computing $\sum_B f_1\left(B^c\right)\left(-2\right)^{k(B)}$, there are only seven summands. They come from the seven Eulerian subgraphs of *G* with maxdeg 2, namely the empty graph, and the six different copies of a subgraph consisting of the two vertices and two of the parallel edges.

Similar identities can be achieved by combining other known results with the differentiation formulas. However, much work remains to be done in this area. For example, what combinatorial information can be extracted from the roots of the derivatives? Or even the roots of the polynomial itself other than those given by (*J, zeros*)?

5. BIBLIOGRAPHY

[B.I] A. BOUCHET, Isotropic Systems, *Europ. J. Combinatorics*, 8 (1987) 231-244.

[B.T] A. BOUCHET, Tutte-Martin Polynomials and Orienting Vectors of Isotropic Systems, *Graphs and Combinatorics*, 7 (1991) 235-252.

[E-M.M] J. A. ELLIS-MONAGHAN, Martin Polynomial Miscellanea, *Cong. Num.* 137 (1999) 19-31.

[E-M.N] J. A. ELLIS-MONAGHAN, New Results for the Martin Polynomial, *Journal of Combinatorial Theory,* Series B, **74** (1998) 326-352.

[E-M.U] J. A. ELLIS-MONAGHAN, *A unique, universal graph polynomial and its Hopf algebraic properties, with applications to the Martin polynomial,* Dissertation, University of North Carolina at Chapel Hill, 1995.

[J] F. JAEGER, On transition polynomials of 4-regular graphs, *Cycles and Rays (Montreal, PQ, 1987)* 123-150, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 301, *Kluwer Acad. Publ., Dordrecht,* 1990.

[LV] M. LAS VERGNAS, Le polynôme de Martin d'un graphe Eulerien, *Ann. Discrete Math.* **17** (1983) 397-411.

[M] P. MARTIN, *Enumerations Euleriennes dans le mulitgraphs et invariants de Tutte-Grothendieck,* Thesis, Grenoble, 1977.

Proof: By theorem 4.3, $J^{(n)}(G;1) = \sum_{\bar{p} \in P_n} \prod_{v \in \bar{p}^c} \left(\deg_{\bar{p}^c} v - 1\right)!!$. However, if G has maxdeg 4, then $\deg_{\bar{p}} v = 2$ or 0 (and hence $\left(\deg_{\bar{p}} v - 1\right)!! = 1$) unless v is missed by all the paths in $\bar{p}$, in which case $\left(\deg_{\bar{p}} v - 1\right)!! = 3$. ///

THEOREM 4.5.

$$J^{(n)}(G;2) = \sum_{k=0} \frac{(k+n)!}{n!} f_{k+n}(G) \cdot 2^k = \sum_A n!\, f_n(A) F\left(A^c\right) \prod_{v \in A^c} \left(\frac{\deg_{A^c} v}{2}\right)!$$

where F(G) is the number of Eulerian orientations of G.

Proof: This follows from theorem 3.1 and result (J,2), as in theorem 4.3. ///

THEOREM 4.6.

$$J'(G;-2) = \sum_{k=0} (k+1) f_{k+1}(G) \cdot (-2)^k = \sum_B f_1\left(B^c\right)(-2)^{k(B)},$$

where the right-most sum is over all subsets $B \subseteq E(G)$ *such that* $G|_B$ *has maxdeg* 2, *i.e. is a disjoint union of cycles. Here* $k(B)$ *is the number of non-empty cycles in* $G|_B$.

Proof: By theorem 3.1, $J'(G;-2) = \sum_{k=0} (k+1) f_{k+1}(G) \cdot (-2)^k$ $= \sum_A f_1(A) J\left(A^c;-2\right)$. But by result ($J$, -2), it follows that $J\left(A^c;-2\right) = 0$ unless maxdeg A^c = 2, in which case $J\left(A^c;-2\right) = k\left(A^c\right)$. Writing B for A^c when maxdeg A^c = 2 gives the form of the right-most sum as claimed. ///

Theorem 4.6 is surprising in that the two sums do not even sum over the same number of objects. For example, let G be the graph consisting of two vertices with four parallel edges between them. Then $J(G;x) = 6x + 3x^2$, since there are six Eulerian 1-partitions and three Eulerian 2-partitions of G. Thus there are nine summands in $\sum_{k=0} (k+1) f_{k+1}(G) \cdot (-2)^k$. However, when

PROPOSITION 4.1. $J^{(n)}(G;0) = n!\, f_n(G)$.

Proof: This trivial, but tidy, result follows immediately from theorem 3.1 and result (*J*, *zeros*), since $J\left(A^c;0\right)$ is nonzero (in fact equals 1) exactly when $A = G$, and $A^c = E$, the empty graph.

