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Identities for circuit partition polynomials, with applications to the Tutte polynomial

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Abstract

The Martin polynomials, introduced by Martin in his 1977 thesis, encode information about the families of circuits in Eulerian graphs and digraphs. The circuit partition polynomials, $J(G;x)$ and $j(\vec{G};x)$, are simple transformations of the Martin polynomials. We give new identities for these polynomials, analogous to Tutte's identity for the chromatic polynomial. Following a useful expansion of Bollobás [J. Combin. Theory Ser. B 85 (2002) 261–268], these formulas give combinatorial interpretations for all integer evaluations of the circuit partition and Martin polynomials. Selected evaluations of the formulas give combinatorial identities that expose the structure and relations of Eulerian graphs and digraphs. New identities and combinatorial interpretations for all integer values of the Tutte polynomial of a planar graph along the line $y = x$ also follow from these results.

Keywords: Martin polynomial; Circuit partition polynomial; Tutte polynomial; Graph invariants; Oriented graphs; Digraphs; Eulerian graphs; Eulerian orientations; Graph polynomials

1. Introduction

In his 1977 thesis [13], Martin recursively defined polynomials that encode information about the families of circuits in 4-regular Eulerian graphs and digraphs. Las Vergnas found a closed form for these polynomials and also extended their properties to general Eulerian graphs and digraphs and further developed their theory (see [10–12]). Bouchet took the natural step of examining the possible application of these polynomials to matroids, via isotropic systems (see [4,5]). Both Martin and Las Vergnas were able to establish

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combinatorial interpretations for some small integer evaluations of the polynomials. Combinatorial interpretations for some derivatives and for evaluations at powers of two were given in [6–8]. An expansion in [2] then gave formulas for interpretations of all integer evaluations of these polynomials. Transforms of the Martin polynomials, $J(G;x)$ and $j(\vec{G};x)$, given in [8], and then aptly named circuit partition polynomials in [1], facilitate these computations. For the oriented and unoriented (respectively) versions of the Martin polynomial, these transforms are:

$$(1)\qquad J(G;x)=xM(G;x+2)\quad\text{and}\quad j\big(\vec{G};x\big)=xm\big(\vec{G};x+1\big).$$

In the current paper, a simple combinatorial proof is given that

$$J(G;x+y)=\sum J(G|_A;x)J(G|_{A^c};y),$$

where the sum is over all subsets A of the edge set $E(G)$ such that the restrictions $G|_A$ and $G|_{A^c}$ of G to A and to A^c are both Eulerian. A similar proof holds for $j(\vec{G};x)$. Recall that Tutte's identity for the chromatic polynomial is $P(G;x+y)=\sum P(G|_A,x)P(G|_{A^c},y)$, where the sum is over all subsets A of the vertex set $V(G)$. The similarity between the formulas is not coincidental, and in fact the proof in the current paper is an adaptation of the proof for the chromatic identity. Like the chromatic polynomial, $J(G;x)$ is a one-variable specialization of a more general polynomial (see [8]).

Because combinatorial interpretations are known for various evaluations of $J(G;x)$ and $j(\vec{G};x)$, manipulations of the identities above reveal relationships in the structures of Eulerian graphs and digraphs. Furthermore, Martin showed that if G is a planar graph, and if a particular orientation is given to its medial graph, G_m, then the Martin polynomial of G_m is equal to the Tutte polynomial of G, with $y=x$ (see [10] for further generalizations as well). This led to surprising relations between valuations of the Tutte polynomial and anticircuits in medial graphs in [11,14]. Further interpretations for the Tutte polynomial of a planar graph, and some of its derivatives, along the line $y=x$ were found in [7]. The Tutte–Martin relation means that the identities for the circuit partition polynomials in the current paper immediately give new information about the Tutte polynomial of a planar graph.

The following conventions are used throughout this paper. Graphs may have loops and multiple edges. A graph is said to be Eulerian if all its vertices have even degrees, but connectedness is not required. An oriented graph, or *digraph*, has a direction assigned to each edge, and will be denoted by $\vec{G}$. An orientation of a graph is called Eulerian if the in-degree equals the out-degree at every vertex, and again connectedness is not required.

