.9. *If Q is a skein-type graph invariant, then*

$$Q(G) = \sum_{S \in S(G)} \prod_{v \in v(G)} \alpha(v, S)\, Q(S),$$

recalling that $S(G)$ is the set of graph states of G.

Proof. This is just Lemma 3.8 with $s = m = |v(G)|$ since by Lemma 3.8 $Q(G) = Q(V^{\bar{v}, \bar{n}}(G, \overline{x_s}))$, for any s with $1 \leqslant s \leqslant m$, so $Q(G) = Q(V^{\bar{v}, \bar{n}}(G, \overline{x_m}))$. But, as observed in Definition 3.6,

$$V^{\bar{v}, \bar{n}}(G, \overline{x_m}) = \sum_{S \in S(G)} \left\{ \prod_{v \in v(G)} \alpha(v, S) \right\} S,$$

so by the linearity of Q it follows that

$$Q(G) = \sum_{S \in S(G)} \prod_{v \in v(G)} \alpha(v, S)\, Q(S). \quad \blacksquare$$

In particular, Corollary 3.9 means that a skein-type graph invariant is independent of the order in which the vertex decompositions are applied.

With the definitions and lemmas just given, the following three results are virtually self-evident.

THEOREM 3.10. *If Q is a linear graph invariant given by skein-type linear recurrence relations as in Definition* 3.7, *then Q is completely determined by its action on disjoint unions of cycles and tails and by the function $s(n, j)$.*

LEMMA 3.11. *If Q is linear graph invariant given by skein-type linear recurrence relations as in Definition* 3.7, *then $Q(GH) = Q(G)\, Q(H)$ for all graphs G and $H \Leftrightarrow Q(GH) = Q(G)\, Q(H)$ whenever G is a cycle or a tail and H is also either a cycle or a tail.*

THEOREM 3.12. *If Q is a linear graph invariant given by skein-type linear recurrence relations as in Definition* 3.7 *and Q is an algebra map on the space of cycles and tails, then Q is completely determined by its action on cycles and tails* (*the sub-Hopf algebra $\mathbf{X} \subseteq \Gamma$*) *and by the function $s(n, j)$.*

The next theorem, which says that if Q is in addition a Hopf map then the function $s(n, j)$ is uniquely determined, is however far from self-evident. This is the central result which makes the applications of $P(G)$ below possible. The proof however is very long and technical and forms the bulk of [2]. An abridged version of the proof can be found in [3], while only a sparse outline is given below.

THEOREM 3.13. *Let Ω be a Hopf Algebra. If Q is a skein-type linear graph invariant given so that $Q\colon \Gamma \to \Omega$ is a Hopf map, then Q is entirely determined by its action on the sub-Hopf Algebra $\mathbf{X} \subseteq \Gamma$. Specifically,*

$$Q(G) = \sum_{S \in S(G)} \prod_{v \in v(G)} h(\deg v - 2j(v, S))\, Q(S),$$

where $S(G)$ is the set of states of G, $j(v, S)$ is the number of pairs of edges in the vertex state of v appearing in the graph state S, and $h(w) = 2^{-w/2}H_w(2^{-1/2})$ where $H_w(x)$ is the w^{th} Hermite polynomial

$$H_w(x) = \sum_{k=0}^{[w/2]} (-1)^k \frac{w!}{k!\,(w-2k)!}(2x)^{w-2k}.$$

Proof. The following is a brief outline of the steps involved in this proof.

Step 1. Show that if Q is a skein-type linear graph invariant so that $Q: \Gamma \to \Omega$ is a Hopf map, then

$$\sum_{F \subseteq e(G)} \sum_{j,k,i,m} g(d(v,F), j, k)\, Q(D_{2j,i}^{v,d(v,F)}(G|_F)) \otimes Q(D_{2k,m}^{v,d(v,F^c)}(G|_{F^c})) = 0 \otimes 0,$$

where

$$g(d(v,F), j, k) = s(d(v,F), j)\, s(d(v,F^c), k) - \sum_{a=0} a! \binom{d(v,F)-2j}{a} \times \binom{d(v,F^c)-2k}{a} s(d(v,F)+d(v,F^c), j+k+a).$$

Step 2. There are then several cases to consider. The various cases arise depending on how Q acts on the cycles and tails; for example, Q may map all the tails to zero, or Q may agree on t_1^2 and t_2, etc. In each case, an appropriate graph G is chosen, typically a graph with only one vertex of degree greater than two. Then the equation in Step 1 is applied to G with the skein decomposition applied at the vertex with degree greater than two. In each case, there is some specific information about how Q acts on certain cycles and tails, and about the linear relationships among their images. The graph G is chosen so that when the vertex decomposition is applied, the resultant states are products of these particular cycles and tails.

Step 3. Characterize $D_{2j,i}^{v,d(v,F)}(G|_F) \in \Gamma$. Since the decomposition will be applied at the only vertex of G with degree greater than two, this will be a disjoint union of cycles and tails of specific lengths.

Step 4. Characterize the elements appearing in $Q(D_{2j,i}^{v,d(v,F)}(G|_F)) \in \Omega$. Find a basis for these elements, call it U, and note that since U is linearly independent, the tensor product $U \otimes U$ is also linearly independent.

Step 5. Make a nice choice of $u_1, u_2 \in U$ and isolate the coefficient of $u_1 \otimes u_2$ in

$$\sum_{F \subseteq e(G)} \sum_{j,k,i,m} g(d(v,F), j, k)\, Q(D_{2j,i}^{v,d(v,F)}(G|_F)) \otimes Q(D_{2k,m}^{v,d(v,F^c)}(G|_{F^c})) = 0 \otimes 0.$$

The coefficient will be some expression involving $g(d(v, F), j, k)$ for various F, j and k.