DEFINITION 4.2. Let

$$P_n(G) = \left\{(p_1,\ldots,p_n), \text{ where the } p_i \text{ are edge-disjoint closed paths in } G\right\}.$$

To simplify notation, write $\overline{p}$ for $(p_1,\ldots,p_n)$, and write P_n for $P_n(G)$ when *G* is clear from context. Also write $\overline{p}^c$ for those edges of *G* which are not in any path of $\overline{p}$.

Note that each $\overline{p} \in P_n(G)$ is an ordering of the *n* components of an Eulerian *n*-partition of $G\big|_A$, where *A* is all the edges that appear in any of the p_i's. Thus, $n!\, f_n(A) = \#\left\{(p_1,\ldots,p_n) \text{ such that } \bigcup_i E(p_i) = A\right\}$, and $\overline{p}^c = A^c$. With this observation, $\sum_A n!\, f_n(A) J\left(A^c;x\right)$ can be written as $\sum_{\overline{p}\in P_n} J\left(\overline{p}^c;x\right)$.

THEOREM 4.3.

$$J^{(n)}(G;1) = \sum_{k=0} \frac{(k+n)!}{n!} f_{k+n}(G) = \sum_{\overline{p}\in P_n} \prod_{v\in \overline{p}^c} \left(\deg_{\overline{p}^c} v - 1\right)!!\,.$$

Proof: Letting $x = 1$ in theorem 3.1 gives that $\sum_{k=0} \frac{(k+n)!}{n!} f_{k+n}(G)\cdot 1^k = \sum_A n!\, f_n(A) J\left(A^c;1\right)$. But as noted above, the RHS can be written as $\sum_{\overline{p}\in P_n} J\left(\overline{p}^c;1\right)$. Combining this with result (*J*,1) gives the formula. ///

COROLLARY 4.4. *If G has maxdeg* 4, *then* $J^{(n)}(G;1) = \sum_{\overline{p}\in P_n} 3^{m(\overline{p})}$, *where* $m(\overline{p})$ *is the number of vertices of G missed by the paths in* $\overline{p}$.

COROLLARY 3.3 (STATE MODEL FORM OF THE DERIVATIVE).

$$\sum_S k(S)x^{k(S)-1} = \sum_A f_1(A)J\left(A^c;x\right), \textit{ and}$$
$$\sum_S \frac{k(S)!}{\left(k(S)-n\right)!}x^{k(S)} = x^n \sum_A f_n(A)J\left(A^c;x\right),$$

where the left-hand sums are over all graph states S of G, and k(S) is the number of components in the graph state S.

Proof: This follows immediately from the derivative formula of theorem 3.1 and differentiating the state model definition 2.4. ///

COROLLARY 3.4. (TRANSLATION TO THE ORIGINAL MARTIN POLYNOMIAL).

$$M^{(n)}(G;x)+\frac{n}{x-2}M^{(n-1)}(G;x) = \sum_A n!\,f_n(A)M\left(A^c;x\right),$$

where the sum is over subsets A of E(G) such that such that G restricted to both A and A^c is Eulerian.

Proof: By theorem 3.1, $J^{(n)}(G;x) = \sum_A n!\,f_n(A)J\left(A^c;x\right)$. Using the transformation identity $J(G;x) = xM(G;x+2)$ from proposition 2.4, this becomes

$$\frac{d^n}{dx^n}xM(G;x+2) = \sum_A n!\,f_n(A)xM\left(A^c;x+2\right), \text{ or}$$

$$xM^{(n)}(G;x+2)+nM^{(n-1)}(G;x+2) = \sum_A n!\,f_n(A)xM\left(A^c;x+2\right).$$

Dividing by *x*, and then substituting $x-2$ for *x* gives the result. ///

4. NEW IDENTITIES FOR EULARIAN GRAPHS

The following is a sampling of combinatorial identities which can be found by combining the new derivative formulas with the known results for $J(G;x)$ given in section 2.

Now, $J^{(n)}(G;x) = \frac{d}{dx} J^{(n-1)}(G;x) = \frac{d}{dx} \sum_A (n-1)!\, f_{n-1}(A) J\left(A^c;x\right)$

$= \sum_A (n-1)!\, f_{n-1}(A) J'\left(A^c;x\right) = \sum_A (n-1)!\, f_{n-1}(A)\left(\sum_B f_1(B) J\left(B^c;x\right)\right)$, where the inner sum is over subsets B of A^c such that such that G restricted to both B and $A^c - B$ is Eulerian.