Following [3], a trail may revisit a vertex, but not retrace an edge. A circuit is a closed trail, and a cycle is a circuit that does not revisit any vertices. Trails, circuits and cycles in a digraph are similarly defined, except that the edges must be consistently oriented as the trail is traversed. An *anticircuit* in a digraph is a closed trail so that the directions of the edges alternate as the trail passes through any vertex of degree greater than 2. However, direction is not required to change when passing through a vertex of degree 2, since this in fact is not possible in a graph with an Eulerian orientation. Note that in a 4-regular Eulerian digraph, the set of anticircuits can be found by pairing the two incoming edges and the two outgoing edges at each vertex.

We will count the number of connected components in two different ways. When evaluating the circuit partition polynomials, we will usually use the number $k(G)$ of connected components of G which are not isolated vertices. Similarly, $k(\vec{G})$ does not count isolated vertices. However, when evaluating the Tutte polynomial, we will use the number $c(G)$ of all components, including isolated vertices.

When there is no danger of confusion, A will be written for $G|_A$, and in the oriented case $\vec{A}$ will be written for $\vec{G}|_{\vec{A}}$.

2. The circuit partition polynomials and principal results

Definition 2.1. An *Eulerian graph state* of a graph G is the result of replacing each $2n$-valent vertex v of G with n 2-valent vertices joining pairs of edges originally adjacent to v. Note that a Eulerian graph state is a disjoint union of cycles.

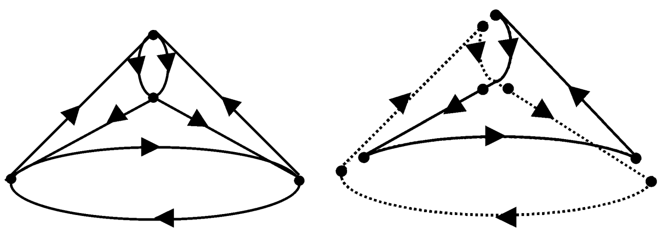
An Eulerian graph state of an Eulerian digraph $\vec{G}$ is defined similarly, except here each incoming edge must be paired with an outgoing edge.

For example, see Fig. 1.

Definition 2.2. The *circuit partition polynomial* $J(G;x)$ of an Eulerian graph G can be given by $J(G;x) = \sum_{k\geqslant 0} f_k(G)x^k$, where $f_k(G)$ is the number of Eulerian graph states of G with k components, defining $f_0(G)$ to be 1 if G has no edges, and 0 otherwise. The circuit partition polynomial is defined similarly for Eulerian digraphs as $j(\vec{G};x) = \sum_{k\geqslant 0} f_k(\vec{G})x^k$, where $f_k(\vec{G})$ is the number of Eulerian graph states of $\vec{G}$ with k components. (See [7] for other forms.)

For example, in Fig. 1, since $\vec{G}$ has 6 states with 1 component, 8 states with 2 components, and 2 states with 3 components, its polynomial is $j(\vec{G};x) = 6x + 8x^2 + 2x^3$. Both $J(G;x)$ and $j(\vec{G};x)$ map non-Eulerian (di)graphs to zero.

Theorem 2.3. *$J(G;x+y) = \sum J(A;x)J(A^c;y)$, where the sum is over all subsets $A \subseteq E(G)$ such that G restricted to both A and A^c is Eulerian. Also, $j(\vec{G};x+y) =$*



(a) (b)

Fig. 1. (a) An Eulerian digraph $\vec{G}$; (b) An Eulerian graph state of $\vec{G}$ with 2 components.

$\sum j(\vec{A}; x) j(\vec{A}^c; y)$*, where the sum is over all subsets* $\vec{A} \subseteq E(\vec{G})$ *such that* $\vec{G}$ *restricted to both* $\vec{A}$ *and* $\vec{A}^c$ *is an Eulerian digraph.*

At the expense of some algebraic machinery, this follows from the fact that $J(G; x)$ is a Hopf map from a Hopf algebra of graphs to the binomial bialgebra, and hence $J(G; 1 \otimes x + x \otimes 1) = \sum J(A; x) \otimes J(A^c; x)$—see [7], proof of Theorem 5.2. Substituting x for $1 \otimes x$ and y for $x \otimes 1$ gives the result. As an alternative, we provide a simple combinatorial proof below, which mimics the coloring proof for the chromatic polynomial. The proof uses the following somewhat contrived definition.

