Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Discrete Mathematics 281 (2004) 173–187

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Exploring the Tutte–Martin connection

Joanna A. Ellis-Monaghan*

Department of Mathematics, Saint Michael's College, 1 Winooski Park, Colchester, VT 05439, USA

Received 8 January 2003; received in revised form 27 October 2003; accepted 30 October 2003

Abstract

The Martin polynomial of an oriented Eulerian graph encodes information about families of cycles in the graph. This paper uses a transformation of the Martin polynomial that facilitates standard combinatorial manipulations. These manipulations result in several new identities for the Martin polynomial, including a differentiation formula. These identities are then applied to get new combinatorial interpretations for valuations of the Martin polynomial, revealing properties of oriented Eulerian graphs. Furthermore, Martin (Thesis, Grenoble, 1977; J. Combin. Theory, Ser. B 24 (1978) 318) and Las Vergnas (Graph Theory and Combinatorics, Research Notes in Mathematics, Vol. 34, Pitman, Boston, 1979; J. Combin. Theory, Ser. B 44 (1988) 367), discovered that if G_m, the medial graph of a connected planar graph G, is given an appropriate orientation, then $m(\vec{G}_m;x) = t(G;x,x)$, where $m(\vec{G};x)$ is the Martin polynomial of an oriented graph, and $t(G;x,y)$ is the Tutte, or dichromatic, polynomial. This relationship, combined with the new evaluations of the Martin polynomial, reveals some surprising properties of the Tutte polynomial of a planar graph along the diagonal $x=y$. For small values of x and y that correspond to points where interpretations of the Tutte polynomial are known, this leads to some interesting combinatorial identities, including a new interpretation for the beta invariant of a planar graph. Combinatorial interpretations for some values of the derivatives of $t(G;x,x)$ for a planar graph G are also found.

MSC: 05C38; 05C45

Keywords: Tutte polynomial; Martin polynomial; Graph polynomials; Graph invariants; Beta invariant; Planar graphs; Oriented graphs; Digraphs; Eulerian graphs; Skein decompositions

* Tel.:+1-802-654-2660; fax: +1-802-654-2610.
E-mail address: joellis@emba.uvm.edu (J.A. Ellis-Monaghan).

doi:10.1016/j.disc.2003.10.018

1. Introduction

In his 1977 thesis, Martin developed polynomials for investigating families of cycles in 4-regular Eulerian graphs. One polynomial, $M(G;x)$, encodes information about unoriented Eulerian graphs, and another, $m(\vec{G};x)$, encodes information about oriented Eulerian graphs. Martin [17,18] found that if the medial graph G_m of a connected planar graph G is given an appropriate orientation, then $m(\vec{G}_m;x) = t(G;x,x)$, where $t(G;x,y)$ is the Tutte, or dichromatic, polynomial.

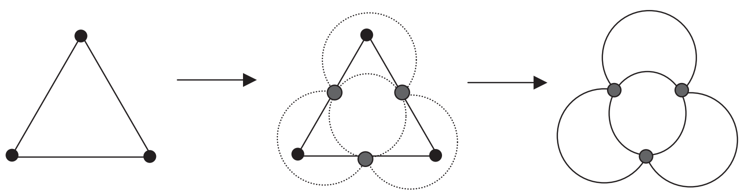
Considerable further work on the Martin polynomials was done by Las Vergnas [12,14], including finding closed formulas for Martin's recursive definitions, and generalizing them to Eulerian graphs with arbitrary (even) degrees. This led to several new insights into valuations of the Tutte polynomial (see [15,18]), including surprising relations between valuations of the Tutte polynomial at $(-1,-1)$ and (3, 3) and anticircuits in medial graphs.

In the current paper, by transforming the Martin polynomial, we derive several new identities and combinatorial interpretations for various evaluations of the shifted polynomial and its derivatives. These new results are combined with the identity of Martin and Las Vergnas to reveal properties of the Tutte polynomial of a planar graph along the diagonal $x = y$.

The following conventions are used throughout this paper. Graphs may have loops and multiple edges. A graph is called Eulerian if all the vertices have even degree, but connectedness is not required. An orientation of a graph is called Eulerian if the in-degree equals the out-degree at every vertex. An oriented graph will be called a digraph. A graph with no edges is denoted by $\mathscr{E}$, regardless of the number of vertices. The number of connected components of a graph is denoted $c(G)$ if isolated vertices are counted, and $k(G)$ if they are not.

Following Bollobás [2], a trail is a path that may revisit a vertex, but not retrace an edge. A circuit is a closed trail, and a cycle is a circuit that does not revisit any vertices. In the case of a digraph, the edges of a circuit must be consistently oriented. An anticircuit in a digraph is a closed trail so that the directions of the edges alternate as the trail passes through any vertex of degree greater than 2. However, direction is not required to change when passing through a vertex of degree 2, since this is in fact not possible in a graph with an Eulerian orientation. Note that in a 4-regular Eulerian digraph, the set of anticircuits can be found by pairing the two incoming edges and the two outgoing edges at each vertex.