However, since

$$\sum_{F \subseteq e(G)} \sum_{j, k, i, m} g(d(v, F), j, k)\, Q(D_{2j, i}^{v, d(v, F)}(G|_F)) \otimes Q(D_{2k, m}^{v, d(v, F^c)}(G|_{F^c})) = 0 \otimes 0.$$

and $U \otimes U$ is linearly independent, the coefficient of $u_1 \otimes u_2$ must be 0. This gives an equation involving $g(d(v, F), j, k)$ for various F, j and k.

Step 6. Note that the equation in $g(d(v, F), j, k)$ is really an equation in the $s(n, w)$ for various n and w since from above $g(d(v, F), j, k)$ is an expression in $s(n, w)$. Use this equation to determine that $s(n, w) = 2^{-(n-2w)/2} H_{n-2w}(2^{-1/2}) = h(n-2w)$ for all n and w, where $H_m(x)$ is the m^{th} Hermite polynomial.

Step 7. This will complete the proof of Theorem 3.13, since by Corollary 3.9, Q has the form

$$Q(G) = \sum_{S \in S(G)} \prod_{v \in v(G)} s(\deg v,\, j(v, S))\, Q(S).$$

But since $s(n, w) = h(n-2w)$, it follows that

$$Q(G) = \sum_{S \in S(G)} \prod_{v \in v(G)} h(\deg v - 2j(v, S))\, Q(S)$$

as claimed. In other words, the coefficients are fixed, and so since for each graph G, $S(G) \subseteq \mathbf{X}$, it follows that Q is completely determined by its action on $\mathbf{X}$.

The following two theorems give the universal polynomial $P(G)$ and its unique extension property. These will be used to get new results for the Martin polynomial.

THEOREM 3.14 (A Unique, Universal Skein-Type Graph Polynomial). *Let $P: \Gamma \to \mathbf{X}$ be the skein-type graph polynomial given by $P(G) = \sum_{j=0}^{n/2} h(n-2j)\, P(D_{2j}^{v, n}(G))$, where P is the identity map on $\mathbf{X}$, the sub-Hopf algebra of Γ generated by the cycles and tails, and where $h(n-2j) = 2^{-(n-2j)/2} H_{n-2j}(2^{-1/2})$. Then $P: \Gamma \to \mathbf{X}$ is a Hopf map and P is unique and universal in that it is the unique Hopf map skein-type graph polynomial with the universal property that any other skein-type graph polynomial $R: \Gamma \to \Omega$ for some Hopf algebra Ω must be a valuation of P. In other words, $R(G) = R|_X(P(G))$, so that $R(G)$ is found by computing $P(G)$ and then evaluating c_j at $R(c_j)$, and t_j at $R(t_j)$ for all j.*

Despite the bulky notation, $P(G)$ is easy to see pictorially, and an example is given in Section 6.

Proof. Since by [10, Lemma 4.04] any bialgebra map between two Hopf algebras is automatically a Hopf map, it suffices to show that P is a bialgebra map. That P is an algebra map follows immediately from Lemma 3.12 since P is the identity, and hence multiplicative, on X.

Showing that P is a coalgebra map involves an identity on the Hermite polynomials and the fact that P is idempotent. Again the details can be found in [2, Theorem 3.23].

The universality of $P(G)$ then follows almost immediately from Theorem 3.13. By Corollary 3.9,

$$P(G) = \sum_{S \in S(G)} \prod_{v \in v(G)} s(\deg v, j(v, S))\, P(S)$$
$$= \sum_{S \in S(G)} \prod_{v \in v(G)} h(\deg v - 2j(v, S))\, P(S)$$

which equals $\sum_{S \in S(G)} \prod_{v \in v(G)} h(\deg v - 2j(v, S))\, S$ since P is the identity on $\mathbf{X}$ and, since, as a disjoint union of cycles and tails, each state S is an element of $\mathbf{X}$.

Now suppose R: $\Gamma \to \Omega$ is a skein-type Hopf map. Then by Theorem 3.13,

$$R(G) = \sum_{S \in S(G)} \prod_{v \in v(G)} h(\deg v - 2j(v, S))\, R(S).$$

But since R is an algebra map and each state S is an element of $\mathbf{X}$, this is just $(R|_X \circ P)(G)$, as claimed.

The uniqueness argument is standard: If Q: $\Gamma \to X$ is a skein-type graph polynomial which also has this universal property, then $P(G) = P|_X(Q(G))$, but $P|_X$ is the identity, so $P(G) = Q(G)$.

THEOREM 3.15 (Unique Extension). *If B: $X \to \Omega$ is a Hopf map, then it can be extended uniquely to a skein-type graph polynomial B' which is a Hopf map.*

Proof. Existence: Let $B' = B \circ P$. Since both B and P are Hopf maps, so is B'. Furthermore

$$B'(G) = B \circ P(G) = B\left(\sum_{j=0}^{n/2} h(n-2j)\, P(D_{2j}^{v,n}(G))\right)$$
$$= \sum_{j=0}^{n/2} h(n-2j)\, B \circ P(D_{2j}^{v,n}(G))$$

for any $G \in \Gamma$ since $\sum_{j=0}^{n/2} h(n-2j)\, P(D_{2j}^{v,n}(G)) \in \mathbf{X}$, and B is linear on $\mathbf{X}$. But

$$\sum_{j=0}^{n/2} h(n-2j)\, B \circ P(D_{2j}^{v,n}(G)) = \sum_{j=0}^{n/2} h(n-2j)\, B'(D_{2j}^{v,n}(G)),$$

and so it follows that $B'(G)=\sum_{j=0}^{n/2} h(n-2j)\, B'(D_{2j}^{v_i n}(G))$. Thus $B'=B\circ P$ is a skein-type graph polynomial which is a Hopf map. Also, if $\prod_i c_i^{\alpha_i} \prod_j t_j^{\beta_j}$ is an arbitrary element of X, then

$$B'\left(\prod_i c_i^{\alpha_i} \prod_j t_j^{\beta_j}\right)=B\circ P\left(\prod_i c_i^{\alpha_i} \prod_j t_j^{\beta_j}\right)=B\left(\prod_i c_i^{\alpha_i} \prod_j t_j^{\beta_j}\right),$$

so B' does extend B.