Definition 2.4. An *n-circuit coloring* of an Eulerian graph G is a one-covering of the edges of G by circuits with each circuit assigned one of the n colors (colors may be repeated, and not all colors must be used).

Thus for example, two different Eulerian circuits of a connected graph G, both colored green, would constitute different 1-circuit colorings of G, since although the color on all the edges is the same, the circuits are different.

Lemma 2.5. $J(G; n) =$ *the number of* n*-circuit colorings of* G.

Proof. Since $f_k(G)$ is the number of one-coverings of G by k circuits, and n^k is the number of ways to color the k circuits with n colors when repeats are allowed, it follows that $J(G; n) = \sum_{k \geqslant 0} f_k(G) n^k$ is the number of n-circuit colorings of G. □

Proof of Theorem 2.3. Given an $(m + n)$-circuit coloring of G, let A be the edges colored by the first m colors. Then an $(m + n)$-circuit coloring of G decomposes into an m-circuit coloring of A and an n-circuit coloring of A^c. Note that both A and A^c are necessarily Eulerian. Thus, for any two nonnegative numbers m and n, it follows that $J(G; m + n) = \sum J(A; m) J(A^c; n)$, where the sum is over all subsets $A \subseteq E(G)$ such that G restricted to both A and A^c is Eulerian. Since the expressions involve finite polynomials, this establishes the result for indeterminates x and y. The analogous result clearly holds for $j(\vec{G}; x)$ in the oriented case. □

Corollary 2.6. *Let n be any positive integer. Then* $J(G; w) = \sum \prod_{i=1}^{n} J(A_i; w_i)$*, where the sum is over all ordered partitions* $(A_1, \ldots, A_n)$ *of* $E(G)$ *so that G restricted to* A_i *is an Eulerian graph for all i, and where* $\sum_{i=1}^{n} w_i = w$.

Proof. This follows easily by induction on n from Theorem 2.3, which gives the case when $n = 2$. □

Again, the analogous result holds in the oriented case, namely that $j(\vec{G}; w) = \sum \prod_{i=1}^{n} j(\vec{A}_i; w_i)$, where the sum is over ordered partitions $(\vec{A}_1, \ldots, \vec{A}_n)$ of $E(\vec{G})$ so that $\vec{G}$ restricted to $\vec{A}_i$ is an Eulerian digraph for all i, and where $\sum_{i=1}^{n} w_i = w$.

A direct proof of the following especially useful special case can be found in [2].

Corollary 2.7. *Let n be any positive integer. Then $J(G;nx) = \sum \prod_{i=1}^{n} J(A_i;x)$, where the sum is over all ordered partitions $(A_1, \ldots, A_n)$ of $E(G)$ so that G restricted to A_i is an Eulerian graph for all i.*

Proof. This follows by letting $w_i = x$ for all i. □

Again, similarly, $j(\vec{G};nx) = \sum \prod_{i=1}^{n} j(\vec{A}_i;x)$ where the sum is over ordered partitions $(\vec{A}_1, \ldots, \vec{A}_n)$ of $E(\vec{G})$ so that $\vec{G}$ restricted to $\vec{A}_i$ is an Eulerian digraph for all i.

3. Applications to the circuit partition polynomials

Theorem 2.3 and its corollaries can now be used to give a variety of interpretations for integer values of the circuit partition polynomials. As noted in [2], this extends the results of [8] from powers of 2 to all integers.

First we give a few definitions to simplify later notation.

Let $A_n(G) = \{(A_1, \ldots, A_n)\}$, where $(A_1, \ldots, A_n)$ is an ordered partition of $E(G)$ into n subsets such that G restricted to A_i is Eulerian for all i. $\vec{A}_n(\vec{G})$ is similarly defined.

Let $D_n(G) = \{(D_1, \ldots, D_n)\}$, where $(D_1, \ldots, D_n)$ is an ordered partition of $E(G)$ into n subsets such that G restricted to D_i is 2-regular for all i. $\vec{D}_n(\vec{G})$ is similarly defined, with the additional restriction that the $\vec{D}_i$'s be consistently oriented.