Recall that the medial graph of a connected planar graph G is constructed by putting a vertex on each edge of G and drawing edges around the faces of G. For example



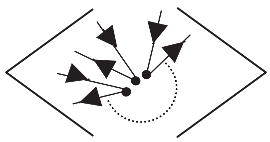
The graph G The medial graph of G

2. New identities for the Martin polynomial of an oriented Eulerian graph

A reduction identity for the Martin polynomial of unoriented Eulerian graphs was derived in [8] using Hopf-algebraic techniques and the fact that, if transformed appropriately, the Martin polynomial is essentially an evaluation of a unique universal graph polynomial. Once the appropriate modification had been motivated by the algebraic structure, results could then be generated by standard combinatorial techniques. See [9,10]. These techniques were also adapted for the oriented version in [1] and in [10], where the transformed Martin polynomial is called the circuit partition polynomial. Here, we exploit the reduction identity in the oriented case to find new evaluations and derivative formulas.

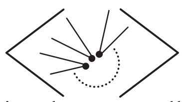
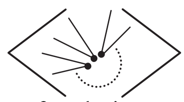
Detailed discussions of the following notions of vertex states and graph states (which were appropriated from knot theory) can be found in [8]. The definitions for Eulerian k-partitions generally follow those given in [18].

Definition 2.1. An *Eulerian vertex state* is a choice of reconfiguration at a vertex of an Eulerian graph G. The reconfiguration consists of replacing a $2n$-valent vertex v with n 2-valent vertices joining pairs of edges originally adjacent to v. In the case of an Eulerian digraph, each incoming edge must be paired with an outgoing edge. A choice of vertex state may be represented pictorially as



Here the angle brackets indicate a graph that is identical to the original graph $\vec{G}$ except in a small neighborhood of v. The small neighborhood of v is replaced by the depicted vertex state in the new graph.

Definition 2.2. An Eulerian graph state of an Eulerian digraph $\vec{G}$ is the result of choosing one vertex state at each vertex of $\vec{G}$. Note that a graph state is a disjoint union of cycles. Let S denote the set of Eulerian graph states of an Eulerian digraph $\vec{G}$. Note that S is not "up to isomorphism", so that each individual state is listed.

Definition 2.3. Let $D^v(\vec{G}) = \sum$ . Here,  is a choice of states at the vertex v. The sum is taken over all choices of Eulerian vertex states at v, so $D^v(\vec{G})$ is a formal sum of graphs, each identical to the original graph $\vec{G}$ except in a small neighborhood of v.

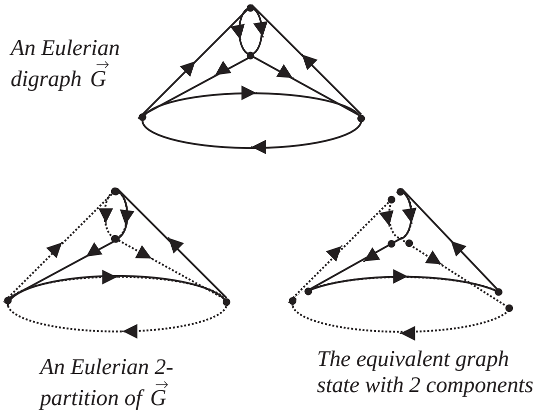


Fig. 1. (a) An Eulerian diagraph $\vec{G}$. (b) An Eulerian 2-partition of $\vec{G}$. (c) The equivalent graph state with 2 components.

Definition 2.4. An *Eulerian k-partition* of an Eulerian digraph $\vec{G}$ is a partition of the edge set of $\vec{G}$ into $k \geqslant 0$ consistently oriented closed trails.

Note that an Eulerian k-partition is equivalent to a graph state with k components (see [18]), as shown in Fig. 1.

Definition 2.5. (*Generating Function Form*). The polynomial $j(\vec{G};x)$ of an Eulerian digraph $\vec{G}$ is given by $j(\vec{G};x)=\sum_{k\geqslant 0} f_k(\vec{G})x^k$, where $f_k(\vec{G})$ is the number of graph states of $\vec{G}$ with k components (or Eulerian k-partitions). In [1], $j(\vec{G};x)$ is called the *circuit partition polynomial.*

For example, since the digraph $\vec{G}$ in Fig. 1 has 6 states with 1 component, 8 states with 2 components, and 2 states with 3 components, its polynomial is $j(\vec{G};x) = 6x + 8x^2 + 2x^3$.

Definition 2.6. (*State Model Form*). Since $f_k(\vec{G})$ is the number of graph states with k components, $j(\vec{G};x)$ can be defined equivalently as $j(\vec{G};x) = \sum_S x^{k(S)}$, where the sum is over all graph states S of $\vec{G}$, and $k(S)$ is the number of components in the state S.

Definition 2.7. (*Linear Recursion Form*). The polynomial $j(\vec{G};x)$ is given by the following recursion relations: If v is any vertex with $\deg(v) \geqslant 4$, then $j(\vec{G};x)=j(D^v(\vec{G});x)$, with $j(c_n;x) = x\ \forall n \geqslant 0$, where c_n is a cycle with n vertices. (Note that n may be 0. Such a cycle is also called a *free loop*.) Also, $j(\mathscr{E};x) = 1$.

Recall that the Martin polynomial of an Eulerian digraph has the closed form $m(\vec{G};x)=\sum_{k\geqslant 0} f_{k+1}(\vec{G})(x-1)^k$ given in [14], where $f_k(\vec{G})$ is the number of Eulerian k-partitions of $\vec{G}$. Thus, we have the following identity.