Uniqueness: Let $B''\colon \Gamma\to\Omega$ be a skein-type Hopf map so that B'' also extended B, i.e., so that $B''|_X=B$. Then since $B''\colon \Gamma\to\Omega$ is a skein-type Hopf map, by Theorem 3.14, $B''=B''|_X\circ P$. But then $B''=B''|_X\circ P=B\circ P=B'$, so B' is unique. ▮

4. THE MARTIN POLYNOMIAL AS A TRANSLATION OF $P(G)$

The definition below of the Martin polynomial of an unoriented Eulerian graph is from [8], and formalizes modifications, such as the extension to non-connected graphs, and the weighting necessary for graphs with loops, which are simply given verbally in [8].

Definition 4.1. The Martin polynomial of a graph G is given by

$$M(G;x)=\sum_{\substack{k=0\\ m=0}} f_{(k+1,m)}(G)\cdot 2^{b(G)-m}(x-2)^k,$$

where x is the independent variable, $b(G)$ is the number of loops of G, and $f_{(k,m)}(G)$ is the number of Eulerian k-partitions of G with m closed paths of length one (copies of c_1 in the terminology from the preceding sections).

Definition 4.2. The binomial bialgebra, **B**, is the polynomial algebra $\mathbf{C}[x]$ with coalgebra structure given by specifying x to be primitive so that $\Delta x^n=\sum_{j=0}^{n}\binom{n}{j} x^j\otimes x^{n-j}$.There is also an antipode given by $\mathscr{S}(x)=-x$, so that **B** is actually a Hopf algebra. See [6] for more about the binomial bialgebra.

Definition 4.3. The polynomial $J\colon \mathbf{X}\to\mathbf{B}$ is the algebra map given by $J(c_1)=x/2$, $J(c_i)=x$ for all $i\geqslant 2$, $J(t_i)=0$ for all $i\geqslant 1$, and $J(E)=1$.

Lemma 4.4. *J is a Hopf map.*

Proof. It suffices to show that J is a coalgebra map, since by [10, Theorem 4.04] a map which is both an algebra and a coalgebra map between two Hopf algebras is automatically a Hopf map.

That J is a coalgebra map is verified by the following equations:

$$(J\otimes J)\,\Delta_{\mathbf{X}}(c_1)=(J\otimes J)(1\otimes c_1+c_1\otimes 1)=1\otimes\frac{x}{2}+\frac{x}{2}\otimes 1=\Delta_{\mathbf{B}}\frac{x}{2}=\Delta_{\mathbf{B}}\,J(c_1).$$

Furthermore, if $j\geqslant 2$, then $(J\otimes J)\,\Delta_{\mathbf{X}}(c_j)=(J\otimes J)(1\otimes c_j+c_j\otimes 1+A)$, where, as seen in Lemma 2.3, the term A is a sum of tensor products of products of tails, and hence is mapped to $0\otimes 0$ by $J\otimes J$. Thus this becomes $1\otimes x+x\otimes 1+0\otimes 0=\Delta_{\mathbf{B}}\,x=\Delta_{\mathbf{B}}\,J(c_j)$.

Also, since the comultiplication takes a tail to a sum of tensor product of products of tails, which is then mapped to $0\otimes 0$ by $J\otimes J$, it follows that $(J\otimes J)\,\Delta_{\mathbf{X}}(t_i)=0\otimes 0=\Delta_{\mathbf{B}}\,0=\Delta_{\mathbf{B}}\,J(t_i)$ for all $i\geqslant 1$.

Lastly, $(J\otimes J)\,\Delta_{\mathbf{X}}(E)=(J\otimes J)(E\otimes E)=1\otimes 1=\Delta_{\mathbf{B}}\,1=\Delta_{\mathbf{B}}\,J(E)$.

To see that the requirement on the counit is satisfied, note that E is the only element of $\mathbf{X}$ whose image has a non-zero constant term. Thus $w\in\ker J$ implies that E is not a summand of w, and thus $\varepsilon_{\mathbf{X}}(w)=0$.

Thus J is a coalgebra map, and hence a Hopf map. ∎

LEMMA 4.5. *J may be extended to Γ in such a way that $J\colon\Gamma\to\mathbf{B}$ is a skein-type graph polynomial and Hopf map.*

Proof. Since by Lemma 4.4 $J\colon\mathbf{X}\to\mathbf{B}$ is a Hopf map, Theorem 3.15 applies, giving that there exists a unique skein-type graph polynomial J' given by $J\circ P(G)$, or equivalently by $J'(G)=\sum_{j=0}^{n/2} h(n-2j)\,J'(D_{2j}^{v,n}(G))$. ∎

To simplify notation, J' will be written simply as J, and to facilitate using the independent variable x, $J(G)$ will be written as $J(G;x)$.