Let $\mathrm{Eul}(G)$ denote the number of Eulerian orientations of G, i.e., the number of ways to orient the edges of G so that the result is an Eulerian digraph. Let $a_k(G)$ denote the number of Eulerian orientations of G with exactly k anticircuits. Following Bollobás [2], define the *anticircuit generating function* as $a(G;x) = \sum_k a_k(G)x^k$.

If G or $\vec{G}$ is edge-colored, a *monochromatic vertex* is a vertex with all its incident edges of the same color. When n is odd, let $n!! = n \cdot (n-2) \cdot \ldots \cdot 3 \cdot 1$.

Derivations of formulas (2) through (6) below, achieved by using Corollary 2.7 with known evaluations, can be found in [2]. They are listed here without proof for completeness and for later applications.

$$j(\vec{G};-n) = \sum_{\vec{D}_n(\vec{G})} (-1)^{g(\vec{D}_n(\vec{G}))}, \quad \text{where } g\big(\vec{D}_n(\vec{G})\big) = \sum_{i=1}^{n} k\big(\vec{D}_i\big). \tag{2}$$

$$J(G;-2n) = \sum_{D_n(G)} (-2)^{g(D_n(G))}, \quad \text{where } g\big(D_n(G)\big) = \sum_{i=1}^{n} k(D_i). \tag{3}$$

$$J(G;n) = \sum_{A_n(G)} \prod_{i=1}^{n} \prod_{v \in V(A_i)} \big(\deg_{A_i}(v) - 1\big)!!. \tag{4}$$

$$J(G;2n) = \sum_{A_n(G)} \prod_{i=1}^{n} \mathrm{Eul}(A_i) \prod_{v \in V(A_i)} \left(\frac{\deg_{A_i}(v)}{2}\right)!. \tag{5}$$

(6) If G is $4n$-regular, with m vertices, then

$$J(G; -4n) = (-1)^{mn} \sum_{A_n(G)} \prod_{i=1}^{n} a(A_i; -2).$$

Results (7) and (8) below also follow Corollary 2.7, and can be found in [7].

(7) $$j(\vec{G}; n) = \sum_{\vec{A}_n} \prod_{i=1}^{n} \prod_{v \in V(\vec{A}_i)} \left(\frac{\deg_{\vec{A}_i}(v)}{2} \right)!.$$

(8) A special case of (7). If $\max\deg(\vec{G}) = 4$, then $j(\vec{G}; n) = \sum 2^{m(c)}$, where the sum is over all edge colorings c of $\vec{G}$ with n colors so that each (possibly empty) set of monochromatic edges forms an Eulerian digraph, and $m(c)$ is the number of monochromatic vertices in the coloring c.

Mimicking Bollobás' proof in [2] of result (6) and using the evaluations (9) and (10) below (see [12] for (9) and (10) in terms of the original Martin polynomial) give result (11).

(9) $j(\vec{G}; r - d) = 0$ for $r = 1, \ldots, d$ if $\max\deg(\vec{G}) = 2d \geqslant 2$.

(10) If $\vec{G}$ has $\max\deg = 4$, then $j(\vec{G}; -2) = (-1)^n (-2)^{h(\vec{G})+C}$, where n is the number of vertices of degree 4 in $\vec{G}$, and $h(\vec{G})$ is the number of anticircuits in $\vec{G}$, and C is the number of components of $\vec{G}$ with $\max\deg = 2$. This minor generalization of the result in [13] is achieved by the slight broadening of the definition of an anticircuit combined with the multiplicative property of the circuit partition polynomial.

(11) If $\vec{G}$ is $4n$-regular, with m vertices, then

$$j(\vec{G}; -4n) = (-1)^{mn} \sum_{\vec{A}_n(\vec{G})} \prod_{i=1}^{n} (-2)^{h(\vec{A}_i)}.$$

Various combinations of these formulas reveal relations in the structures of Eulerian (di)graphs. For example, Lemma 2.5 combined with formula (4) means that the number of n-circuit colorings of a graph G is equal to $\sum_{A_n(G)} \prod_{i=1}^{n} \prod_{v \in V(A_i)} (\deg_{A_i}(v) - 1)!!$.

Formulas (4) and (5) combined together give that

$$\sum_{A_{2n}(G)} \prod_{i=1}^{2n} \prod_{v \in V(A_i)} \left(\deg_{A_i}(v) - 1\right)!! = \sum_{A_n(G)} \prod_{i=1}^{n} \mathrm{Eul}(A_i) \prod_{v \in V(A_i)} \left(\frac{\deg_{A_i}(v)}{2} \right)!,$$

etc. While these are the result of rather facile algebraic manipulations, they reveal relations within Eulerian graphs that may merit further study.