Identity 2.8. The polynomial j is a simple transform of the original Martin polynomial: $j(\vec{G};x) = xm(\vec{G};x+1)$ and $m(\vec{G};x) = (1/(x-1))j(\vec{G};x-1)$.

It is understood that any division in Identity 2.8 is performed before any evaluation. Thus, except for the Martin polynomial on an edgeless graph $\mathscr{E}$, which is now defined to be $1/(x-1)$, the expressions in Identity 2.8 are all polynomials.

Since the identities that follow have simpler forms when given in terms of $j(\vec{G};x)$, this polynomial will be used throughout, leaving to the reader the translations to the original Martin polynomial via Identity 2.8.

There is a subtle difference between the linear recursion processes used to compute $m(\vec{G};x)$ and $j(\vec{G};x)$. The Martin polynomial (as given in [18] and nicely illustrated in [11], for example) suppresses vertices of degree 2, while the polynomial j retains them. However, since both polynomials are invariant under the addition or suppression of vertices of degree 2, this does not alter any end results.

Note that the linear recursion form of Definition 2.7 generalizes one of Jaeger's transition polynomials, $Q(\vec{G},A,x)$, defined on 4-regular digraphs, to Eulerian digraphs with arbitrary (even) degrees. If $\vec{G}$ is 4-regular, then $j(\vec{G};x) = xQ(\vec{G},A,x)$ where A is a function that assigns an appropriate weight to each transition at the vertices. See [11], examples 2 and 3, for the details.

3. Known relationships and evaluations

Unless otherwise noted, the identities below can be found, in terms of the original Martin polynomial, in [14].

Identity 3.1. If $\overrightarrow{GH}$ is the disjoint union of $\vec{G}$ and $\vec{H}$, then $j(\overrightarrow{GH};x) = j(\vec{G};x)j(\vec{H};x)$.

Identity 3.2. Let $B(d,x) = \prod_{i=0}^{d-1}(x+i) = \sum_{k=1} s(d,k)x^k$, where $s(d,k)$ are the unsigned Stirling numbers of the first kind (cf. [14]). Let $\vec{G}_1,\ldots,\vec{G}_n$ be graphs with specified vertices $v_1,\ldots,v_n$, which have degrees $2d_1,\ldots,2d_n$, respectively. Let $\vec{G}$ be the graph that results from identifying the vertices $v_1,\ldots,v_n$ to a single cut-vertex v with degree $\sum_{i=1}^{n} 2d_i$. Then there is the following reduction for cut-vertices:

$$j(\vec{G};x) = \frac{B\left(\sum_{i=1}^{n} d_i, x\right)}{\prod_{i=1}^{n} B(d_i,x)} \prod_{i=1}^{n} j(\vec{G}_i;x).$$

The following is a list of some valuations for the Martin polynomial on digraphs for which combinatorial interpretations are known.

Evaluation 3.3. $j(\vec{G};1) = \prod_v ((\deg v)/2)!$.

Evaluation 3.4. (cf. Martin [17]). If $\vec{G}$ has $\max\deg = 4$, then $j(\vec{G};-2) = (-1)^n(-2)^{a(\vec{G})+C}$, where n is the number of vertices of $\vec{G}$ with degree 4, and $a(\vec{G})$ is the number of anticircuits in $\vec{G}$, and C is the number of 2-regular components of $\vec{G}$. This minor generalization of the result for the original Martin polynomial is achieved by the slight broadening of the definition of an anticircuit to accommodate vertices of degree 2, combined with multiplicative Identity 3.1.

Evaluation 3.5. If $\max\deg(\vec{G}) = 2d \geqslant 2$, then $j(\vec{G}; r-d) = 0$ for $r = 1, \ldots, d$.

Note that Evaluation 3.5 combined with Identity 3.1 means that if $\max\deg(\vec{G}) = 2d \geqslant 4$, then

$$j(\vec{G}; -1) = \begin{cases} 0 & \text{if } \max\deg(\vec{G}) > 2 \\ (-1)^{k(\vec{G})} & \text{else.} \end{cases}$$

This will be useful in later identities.

4. The splitting formula and some of its applications

The following splitting formula from [10] is the primary device used to generate the new identities for the oriented Martin polynomial that follow. Although the proof in [10] is purely combinatorial, the formula actually reflects the Hopf algebraic structure mentioned previously, hence the terminology "splitting".

Applications of the following results to planar graphs with maximum degree 4 are particularly useful for analyzing the Tutte polynomial at diagonal values in Section 6. Because of this and other graph theoretical questions involving 4-regular graphs (see, e.g. [4,11,15]), specializations to 4-regular graphs are provided, both in this section and in Section 5.

When there is no danger of confusion, $\vec{A}$ will be written for $\vec{G}|_A$, the restriction of $\vec{G}$ to the edges of the subset A of the edge set $E(\vec{G})$, and $\vec{A}^c$ will be written for $\vec{G}$ restricted to $E(\vec{G}) - A$.

Theorem 4.1. $j(\vec{G}; w) = \sum \prod_{i=1}^{n} j(\vec{A}_i; w_i)$, *where the sum is over ordered partitions* $(\vec{A}_1, \ldots, \vec{A}_n)$ *of* $E(\vec{G})$ *so that* $\vec{G}$ *restricted to* $\vec{A}$ *is an Eulerian digraph* (*possibly empty*) *for all* i, *and where* $w = \sum_{i=1}^{n} w_i$.