THEOREM 4.6. *$J(G;x)=(x/2^{b(G)})\,M(G;x+2)$ if G is an Eulerian graph.*

Proof. Since $J\colon\Gamma\to\mathbf{B}$ is a skein-type graph polynomial and Hopf map, by Corollary 3.9, $J(G;x)$ can be written as

$$\sum_{S\in S(G)}\prod_{v\in v(G)} h(\deg v-2j(v,S))\,J(S;x).$$

But since $J(t_i;x)=0$ for all tails, the only states of G which do not map to zero are those states which consist entirely of cycles. However, in such states, $j(v,S)$ must be $(\deg v)/2$, and $\deg v$ must be even, which it is when G is Eulerian. Furthermore, $h(\deg v-2\,(\deg v)/2)=h(0)=1$, so $J(G;x)=\sum_{S\in C(G)} J(S;x)$, where $C(G)\subseteq S(G)$ is the set of cycle states of G, i. e. those states of G which are composed entirely of cycles. Thus $S \in C(G)$ has the form $c_1^{a_1(S)} \prod_{i=2} c_i^{a_i(S)}$. With this,

$$J(G; x) = \sum_{S \in C(G)} J(c_1^{a_1(S)}; x)\, J\left(\prod_{i=2} c_i^{a_i(S)}; x\right)$$

$$= \sum_{S \in C(G)} \left(\frac{x}{2}\right)^{a_1(S)} x^{\sum_{i=2} a_i(S)} = \sum_{S \in C(G)} (2)^{-a_1(S)}\, x^{\sum_{i=1} a_i(S)}.$$

However, note that $\sum_{i=1} a_i(S)$ is just the number of cycles in the cycle state S. In particular, each Eulerian k-repartition of G corresponds to a cycle state S with $\sum_{i=1} a_i(S) = k$. Also, $a_1(S) = m$, the number of closed paths of length one in the k-partition. Thus, by collecting powers of x,

$$J(G; x) = \sum_{\substack{k=1\\ m=0}} f_{(k,m)}(G) \cdot 2^{-m} x^k = \sum_{\substack{k=0\\ m=0}} f_{(k+1,m)}(G) \cdot 2^{-m} x^{k+1}$$

$$= \frac{x}{2^{b(G)}} \sum_{\substack{k=0\\ m=0}} f_{(k+1,m)}(G) \cdot 2^{b(G)-m} x^{k+1} = \frac{x}{2^{b(G)}} M(G; x+2),$$

as claimed. ∎

Note that if G is non-empty and not Eulerian, then every state has a tail, and hence $J(G; x) = 0$. Thus, if $M(E; x)$ is set to be $1/(x-2)$, then by Theorem 4.6 the Martin polynomial's domain can be extended to all graphs, including the empty graph, by redefining the Martin polynomial as $M(G; x) = (2^{b(G)}/(x-2))\, J(G; x-2)$. The technical detail that Γ is actually the space of equivalence classes of graphs under graph isomorphism and ignoring isolated vertices creates no concerns here, since the Martin polynomial is constant on these isomorphism classes. Since $J(G; x)$ is a polynomial with no constant term, then $(2^{b(G)}/(x-2))\, J(G; x-2)$ is a polynomial whenever $G \neq E$.

5. NEW RESULTS FOR THE MARTIN POLYNOMIAL

Having established the Martin polynomial as a translations of a skein-type Hopf map, all the Hopf-algebraic machinery can now be used to extract combinatorial information from it.

THEOREM 5.1. *$M(GH; x) = (x-2)\, M(G; x)\, M(H; x)$.*

Proof. A combinatorial proof for a similar result on oriented Eulerian graphs is given in [8, p. 400], and can be mimicked to get this result. The result can also be achieved for the special case where G is 4-regular (i.e., where all vertices have degree 4) by combining propositions 5 and 2(i) in [5].

Here, however, the result for all graphs follows immediately from J being an algebra map:

$$M(GH;x) = \frac{2^{b(GH)}}{x-2}\,J(GH;x-2) = \frac{(x-2)\,2^{b(G)}2^{b(H)}}{(x-2)(x-2)}\,J(G;x-2)\,J(H;x-2)$$
$$= (x-2)\,M(G;x)\,M(H;x). \quad \blacksquare$$

THEOREM 5.2.

$$M(G;x)$$
$$= \frac{x-2}{4} \sum_{A \in U(G)} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}; \frac{x+2}{2}\right)$$
$$= M\left(G; \frac{x+2}{2}\right) + \frac{x-2}{4} \sum_{\substack{A \in U(G) \\ A \neq E,\, G}} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}; \frac{x+2}{2}\right),$$

where

$$U(G) = \{A \subseteq e(G) \text{ such that both } G|_A \text{ and } G|_{A^c} \text{ are Eulerian, where } A^c = e(G) - A\}.$$

Proof. Since J is a Hopf map, $\Delta_{\mathbf{B}}\,J(G;x) = (J \otimes J)\,\Delta_\Gamma\,G$. This implies that

$$J(G; 1 \otimes x + x \otimes 1) = (J \otimes J)\left(\sum_{F \subseteq e(G)} G|_F \otimes G|_{F^c}\right)$$
$$= \sum_{F \subseteq e(G)} J(G|_F;x) \otimes J(G|_{F^c};x).$$

However, unless $F \in U(G)$, then at least one of $G|_F$ or $G|_{F^c}$ is not Eulerian, and hence $J(G|_F;x) \otimes J(G|_{F^c};x) = 0 \otimes 0$. Thus

$$J(G; 1 \otimes x + x \otimes 1) = \sum_{A \in U(G)} J(G|_A;x) \otimes J(G|_{A^c};x).$$

Applying the multiplication map m to both sides of this equation gives that

$$J(G;2x) = \sum_{A \in U(G)} J(G|_A;x)\,J(G|_{A^c};x).$$

Now by Theorem 4.6, this implies that

$$\frac{2x}{2^{b(G)}} M(G; 2x+2) = \sum_{A \in U(G)} \frac{x}{2^{b(G|_A)}} M(G|_A; x+2) \frac{x}{2^{b(G|_{A^c})}} M(G|_{A^c}; x+2)$$

But $b(G) = b(G|_A) + b(G|_{A^c})$ for all A, so this becomes

$$M(G; 2x+2) = \frac{x}{2} \sum_{A \in U(G)} M(G|_A; x+2)\, M(G|_{A^c}; x+2).$$

Substituting x for $2x+2$ gives the first form of $M(G; x)$ in the statement.