Theorem 2.3 can also be used to explore relationships in Eulerian graphs, as in (13) below. This requires a result that can be found in [12]:

(12) $$j(\vec{G};1)=\prod_v\left(\frac{\deg v}{2}\right)!.$$

Also note that by (9) above, if $\max\deg(\vec{G})=2d\geqslant 4$, then the alternating sum of the numbers of Eulerian k-partitions, $j(\vec{G};-1)=\sum_{k=0}f_k(\vec{G})(-1)^k$, is zero. If however $\max\deg(\vec{G})=2$, so that $\vec{G}$ is a disjoint union of cycles, then $j(\vec{G};-1)=(-1)^{k(\vec{G})}$.

From Theorem 2.3 it then follows that

$$0=j(\vec{G};0)=j(\vec{G};-1+1)=\sum j(\vec{A};-1)J(\vec{A}^c;1).$$

But, as noted above, $j(\vec{A};-1)=0$ unless $\max\deg(\vec{A})=2$, in which case $j(\vec{A};-1)=(-1)^{k(\vec{A})}$. This, combined with identity (12) gives (13):

(13) For any digraph $\vec{G}$, it follows that

$$\sum(-1)^{k(\vec{D})}\prod_{v\in V(\vec{D}^c)}\left(\frac{\deg_{\vec{D}^c}(v)}{2}\right)!=0,$$

where the sum is over all subsets $\vec{D}$ of $E(\vec{G})$ such that $\vec{G}$ restricted to $\vec{D}$ is a 2-regular Eulerian digraph.

These few examples illustrate, but certainly do not exhaust, the types of relations for Eulerian (di)graphs attainable by manipulating the preceding results.

4. Applications to the Tutte polynomial

The preceding formulas now have direct applications to the Tutte polynomial of a planar graph along the line $y=x$. In [13,14], Martin found a relationship between the Tutte polynomial of a connected planar graph and the Martin polynomial of its medial graph when given a specific orientation. This relation was further explored by Las Vergnas in [9–11]. Recall that the *medial graph* of a connected planar graph G is constructed by putting a vertex on each edge of G and drawing edges around the faces of G. The faces of this medial graph are colored black or white, depending on whether they contain or do not contain, respectively, a vertex of the original graph G. This face-two-colors the medial graph. The edges of the medial graph are then directed so that the black face is on the left. For example, see Fig. 2.

Hereafter, the phrase "directed medial graph" always refers to the medial graph with this specific orientation, which will be denoted by $\vec{G}_m$.

Let G be a connected planar graph, and let $\vec{G}_m$ be its directed medial graph. The relation between the Martin polynomial and Tutte polynomial is:

(14) $$m(\vec{G}_m;x)=t(G;x,x).$$

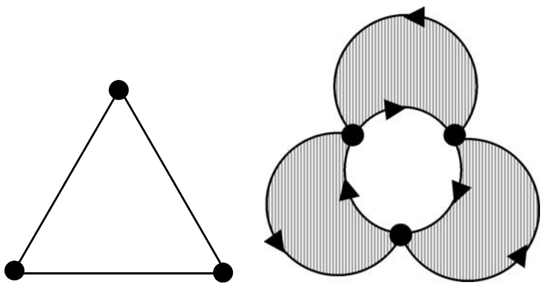


Fig. 2. (a) A planar graph G; (b) G_m with the vertex faces colored black, oriented so that black faces are to the left of each edge.

Combining this with the transformation identity (1) and the fact that the Tutte and circuit partition polynomials are multiplicative on disjoint unions of graphs gives the following relation, for G not necessarily connected:

(15) $j(\vec{G}_m; x) = x^{c(G)} t(G; x+1, x+1)$, where $c(G)$ is the number of components of G, counting isolated vertices. For this to be consistent, define $j(\vec{G}_m; x) = x$ if G is a graph consisting of a single vertex and no edges.

The evaluations of the circuit partition polynomials in Section 3 now give combinatorial interpretations for all integer values of the Tutte polynomial of a planar graph along the line $y = x$.