A proof of Theorem 4.1, similar to that of Tutte's identity for the Chromatic polynomial, can be found in [10]. A direct proof of the useful corollary that $j(\vec{G}; nx) = \sum \prod_{i=1}^{n} j(\vec{A}_i; x)$ can also be found in [1].

Evaluation 4.2. For n a positive integer, $j(\vec{G}; n) = \sum \prod_i g(\vec{A}_i)$, where the sum is over ordered partitions $(\vec{A}, \ldots, \vec{A}_n)$ of $E(\vec{G})$ such that $\vec{G}$ restricted to $\vec{A}_i$ is an Eulerian digraph for all i, and where $g(\vec{A}) = \prod_v ((\deg_{\vec{A}} v)/2)!$.

This follows from Theorem 4.1 and Evaluation 3.3, letting $w_i = 1$ for all i.

Definition 4.3. An *Eulerian n-coloring* of an Eulerian digraph $\vec{G}$ is an edge coloring of $\vec{G}$ with n colors so that each (possibly empty) set of monochromatic edges forms an Eulerian subdigraph.

In an edge coloring, a *monochromatic vertex* is one that has all of its adjacent edges the same color.

Evaluation 4.4. Let n be a positive integer, and suppose $\max\deg(\vec{G})=4$. Then $j(\vec{G};n)=\sum 2^{m(q)}$, where the sum is over all Eulerian n-colorings q, and $m(q)$ is the number of monochromatic vertices in the coloring.

This follows from Evaluation 4.2 together with the fact that if $\vec{G}$ has maximum degree 4, then so does $\vec{A}_i$ for all i. Thus $g(\vec{A}_i)=2^{n(\vec{A}_i)}$, where $n(\vec{A}_i)$ is the number of vertices of degree 4 in $\vec{A}_i$. Assigning each $\vec{A}_i$ a different color gives an Eulerian n-coloring q, and hence the result.

Definition 4.5. A *cyclic n-coloring* of an Eulerian digraph $\vec{G}$ is an edge coloring of $\vec{G}$ with n colors so that each (possibly empty) set of monochromatic edges forms a consistently oriented 2-regular graph, i.e., a disjoint union of consistently oriented cycles.

Evaluation 4.6. Let n be a positive integer. Then $j(\vec{G};-n)=\sum(-1)^{b(d)}$, where the sum is over all cyclic n-colorings d, and $b(d)$ is the total number of cycles in the coloring.

This follows from Theorem 4.1 and Evaluation 3.5, letting $w_i=-1$ for all i.

5. The derivative formula and some of its applications

Theorem 5.1 (The Derivative Formula).

$$j^{(n)}(\vec{G};x)=\sum_{A} n!\,f_n(\vec{A})\,j(\vec{A}^c;x),$$

where the sum is over subsets A of $E(\vec{G})$ such that $\vec{G}$ restricted to both A and A^c is an Eulerian digraph.

Proof. The proof is by induction on n. For $n=1$, the generating function form from Definition 2.5 gives that

$$j'(\vec{G};x)=\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0} f_k(\vec{G})x^k=\sum_{k=0} f_{k+1}(\vec{G})(k+1)x^k,$$

and the coefficient of x^k is $(k+1)f_{k+1}(\vec{G})$. Also,

$$\sum_{A} f_1(\vec{A})j(\vec{A}^c;x)=\sum_{A} f_1(\vec{A})\sum_{k=0} f_k(\vec{A}^c)x^k$$

$$=\sum_{k=0}\left(\sum_{A} f_1(\vec{A})f_k(\vec{A}^c)\right)x^k,$$

so the coefficient of x^k here is $\sum_A f_1(\vec{A})f_k(\vec{A}^c)$. However, given a state of $\vec{G}$ with $k+1$ components, there are $k+1$ choices of which component to let equal A. This component is counted once in $f_1(\vec{A})$, and the remaining k components are counted in $f_k(\vec{A}^c)$. Thus $(k+1)f_{k+1}(\vec{G}) = \sum_A f_1(\vec{A})f_k(\vec{A}^c)$ for all k, so $j'(\vec{G};x) = \sum_A f_1(\vec{A})j(\vec{A}^c;x)$, and the theorem holds when $n=1$.

Assume the derivative formula holds for $n-1$. Then,

$$
\begin{aligned}
j^{(n)}(\vec{G};x) &= \frac{\mathrm{d}}{\mathrm{d}x} j^{(n-1)}(\vec{G};x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_A (n-1)!\, f_{n-1}(\vec{A}) j(\vec{A}^c;x) \\
&= \sum_A (n-1)!\, f_{n-1}(\vec{A}) j'(\vec{A}^c;x) \\
&= \sum_A (n-1)!\, f_{n-1}(\vec{A}) \left(\sum_B f_1(\vec{B}) j(\vec{B}^c;x) \right),
\end{aligned}
$$

where the inner sum is over subsets B of A^c such that $\vec{G}$ restricted to both B and $A^c - B$ is an Eulerian digraph.

However, this can be written as $\sum_{(A_1,A_2,A_3)} (n-1)!\, f_{n-1}(\vec{A}_1) f(\vec{A}_2) j(\vec{A}_3;x)$, where the sum is over ordered partitions (A_1, A_2, A_3) of $E(\vec{G})$ so that $\vec{G}$ restricted to A_i is an Eulerian digraph for $i = 1,2,3$.