However, both G itself and the empty graph E are elements of $U(G)$, so the right hand side of

$$M(G; x) = \frac{x-2}{4} \sum_{A \in U(G)} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}; \frac{x+2}{2}\right)$$

can be written as

$$\frac{x-2}{4} M\left(G; \frac{x+2}{2}\right) M\left(E; \frac{x+2}{2}\right) + \frac{x-2}{4} M\left(E; \frac{x+2}{2}\right) M\left(G; \frac{x+2}{2}\right)$$
$$+ \frac{x-2}{4} \sum_{\substack{A \in U(G) \\ A \neq E,\, G}} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}: \frac{x+2}{2}\right),$$

which is

$$\frac{x-2}{4} 2M\left(G; \frac{x+2}{2}\right) \frac{1}{\frac{x+2}{2} - 2}$$
$$+ \frac{x-2}{4} \sum_{\substack{A \in U(G) \\ A \neq E,\, G}} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}; \frac{x+2}{2}\right),$$

or

$$M\left(G; \frac{x+2}{2}\right) + \frac{x-2}{4} \sum_{\substack{A \in U(G) \\ A \neq E,\, G}} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}; \frac{x+2}{2}\right),$$

which gives the second form in the statement, namely that

$$M(G; x) = M\left(G; \frac{x+2}{2}\right) + \frac{x-2}{4} \sum_{\substack{A \in U(G) \\ A \neq E,\, G}} M\left(G|_A; \frac{x+2}{2}\right) M\left(G|_{A^c}; \frac{x+2}{2}\right).$$

Note that if $G|_A$ does not equal E or G, then both $M(G|_A; (x+2)/2)$ and $M(G|_{A^c}; (x+2)/2)$ are polynomial. Thus, when $x=2$, Theorem 5.2 simply gives the tautology that $M(G; 2) = M(G; 2)$. ∎

All of the propositions below giving combinatorial interpretations for the Martin polynomial derive from this theorem which makes the induction and recursion possible.

The following is a list of some valuations for the Martin polynomial on unoriented graphs for which combinatorial interpretations are known. Its brevity, only three nontrivial results, and one of those on a very restricted class of graphs, reflects the difficulty, inherent in almost all graph polynomials, of extracting combinatorial information encoded in an algebraic object. All the graphs G below are Eulerian.

Result 1. If G is a 4-regular graph, with no loops, then

$$M(G; -2) = \frac{(-1)^{|v(G)|}}{2} \sum (-2)^{k-1} a_k(G),$$

where $a_k(G)$ is the number of Eulerian orientations of G with k anticircuits. [8, p. 408].

Result 2. If $\max \deg(G) \geqslant 4$ and G has no loops, then $M(G; 0) = 0$, where $\max \deg(G) = \max_{v \in v(G)}(\deg v)$. This follows from [8, Theorem 4.2, p. 405] which implies that if $\max \deg(G) \geqslant 4$, then $M(G; x)$ is a polynomial with non-negative coefficients and no constant term.

Result 3. If $\max \deg(G) \leqslant 2$, then $M(G; 0) = (-2)^{K(G)-1}$, where $K(G)$ is the number of components of G. This is because, if $\max \deg(G) \leqslant 2$, then G must be a disjoint union of $K(G)$ cycles, so $G = \prod_{i=1} c_i^{a_i}$ with $\sum_{i=1} a_i = K(G)$. Thus, by Theorem 5.1,

$$M(G; 0) = M\left(\prod_{i=1} c_i^{a_i}; 0\right) = (0-2)^{\sum_{i=1} a_i - 1} \prod_{i=1} M(c_i^{a_i}; 0).$$

However, since $M(c_i; 0) = 1$ for all i, this gives that $M(G; 0) = (-2)^{K(G)-1}$.

Result 4. $M(G; 2) = p(G)$ where $p(G)$ is the number of closed Eulerian paths of G. (A closed Eulerian path is a closed path that traverses each of the edges of G exactly once.) This result follows from observing that $M(G; 2) = f_{(1,0)}(G) + f_{(1,1)}(G)$. Thus if G is not connected, $M(G; 2) = 0$. If G is connected, and $G \neq c_1$, then $f_{(1,1)}(G) = 0$, and so $M(G; 2) = f_{(1,0)}(G) = p(G)$. And if $G = c_1$, then $M(c_1; 2) = 1 = p(c_1)$.

Result 5. If G has no loops, then $M(G; 3) = \sum_{v \in v(G)} g(v)$, where $g(v) = (\deg v - 1)(\deg v - 3) \cdots 3 \cdot 1$. This is [8, Prop. 4.3], where the notation $(\deg v - 1)!!$ is used for $g(v)$.

Result 6. $M(G; 4) = 2^{b(G)-1} f(G)\, F(G)$ where $f(G) = \prod_{v \in v(G)} (\deg v/2)!$, and $F(G)$ is the number of Eulerian orientations of G. This is [8, Theorem 5.2]. Since in this paper, the Martin polynomial is defined on E, the number $F(E)$ is defined to be one, which is consistent with this result.