Identity 4.1. Let G be a planar graph with oriented medial graph $\vec{G}_m$. Then:

(16) $(-n)^{c(G)} t(G; 1-n, 1-n) = j(\vec{G}_m; -n) = \sum_{\vec{D}_n(\vec{G}_m)} (-1)^{g(\vec{D}_n(\vec{G}_m))}$, where $g(\vec{D}_n(\vec{G}_m)) = \sum_{i=1}^{n} k(\vec{D}_i)$, and

(17) $n^{c(G)} t(G; 1+n, 1+n) = j(\vec{G}_m; n) = \sum 2^{m(c)}$, where the sum is over all edge colorings c of $\vec{G}_m$ with n colors so that each (possibly empty) set of monochromatic edges forms an Eulerian digraph, and where $m(c)$ is the number of monochromatic vertices in the coloring c.

Proof. Identity (16) follows from relation (15) combined with (2) in Section 3. Since $\vec{G}_m$ is 4-regular, (17) follows from relation (15) combined with (7) in Section 3. □

Theorem 2.3 and Corollary 2.6 also give the following three identities for the Tutte polynomial of a planar graph.

Identity 4.2. Let G be a planar graph, and let $\vec{G}_m$ be its oriented medial graph. Then

$$(x+y)^{c(G)} t(G; x+y+1, x+y+1) = \sum j(\vec{A}; x) j(\vec{A}^c; y),$$

where the sum is over all subsets $\vec{A} \subseteq E(\vec{G}_m)$ such that $\vec{G}_m$ restricted to both $\vec{A}$ and $\vec{A}^c$ is an Eulerian digraph.

Proof. This follows directly from Theorem 2.3 and relation (15). □

Identity 4.3. Let G be a planar graph and let $\vec{G}_m$ be its oriented medial graph. Then

$$(x+y-1)^{c(G)} t(G; x+y, x+y) = \sum_{(\vec{D}, \vec{A}_1, \vec{A}_2)} (-1)^{k(\vec{D})} j(\vec{A}_1; x) j(\vec{A}_2; y),$$

where the sum is over all partitions $(\vec{D}, \vec{A}_1, \vec{A}_2)$ of $E(\vec{G}_m)$ such that $\vec{G}_m$ restricted to each of $\vec{D}$, $\vec{A}_1$, and $\vec{A}_2$ is an Eulerian digraph, and furthermore, $\vec{G}_m$ restricted to $\vec{D}$ is 2-regular.

Proof. Theorem 2.3 and relation (15) give that

$$(x+y-1)^{c(G)} t(G; x+y, x+y) = \sum_{(\vec{A}_1, \vec{A}_2, \vec{A}_3)} j(\vec{A}_1; x) j(\vec{A}_2; y) j(\vec{A}_3; -1).$$

However, by (9), $j(\vec{A}_3, -1) = 0$ unless $\vec{A}_3$ is 2-regular, and in this case $j(\vec{A}_3, -1) = (-1)^{k(\vec{A}_3)}$. □

Identity 4.4. Let G be a planar graph, and let $\vec{G}_m$ be its oriented medial graph. Then $(w)^{c(G)} t(G; w+1, w+1) = \sum \prod_{i=1}^{n} j(\vec{A}_i; w_i)$, where the sum is over ordered partitions $(\vec{A}_1, \ldots, \vec{A}_n)$ of $E(\vec{G}_m)$ so that $\vec{G}_m$ restricted to $\vec{A}_i$ is an Eulerian digraph for all i, and where $\sum_{i=1}^{n} w_i = w$.

Proof. This follows directly from Corollary 2.6 and relation (15). □

The identities and formulas of this paper might be generalized to matroids. The work of Bouchet in [4,5] lays a foundation for extending results in this direction. Furthermore, although the results for the Tutte polynomial given here apply to planar graphs, it may be possible to extend them to certain classes of nonplanar graphs. For example, in [9], Las Vergnas explores Tutte–Martin relations in graphs imbedded in surfaces. The evaluations given here were for integer values, and resulted from being able to use Theorem 2.3 to build upon a few known small integer evaluations of the circuit partition polynomials. Because they could similarly be expanded to a whole body of new results, any combinatorial interpretations for any noninteger values of the circuit partition polynomials would be most welcome.

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