This in turn can be written as $\sum_A (n-1)!(\sum_{(A_1,A_2)} f_{n-1}(\vec{A}_1) f_1(\vec{A}_2)) j(\vec{A}^c;x)$, where the inner sum is over ordered partitions (A_1, A_2) of A so that $\vec{G}$ restricted to A_i is an Eulerian digraph for $i = 1,2$.

However, note that $\sum_{(A_1,A_2)} f_{n-1}(\vec{A}_1) f_1(\vec{A}_2) = n f_n(\vec{A})$ since there are n choices of components in one of the partitions counted in $f_n(\vec{A})$ to be counted in $f_1(\vec{A}_2)$, while the other $n-1$ are counted in $f_{n-1}(\vec{A}_1)$. Thus, the expression becomes $\sum_A n!\, f_n(\vec{A}) j(\vec{A}^c)$, which gives the result as claimed. □

Note that $j^{(n)}(\vec{G};x) = \sum (\prod_{i=1}^n f_1(\vec{A}_i)) j(\vec{A}_{n+1};x)$, where the sum is over partitions $(A_1, \ldots, A_{n+1})$ of the edge set $E(\vec{G})$ such that $\vec{G}$ restricted to A_i is an Eulerian digraph for all i, since $n!\, f_n(\vec{A}) = \prod_{i=1}^n f_1(\vec{A}_i)$, where $A = \bigcup_{i=1}^n A_i$ so that $A^c = A_{n+1}$.

Also, Theorem 5.1 and differentiating the state model form of Definition 2.6 give that $\sum_S k(S) x^{k(S)-1} = \sum_A f_1(\vec{A}) j(\vec{A}^c;x)$, and $\sum_S (k(S)!/(k(S)-n)!) x^{k(S)} = x^n \sum_A f_n(\vec{A})$ $j(\vec{A}^c;x)$, where the left-hand sums are over all graph states S of $\vec{G}$, and $k(S)$ is the number of components in the graph state S.

Translation to the original form of the Martin polynomial yields a recursive formula, $m^{(n)}(\vec{G};x) + (n/(x-1)) m^{(n-1)}(\vec{G};x) = \sum_A n!\, f_n(\vec{A}) m(\vec{A}^c;x)$, where the sum is over subsets A of $E(\vec{G})$ such that $\vec{G}$ restricted to both A and A^c is an Eulerian digraph.

The results below give a sampling of combinatorial identities that can be found by combining the new derivative formulas with the known results for $j(\vec{G};x)$ given in Section 3.

Definition 5.2. Let

$$P_n(\vec{G}) = \begin{array}{l}\{\text{ordered } n\text{-tuples } (p_1,\ldots,p_n), \text{ where the } p_i \text{ are} \\ \text{consistently oriented edge-disjoint closed trails in } \vec{G}\}.\end{array}$$

To simplify notation, write $\bar{p}$ for $(p_1,\ldots,p_n)$, and write P_n for $P_n(\vec{G})$ when $\vec{G}$ is clear from context. Also write $\bar{p}^c$ for those edges of $\vec{G}$ which are not in any trail of $\bar{p}$.

Note that each $\bar{p}\in P_n(\vec{G})$ is an ordering of the n components of an Eulerian n-partition of $\vec{G}|_A$, where A is all the edges that appear in any of the p_i's. Thus, $n!f_n(\vec{A})=\#\left\{(p_1,\ldots,p_n) \text{ such that } \bigcup_i E(p_i)=A\right\}$, and $\bar{p}^c=A^c$. With this observation, $\sum_A n!f_n(\vec{A})j(\vec{A}^c;x)$ can be written as $\sum_{\bar{p}\in P_n} j(\bar{p}^c;x)$.

Evaluation 5.3. For n a non-negative integer,

$$j^{(n)}(\vec{G};1)=\sum_{k=0}\frac{(k+n)!}{k!}f_{k+n}(\vec{G})=\sum_{\bar{p}\in P_n}\prod_{v\in\bar{p}^c}\left(\frac{\deg_{\bar{p}^c} v}{2}\right)!.$$

Letting $x=1$ in Theorem 5.1 gives that

$$\sum_{k=0}\frac{(k+n)!}{k!}f_{k+n}(\vec{G})\cdot 1^k=\sum_A n!f_n(\vec{A})j(\vec{A}^c;1)=\sum_{\bar{p}\in P_n} j(\bar{p}^c;1).$$

Combining this with Evaluation 3.3 gives the formula.

Evaluation 5.4. If $\vec{G}$ has maximum degree 4, then $j^{(n)}(\vec{G};1)=\sum_{\bar{p}\in P_n}2^{m(\bar{p})}$, where $m(\bar{p})$ is the number of vertices of $\vec{G}$ not belonging to any of the trails in $\bar{p}$.

This follows from Evaluation 5.3, since if $\vec{G}$ has maximum degree 4, then $\deg_{\bar{p}^c} v=2$ or 0 (and hence $((\deg_{\bar{p}^c} v)/2)!=1$) unless v is missed by all the trails in $\bar{p}$, in which case $((\deg_{\bar{p}^c} v)/2)!=2$.

Evaluation 5.5. $j'(\vec{G};-1)=-\sum_{k=1}kf_k(\vec{G})\cdot(-1)^k=\sum_B f_1(\vec{B}^c)(-1)^{k(\vec{B})}$, where the right-most sum is over all subsets $B\subseteq E(\vec{G})$ such that $\vec{B}$ is a disjoint union of consistently oriented cycles and $\vec{B}^c$ is connected.