Getting combinatorial interpretations for new evaluations of the Martin polynomial is now simply a matter of applying Theorem 5.2 to these results.

PROPOSITION 5.3. *If G has no loops, then*

$$M(G; -2) = \frac{-1}{4} \sum_{B \in D(G)} (-2)^{K(B) + K(B^c)},$$

where $K(B)$ is the number of components of $G|_B$, and

$$D(G) = \{B \subseteq e(G) \text{ such that } G|_B \text{ and } G|_{B^c} \text{ consist of disjoint unions of cycles}\}.$$

Proof. Theorem 5.2 with $x = -2$ gives that

$$M(G; -2) = \frac{-2-2}{4} \sum_{A \in U(G)} M(G|_A; 0)\, M(G|_{A^c}; 0),$$

but by results 2 and 3 above, $M(G|_A; 0)\, M(G|_{A^c}; 0) = 0$ unless both $G|_A$ and $G|_{A^c}$, consist of disjoint cycles. However, if $A = B$ for some $B \in D(G)$, then $M(G|_B; 0)\, M(G|_{B^c}; 0) = (-2)^{k(B)-1} (-2)^{k(B^c)-1}$, and hence

$$M(G; -2) = \frac{-1}{4} \sum_{B \in D(G)} (-2)^{K(B) + K(B^c)}$$

as claimed. ∎

Note that if $D(G) = \varnothing$ then for every $A \in U(G)$ at least one of $G|_A$ and $G|_{A^c}$, must have a vertex with degree greater than 4, and hence $M(G|_A; 0)$ $M(G|_{A^c}; 0) = 0$ for every A. Thus in this case,

$$M(G; -2) = \frac{-2-2}{4} \sum_{A \in U(G)} M(G|_A; 0)\, M(G|_{A^c}; 0) = 0,$$

which is consistent with a sum over an empty set. In particular, -2 is a root of $M(G, x)$ if $\max \deg(G) > 4$, since $D(G) = \varnothing$ in this case. See also Corollary 5.8.

PROPOSITION 5.4. *If G has no loops, then the following formula for the number of Eulerian orientations of G holds.*

$$F(G) = \frac{\sum_{A \in U(G)} \prod_{v \in v(G|_A)} g(v) \prod_{v \in v(G|_{A^c})} g(v)}{f(G)}.$$

Proof. This is immediate from Theorem 5.2 with $x = 4$, and results 5 and 6 above. ▮

COROLLARY 5.5. *In the special case where G is 4-regular, and supposing G has n vertices, then* $F(G) = (1/2^n) \sum_{A \in U(G)} 3^{N(A)}$, *where* $N(A) = n - |\{v \in v(G|_A)$ *such that* $\deg v = 2\}|$.

Proof. This follows from the formula in Proposition 5.4, together with the following observation. If v is a vertex of G, write v' for the corresponding vertex in $G|_A$, and v'' for the one in $G|_{A^c}$. Then if $\deg v' = 2$, it follows that $g(v') = g(v'') = 1$, but otherwise either $\deg v' = 0$ and $\deg v'' = 4$, or else $\deg v' = 4$ and $\deg v'' = 0$, and in both these latter cases $g(v')\, g(v'') = 3$. ▮

While it is true that this result can be achieved in this special case by combinatorial methods (see [2, following Cor. 4.11]), the algebraic techniques are less work. Furthermore, the general formula of Proposition 5.4, the result of a simple evaluation, would be quite intractable (although presumably not impossible) to derive using purely combinatorial methods.

DEFINITION 5.6. A *cyclic edge* 2^{k-1}-partition of G is a partition of $e(G)$ into sequences of 2^{k-1} subsets (possibly improper) of $e(G)$, written as $(y_1, ..., y_{2^{k-1}})$, so that for all i, $G|_{y_i}$ consists of a disjoint collection of cycles. Let $q(y) = \sum_{i=1}^{2^{k-1}} K(y_i)$, where as before $K(y_i)$ is the number of components of $G|_{y_i}$.

PROPOSITION 5.7. *If* $k \geqslant 2$, *then*

$$M(G; 2 - 2^k) = -2^{-k} \sum_{y \in Y_{2^{k-1}}(G)} (-2)^{q(y)},$$

where $y = (y_1, ..., y_{2^{k-1}})$ *is an element of* $Y_{2^{k-1}}(G)$, *which is the set of cyclic edge* 2^{k-1}*-partitions of G.*

This proposition (as well as Proposition 5.10 which follows) gives combinatorial interpretations for the Martin polynomial evaluated on an infinite class of integers.

Proof. First a word of clarification: If two or more of the subsets in $(y_1, ..., y_{2^{k-1}})$ are empty, permuting just these empty subsets does not constitute a new ordering, i. e. a separate element of $Y_{2^{k-1}}(G)$, since such elements would be indistinguishable from each other.

The proof is by induction on k. If $k=2$, then by Proposition 5.3,

$$M(G;2-4)=M(G;-2)=\frac{-1}{4}\sum_{B\in D(G)}(-2)^{K(B)+K(B^c)}.$$

However, B, B^c with $B\in D(G)$ corresponds to $(y_1, y_2)\in Y_2(G)$, so this becomes

$$-2^{-2}\sum_{y\in Y_{2^{2-1}}(G)}(-2)^{q(y)}=M(G;2-2^2),$$

and so the conclusion is satisfied when $k=2$.