From Theorem 5.1, $j'(\vec{G};-1)=-\sum_{k=1}kf_k(\vec{G})\cdot(-1)^k=\sum_A f_1(\vec{A})j(\vec{A}^c;-1)$. By Evaluation 3.5 however, $j(\vec{A}^c;-1)=0$ unless maximum degree $\vec{A}^c=2$, in which case $j(\vec{A}^c;-1)=k(\vec{A}^c)$. Also, $f_1(\vec{A})=0$ unless $\vec{A}$ is connected. Writing $\vec{B}$ for $\vec{A}^c$ when maximum degree $\vec{A}^c=2$ gives the form of the right-most sum.

Evaluation 5.6. If $\vec{G}$ has maximum degree 4, then $j'(\vec{G};-1)=\sum(-1)^{k(P^c)}$, where the sum is over all consistently oriented closed trails P that visit every vertex of $\vec{G}$ at least once.

If $\vec{G}$ has maximum degree 4, then in Evaluation 5.5, for $\vec{B}$ to have maximum degree 2, then $\vec{B}^c$ must contain every vertex of $\vec{G}$. Since $f_1(\vec{B}^c)$ then enumerates consistently oriented closed trails that visit every vertex of $\vec{G}$ at least once, the result follows.

Similar identities can be achieved by combining other known results with the differentiation formulas.

6. Applications to the Tutte polynomial of a planar graph

The preceding results for the Martin polynomial can now be used to garner new information about the Tutte polynomial of a planar graph. The Tutte, or dichromatic, polynomial has a very rich history and a wide range of applications in many areas of combinatorics. As a result, its applications and properties have been much studied and continue to generate a great deal of interest. However, only those definitions and properties actually needed for the current paper will be given here. The interested reader is directed, for example, to [5] or [6] for an in-depth treatment of the Tutte polynomial, including generalizations to matroids.

Definition 6.1. The rank generating function form of the Tutte polynomial of a graph G is given by $t(G;x,y)=\sum_{F\subseteq E}(x-1)^{r(E)-r(F)}(y-1)^{|F|-r(F)}$, where E and V are the sets of edges and vertices, respectively, of the graph G, and $r(F)=|V|-c(F)$, where $c(F)$ is the number of components of F in G, counting isolated vertices.

Along the diagonal line $y=x$, combinatorial interpretations for $t(G;x,x)$ are known for very few values of x. There are three evaluations whose interpretations follow trivially from Definition 6.1:

Evaluation 6.2. $t(G;0,0)=0$.

Evaluation 6.3. $t(G;1,1)=$ the number of spanning trees of G.

Evaluation 6.4. $t(G;2,2)=2^{|E(G)|}$.

Rosenstiehl and Read [20] found an interpretation for when $x=-1$ in terms of the binary bicycle space of a graphic matroid M:

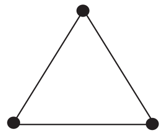
Evaluation 6.5. $t(M;-1,-1)=-1^{|E|}(-2)^{\dim(B)}$, where M is a graphic matroid, and B is the binary bicycle space of M.

Then there are two surprising results for connected planar graphs G from [15,18]:

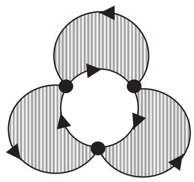
Evaluation 6.6. $t(G;-1,-1)=(-1)^{|E(G)|}(-2)^{a(\vec{G}_m)-1}$, where $a(\vec{G}_m)$ is the number of anticircuits in the directed medial graph.

Evaluation 6.7. $t(G;3,3)=K(2)^{a(\vec{G}_m)-1}$, where K is an odd integer.

In Evaluations 6.6 and 6.7, the medial graph is given a specific orientation. This is achieved by coloring the faces of the medial graph black or white, depending on whether they contain or do not contain, respectively, a vertex of the original graph G. This face-two-colors the medial graph. The edges of the medial graph are then directed so that the black face is on the left. For example,



A planar graph G



G_m with the vertex faces colored black, oriented so that black faces are to the left of each edge.

Hereafter, the phrase "directed medial graph" always refers to the medial graph with this specific orientation.

Let G be a connected planar graph, and let $\vec{G}_m$ be its directed medial graph. Martin [17,18] found the relationship $m(\vec{G}_m;x)=t(G;x,x)$, which was further explored by Las Vergnas in [12,15].

Combining this relationship with the transformation and multiplication identities (Identities 2.8 and 3.1, respectively), and the fact that the Tutte polynomial is also multiplicative on disjoint unions of graphs, gives the following, for G not necessarily connected:

Proposition 6.8. *Let V and E be the vertex and edge sets, respectively, of G. Then* $j(\vec{G}_m;x)=x^{c(G)}t(G;x+1,x+1)=x^{-|V|}\sum_{F\subseteq E}x^{|F|+2c(F)}$.

Proposition 6.8, together with the results of Sections 4 and 5, now gives new combinatorial interpretations for the Tutte polynomial of a planar graph for all integers (see also [10]).

Evaluation 6.9. If G is a planar graph with $c(G)$ components, and n is a positive integer, then $t(G;1+n,1+n)=(1/n^{c(G)})\sum 2^{m(q)}$, where the sum is over all Eulerian n-colorings q of the directed medial graph $\vec{G}_m$ and $m(q)$ is the number of monochromatic vertices in the coloring q.

Since the directed medial graph $\vec{G}_m$ is an Eulerian digraph with $\max\deg(\vec{G}_m)=4$, this evaluation follows immediately from Proposition 6.8 and Corollary 4.4.

Evaluation 6.10. If G is a connected planar graph, then there exists some odd integer K such that

$$2t(G;3,3) = \sum_{F \subseteq E(G)} 2^{|F|+2c(F)-|V(G)|} = \sum_{\substack{\text{Eulerian} \\ \text{2-colorings of } \vec{G}_m}} 2^{m(q)} = 2^{a(\vec{G}_m)}K.$$

This is simply Evaluation 6.7 combined with Proposition 6.8 and Evaluation 6.9 in the case that $n=1$.

As consideration of the simple example of letting G be the complete graph K_3 shows, these relations do not seem to be graph theoretically obvious. In this example, the directed medial graph has only one anticircuit. However, in the sum $\sum_{F \subseteq E(G)} 2^{|F|+2c(F)-|V(G)|}$, there are eight summands from the eight edge-induced subgraphs of G, yet there are 10 summands in $\sum_{\substack{\text{Eulerian} \\ \text{2-colorings of } \vec{G}_m}} 2^{m(q)}$, coming from the 10 Eulerian 2-colorings of $\vec{G}_m$.

Evaluation 6.11. If G is a planar graph with $c(G)$ components, and n is a positive integer, then, $t(G;1-n,1-n) = (1/(-n)^{c(G)}) \sum (-1)^{b(d)}$, where the sum is over all cyclic n-colorings d of the directed medial graph $\vec{G}_m$ and $b(d)$ is the total number of cycles in the coloring d.

Since the directed medial graph $\vec{G}_m$ is an Eulerian digraph, this follows immediately from Evaluation 4.6 and Proposition 6.8.

For example, if G is K_3, then there are only two cyclic 2-colorings of $\vec{G}_m$, each with two cycles, so $j(\vec{G}_m;-2) = (-1)^2 + (-1)^2 = 2 = -2t(K_3;-1,-1)$.

The case that $n=1$ in Evaluation 6.11 gives an additional interpretation for the Tutte polynomial at $(-1,-1)$: If G is a connected planar graph, then

$$-2t(G;-1,-1) = (-1)^{|E(G)|}(-2)^{a(\vec{G}_m)} = \sum_{F \subseteq E(G)} (-2)^{|F|+2c(F)-|V(G)|}$$
$$= \sum_{\substack{d \text{ a cyclic} \\ \text{2-coloring of } \vec{G}_m}} (-1)^{b(d)}.$$

Also, since

$$2^{a(\vec{G}_m)}K = \sum_{\substack{\text{Eulerian} \\ \text{2-coloring of } \vec{G}_m}} 2^{m(q)}, \quad \text{and}$$

$$(-1)^{|E(G)|}(-2)^{a(\vec{G}_m)} = \sum_{\substack{d \text{ a cyclic} \\ \text{2-coloring of } \vec{G}_m}} (-1)^{b(d)},$$

it follows that if $\vec{G}_m$ is the directed medial graph of some connected planar graph G, then

$$\left| \sum_{\substack{d \text{ a cyclic} \\ \text{2-coloring of } \vec{G}_m}} (-1)^{g(d)} \right| \text{ divides } \sum_{\substack{\text{Eulerian} \\ \text{2-colorings of } \vec{H}}} 2^{m(q)},$$

and the quotient is an odd integer.

There are some outstanding conjectures as well as new interest in the derivatives of the Tutte polynomial. For example, in [15], Las Vergnas conjectured that if M is a binary matroid, then $t^{(n)}(G;-1,-1)=K2^{d-n}$, where K is an integer, and $n\in\{0,\dots,d\}$. Recent work includes [16], which gives generating function formulas for the partial derivatives of the Tutte polynomial. The derivative results for the Martin polynomial above provide interpretations for some special cases of derivatives of the Tutte polynomial.

First, note that differentiating the forms given in Proposition 6.8 yields that $j^{(n)}(\vec{G}_m;x)=xt^{(n)}(G;x+1,x+1)+nt^{(n-1)}(G;x+1,x+1)$.

Evaluation 6.12. If G is a connected planar graph, then

$$t^{(n)}(G;2,2)=\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\sum_{\bar{p}\in P_k(\vec{G}_m)}2^{m(\bar{p})}$$

for all non-negative integers n.

This evaluation follows by induction on n. For $n=0$, by Evaluation 6.4, $t(G;2,2)=2^{|E(G)|}=(-1)^0\cdot 1\cdot 2^{|V(\vec{G}_m)|}$, so the hypothesis holds, since if $\bar{p}\in P_0$, then $m(\bar{p})=|V(\vec{G}_m)|$ because all the vertices are missed by the empty trail. Then assume that the hypothesis holds for $n-1$. Since the directed medial graph $\vec{G}_m$ is an Eulerian digraph with maximum degree 4, Evaluation 5.4 applies, so that $\sum_{\bar{p}\in P_n(\vec{G}_m)}2^{m(\bar{p})}=t^{(n)}(G;2,2)+nt^{(n-1)}(G;2,2)$. Induction then gives that

$$\sum_{\bar{p}\in P_n(\vec{G}_m)}2^{m(\bar{p})}-n\sum(-1)^{n-1-k}\frac{(n-1)!}{k!}\sum_{\bar{p}\in P_k(\vec{G}_m)}2^{m(\bar{p})}=t^{(n)}(G;2,2)$$

from which the result follows.