Now suppose the conclusion holds for all integers greater than or equal to 2, and less than k. By Theorem 5.2,

$$M(G;2-2^k)=-2^{k-2}\sum_{A\in U(G)}M(G|_A;2-2^{k-1})\,M(G|_{A^c};2-2^{k-1}).$$

Since $k-1<k$, this becomes

$$-2^{k-2}\sum_{A\in U(G)}\left(-2^{-(k-1)}\sum_{u\in Y_{2^{k-2}}(G|_A)}(-2)^{q(u)}\right)\left(-2^{-(k-1)}\sum_{v\in Y_{2^{k-2}}(G|_{A^c})}(-2)^{q(v)}\right)$$

$$=-2^{k}\sum_{A\in U(G)}\left(\sum_{u\in Y_{2^{k-2}}(G|_A)}(-2)^{q(u)}\right)\left(\sum_{v\in Y_{2^{k-2}}(G|_{A^c})}(-2)^{q(v)}\right).$$

Note, however, that finding A and A^c in $U(G)$ and then choosing cyclic edge 2^{k-2}-partitions $u=(u_1, ..., u_{2^{k-2}})$ and $v=(v_1, ..., v_{2^{k-2}})$ for $G|_A$ and $G|_{A^c}$ is the same as taking a cyclic edge $2\cdot 2^{k-2}=2^{k-1}$-partition $y=(y_1, ..., y_{2^{k-1}})$ of G and splitting it into two sets by the correspondence $u_i\leftrightarrow x_i$ for $1\leqslant i\leqslant 2^{k-2}$ and $v_{i-2^{k-2}}\leftrightarrow y_i$ for $2^{k-2}+1\leqslant i\leqslant 2^{k-1}$. Note that under this correspondence $q(u)+q(v)=q(y)$. With this, the expression above becomes

$$-2^{-k}\sum_{y\in Y_{2^{k-2}}(G)}(-2)^{q(y)}$$

which, together with the induction, completes the proof. ∎

COROLLARY 5.8. *If* max deg$(G)>2^k$ *then* $2-2^k$ *is a root of* $M(G;x)$.

Proof. If $\max\deg(G) > 2^k = 2(2^{k-1})$ then $Y_{2^{k-1}}(G) = \varnothing$ since more than 2^{k-1} disjoint cycles are necessary to cover G. Thus in Propsition 5.7, the sum is over the empty set, and hence $M(G; 2-2^k) = 0$.

DEFINITION 5.9. An *Eulerian edge* 2^{k-1}*-partition* of G is a partition of $e(G)$ into (ordered) sequences of 2^{k-1} subsets (possibly improper) of $e(G)$, written as $(a_1, ...a_{2^{k-1}})$, so that for all i, $G|_{a_i}$ is Eulerian.

Let $W(a) = \prod_{i=1}^{2^{k-1}} f(G|_{a_i})\, F(G|_{a_i})$, where as before $f(G) = \prod_{v \in v(G)} ((\deg v)/2)!$, and $F(G)$ is the number of Eulerian orientations of G.

PROPOSITION 5.10. *If* $k \geqslant 1$, *then*

$$M(G; 2+2^k) = 2^{b(G)-k} \sum_{a \in E_{2^{k-2}}(G)} W(a),$$

where $a = (a_1, ...a_{2^{k-1}})$ *is an element of* $E_{2^{k-2}}(G)$, *which is the set of Eulerian edge* 2^{k-1}*-partitions of* G.

Proof. The proof is by induction on k. If $k=1$, then $M(G; 2+2^1) = M(G; 4) = 2^{b(G)-1} f(G)\, F(G)$ by result 6. Also

$$M(G; 2+2^1) = 2^{b(G)-1} \sum_{a \in E_{2^0}(G)} W(a) = 2^{b(G)-1} f(G)\, F(G),$$

since $E_1(G)$ consists just of $e(G)$ itself. Thus the conclusion is satisfied when $k=1$.

Now suppose the conclusion holds for all integers greater than or equal to 1, and less than k.

By theorem 5.2,

$$M(G; 2+2^k) = 2^{k-2} \sum_{A \in U(G)} M(G|_A; 2+2^{k-1})\, M(G|_{A^c}; 2+2^{k-1}).$$

Since $k-1 < k$, this becomes

$$\begin{aligned} &2^{k-2} \sum_{A \in U(G)} \left(2^{b(G|_A)-(k-1)} \sum_{u \in E_{2^{k-2}}(G|_A)} W(u_i) \right) \\ &\quad \times \left(2^{b(G|_{A^c})-(k-1)} \sum_{v \in E_{2^{k-2}}(G|_{A^c})} W(v_i) \right) \\ &= 2^{b(G)-k} \sum_{A \in U(G)} \left(\sum_{u \in E_{2^{k-2}}(G|_A)} W(u_i) \right) \left(\sum_{v \in E_{2^{k-2}}(G|_{A^c})} W(v_i) \right). \end{aligned}$$

Note, however, that finding A and A^c in $U(G)$ and then choosing Eulerian edge 2^{k-2}-partitions $u=(u_1, ..., u_{2^{k-2}})$ and $v=(v_1, ..., v_{2^{k-2}})$ for $G|_A$ and $G|_{A^c}$ is the same as taking an Eulerian edge $2 \cdot 2^{k-2}=2^{k-1}$-partition $x=(x_1, ..., x_{2^{k-1}})$ of G and splitting it into two sets by the correspondence $u_i \leftrightarrow x_i$ for $1 \leqslant i \leqslant 2^{k-2}$ and $v_{i-2^{k-2}} \leftrightarrow x_i$ for $2^{k-2}+1 \leqslant i \leqslant 2^{k-1}$. With this, the expression above becomes

$$2^{b(G)-k} \sum_{a \in E_2^{k-1}(G)} W(a),$$

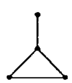
which, together with the induction, completes the proof. ▮

The author notes that the above propositions probably do not exhaust the possibilities for applying these techniques to the Martin polynomial. For example, it should be feasible to repeat the process given here, but on a Hopf algebra of directed graphs. Since there is a version of the Martin polynomial defined for directed graphs, new combinatorial interpretations similar to those given here should also be expected in the directed case. Furthermore, $J(G; x)$ (and hence the Martin polynomial) is just one valuation of the unique universal graph polynomial $P(G)$; results from other valuations could also be possible.

6. EXAMPLES AND COMPUTATIONS

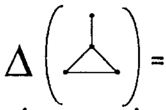
Computing $\Delta(G)$.

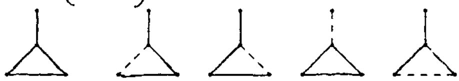
The following is an example of computing the comultiplication on a graph G, where G is the graph



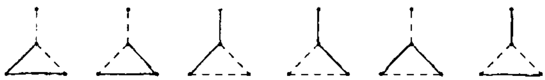
In the computation below, dotted lines are used to indicate the edges in F, a subset of the edge set of G, and solid lines are used to indicate the edges in the complement. For ease of notation, the terms in the result are written as t_i or c_i, where possible, and s_3 is used to represent the graph

With this notation,

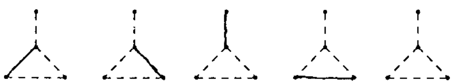




$E \otimes G + t_1 \otimes t_3 + t_1 \otimes t_3 + t_1 \otimes c_3 + t_1 \otimes s_3 +$



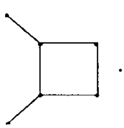
$t_2 \otimes t_2 + t_2 \otimes t_2 + t_2 \otimes t_2 + t_2 \otimes t_2 + t_1^2 \otimes t_2 + t_2 \otimes t_1^2 +$



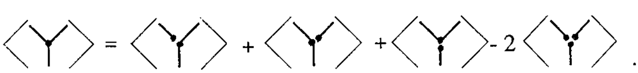
$t_3 \otimes t_1 + t_3 \otimes t_1 + c_3 \otimes t_1 + s_3 \otimes t_1 + G \otimes E$

Computing $P(G)$.

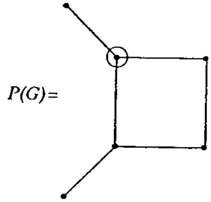
The following is an example of computing $P(G)$, where G is the graph

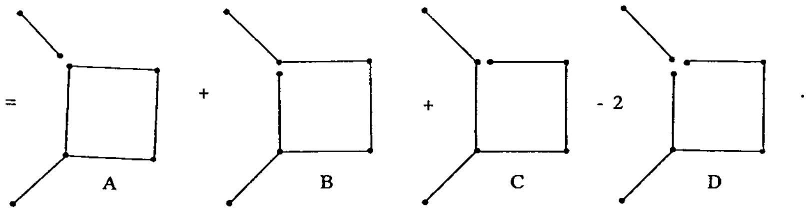


The vertex at which the vertex decomposition will be applied is circled. Also the notation $P(\ldots\ldots)$ is omitted—the graph itself is used to represent the polynomial associated with the graph, with the variables not being assigned until the end. The graphs which result from the first vertex decomposition are labelled A, B, C, D to identify them for the second set of vertex decompositions. Recall that since $h(3-2) = h(1) = 1$ and $h(3-0) = h(3) = -2$, the vertex decomposition at a vertex with degree 3 is given by

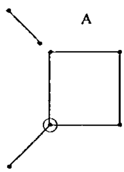


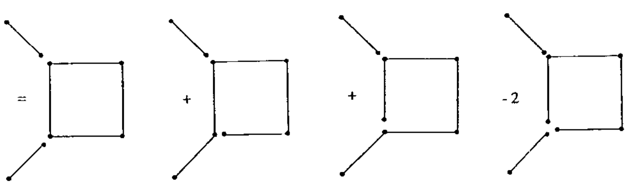
With these conventions,





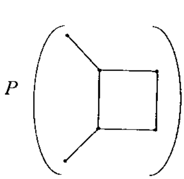
Now the vertex decomposition is applied to the remaining vertex of degree greater than two in each of A, B, C, and D:





$$= t_1^2 c_4 \quad + \quad t_1 t_5 \quad + \quad t_1 t_5 \quad -2 t_1^2 t_4.$$

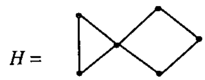
Similarly, B gives $t_1 t_5 + t_2 t_4 + t_1 t_5 - 2t_1^2 t_4$, C gives $t_1 t_5 + t_3^2 + t_2 t_4 - 2t_1 t_2 t_3$, and D, which is multiplied by -2, gives $-2t_1^2 t_4 - 2t_1 t_2 t_3 - 2t_1^2 t_4 + 4t_1^3 t_3$. Thus, collecting terms,



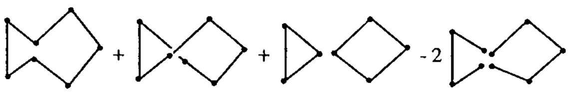
$$= t_1^2 c_4 + 5 t_1 t_5 + 2 t_2 t_4 + 4 t_1^3 t_3 + t_3^2 - 4 t_1 t_2 t_3 - 8 t_1^2 t_4.$$

Computing the Martin Polynomial and $J(G: x)$.

If H is the graph



then applying the vertex decomposition at the 4-valent vertex gives



$= 2x + x^2 + 0.$

Thus $J(H; x) = 2x + x^2$, and $M(H; x) = (1/(x-2))\, J(H; x-2) = x = 2(x-2)^0 + 1(x-2)$, reflecting that H has two Eulerian 1-partitions and one Eulerian 2-partitions.

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