Recall that if G is a connected graph with at least two edges, then the coefficients of x and y in $t(G;x,y)$ are equal, and this value β is called the *beta*, or *chromatic*, invariant of the graph. Evaluation 6.13 provides another interpretation for this important invariant (see also [5,7,19]).

Evaluation 6.13. Let G be a connected planar graph, and let A and B be the coefficients of x and y, respectively, in $t(G;x,y)$. Then $A+B=\sum(-1)^{k(P^c)+1}$, where the sum is over all closed trails P in $\vec{G}_m$ which visit all its vertices at least once. Thus, when G has at least two edges, $\beta=\frac{1}{2}\sum(-1)^{k(P^c)+1}$.

Differentiating Proposition 6.8 and evaluating at $x=-1$ gives that $j'(\vec{G}_m;-1)=-(A+B)$ since $(\mathrm{d}/\mathrm{d}x)t(G;x,x)|_{x=0}=A+B$ and $t(G;0,0)=0$. Since the directed medial graph $\vec{G}_m$ is an Eulerian digraph with maximum degree 4, the result follows from Evaluation 5.6.

7. Conclusion

The identities given here result from algebraic manipulations. This leaves open questions about possible graph theoretical insight into them. Also, these results probably do not exhaust the potential of this approach. Other promising research directions include generalizing the results of this paper to matroids (see, e.g. [3,4]), and extending them to non-planar graphs (see [13] for Tutte–Martin relations analogous to Proposition 6.8 for the torus and projective plane, for example).

References

[1] B. Bollobás, Evaluations of the circuit partition polynomial, J. Combin. Theory, Ser. B 85 (2002) 261–268.
[2] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, Springer, New York, 1998.
[3] A. Bouchet, Isotropic systems, Eur. J. Combin. 8 (1987) 231–244.
[4] A. Bouchet, Tutte–Martin polynomials and orienting vectors of isotropic systems, Graphs Combin. 7 (1991) 235–252.
[5] T. Brylawski, The Tutte polynomial, Proceedings of the Third International Mathematical Summer Centre, 1980, pp. 125–275.
[6] T. Brylawski, J. Oxley. The Tutte polynomial and its applications, Matroid Applications, Encyclopedia of Mathematical Applications, Vol. 40, Cambridge University Press, Cambridge, 1992, pp. 123–225.
[7] H.H. Crapo, A higher invariant for matroids, J. Combin. Theory 2 (1967) 406–417.
[8] J.A. Ellis-Monaghan, New results for the Martin polynomial, J. Combin. Theory, Ser. B 74 (1998) 326–352.
[9] J.A. Ellis-Monaghan, Martin polynomial miscellanea, Congr. Numer. 137 (1999) 19–31.
[10] J.A. Ellis-Monaghan, Identities for the circuit partition polynomial, with applications to the diagonal Tutte polynomial, to appear.
[11] F. Jaeger, On transition polynomials of 4-regular graphs, Cycles and Rays (Montreal, PQ, 1987), NATO Advanced Science Institute, Series C: Mathematical Physical Sciences, Vol. 301, Kluwer Academic Publications, Dordrecht, 1990, pp. 123–150.
[12] M. Las Vergnas, On Eulerian partitions of graphs, Graph Theory and Combinatorics, Proceedings of Conference, Open University, Milton Keynes, 1978, Research Notes in Mathematics, Vol. 34, Pitman, Boston, MA, London, 1979, pp. 62–75.
[13] M. Las Vergnas, Eulerian circuits of 4-valent graphs embedded in surfaces, Algebraic Methods in Graph Theory, Vol. I, II (Szeged, 1978), Colloquia Mathematica Societatis Janos Bolyai, Vol. 25, North-Holland, Amsterdam, New York, 1981, pp. 451–477.
[14] M. Las Vergnas, Le polynôme de Martin d'un graphe eulérien, Combinatorial Mathematics (Marseille-Luming, 1981), North-Holland Mathematics Studies, Vol. 75, North-Holland, Amsterdam, 1983, pp. 397–411.
[15] M. Las Vergnas, On the evaluation at (3,3) of the Tutte polynomial of a graph, J. Combin. Theory, Ser. B 44 (1988) 367–372.
[16] M. Las Vergnas, The Tutte polynomial of a morphism of matroids, IV: derivatives as generating functions, to appear.

[17] P. Martin, Enumerations eulériennes dans le multigraphs et invariants de Tutte-Grothendieck, Thesis, Grenoble, 1977.
[18] P. Martin, Remarkable valuation of the dichromatic polynomial of planar multigraphs, J. Combin. Theory, Ser. B 24 (1978) 318–324.
[19] J.G. Oxley, On Crapo's beta invariant for matroids, Stud. Appl. Math. 66 (3) (1982) 267–277.
[20] P. Rosenstiehl, R.C. Read, On the principal edge tripartition of a graph, Advances in Graph Theory, Cambridge Combinatorial Conference, Trinity College, Cambridge, 1977; Ann. Discrete Math. 3 (1978) 195–226.