But this can be written as $\sum_{(A_1,A_2,A_3)} (n-1)!\, f_{n-1}(A_1) f(A_2) J(A_3;x)$, where the sum is over ordered partitions (A_1,A_2,A_3) of $E(G)$ so that G restricted to A_i is Eulerian for $i = 1,2,3$.

This in turn can be written as $\sum_A (n-1)!\left(\sum_{(A_1,A_2)} f_{n-1}(A_1) f_1(A_2)\right) J\left(A^c;x\right)$, where the inner sum is over ordered partitions (A_1,A_2) of A so that G restricted to A_i is Eulerian for $i = 1,2$.

However, note that $\sum_{(A_1,A_2)} f_{n-1}(A_1) f_1(A_2) = n f_n(A)$ since there are n choices of components in one of the partitions counted in $f_n(A)$ to be counted in $f_1(A_2)$ while the other n-1 are counted in $f_{n-1}(A_1)$. Thus the expression becomes $\sum_A n!\, f_n(A) J\left(A^c\right)$, which gives the result as claimed. ///

COROLLARY 3.2 (ALTERNATE FORM OF THE DERIVATIVE). $J^{(n)}(G;x) = \sum\left(\prod_{i=1}^{n} f_1(A_i)\right) J(A_{n+1};x)$, *where the sum is over partitions* $(A_1 \ldots A_{n+1})$ *of the edge set* $E(G)$ *such that* G *restricted to* A_i *is Eulerian for all* i.

Proof: Note that $n!\, f_n(A) = \prod_{i=1}^{n} f_1(A_i)$, where $A = \bigcup_{i=1}^{n} A_i$ so that $A^c = A_{n+1}$. ///

(*J*,2) $J(G;2) = F(G)\prod_{v}\left(\frac{\deg v}{2}\right)!$, where $F(G)$ is the number of Eulerian orientations of *G*.

3. DIFFERENTIATING THE MARTIN POLYNOMIAL

THEOREM 3.1 (THE DERIVATIVE FORMULA).

$$J^{(n)}(G;x) = \sum_{A} n!\, f_n(A) J\left(A^c;x\right),$$

where the sum is over subsets A of E(*G*) *such that such that G restricted to both A and* A^c *is Eulerian.*

Proof: The proof is by induction on *n*. First show that the statement is true when $n=1$, namely that $J'(G;x) = \sum_{A} f_1(A) J\left(A^c;x\right)$.

Now, $J'(G;x) = \frac{d}{dx}\sum_{k=0} f_k(G)x^k = \sum_{k=0} f_{k+1}(G)(k+1)x^k$, so on the LHS, the coefficient of x^k is $(k+1) f_{k+1}(G)$. On the RHS, $\sum_{A} f_1(A) J\left(A^c;x\right) = \sum_{A} f_1(A)\sum_{k=0} f_k\left(A^c\right)x^k = \sum_{k=0}\left(\sum_{A} f_1(A) f_k\left(A^c\right)\right)x^k$, so the coefficient of x^k is $\sum_{A} f_1(A) f_k\left(A^c\right)$. However, given a state of *G* with $k+1$ components, there are $k+1$ choices of which component to let equal *A*. This component is counted once in $f_1(A)$, and the remaining *k* components are counted in $f_k\left(A^c\right)$. Thus $(k+1) f_{k+1}(G) = \sum_{A} f_1(A) f_k\left(A^c\right)$ for all *k*, and so the theorem holds when $n=1$.

Now assume the derivative formula holds for $n-1$, and consider the n^{th} derivative.

(J, *) $J(GH;x) = J(G;x)J(H;x)$, where GH is the disjoint union of the graphs G and H.

(J, *split*) $J(G;2x) = \sum_A J(A;x)J(A^c;x)$, where the sum is over subsets A of the edge set $E(G)$ such that G restricted to both A and A^c is Eulerian.

(J, *zeros*) If $\max\deg(G) = 2n$, then $-2r$ is a root of $J(G;x)$ for all $r \in \{0,\ldots,n-1\}$.

(J, -2)

$$J(G;-2) = \begin{cases} 0 & \text{if } \max\deg G \geq 4 \\ (-2)^{K(G)} & \text{if } \max\deg G < 4, \\ & \text{where } K(G) \text{ is the number of} \\ & \text{components of } G \end{cases}$$

(J_1, -4) $J(G;-4) = \sum_B (-2)^{K(B)+K(B^c)}$, where the sum is over subsets B of $E(G)$ such that both $G|_B$ and $G|_{B^c}$ are unions of disjoint cycles.

(J_2, -4) If G has maxdeg 4, and is loopless, then $J(G;-4) = (-1)^n \sum_{k=1} (-2)^k a_k(G)$, where n is the number of vertices of G, and $a_k(G)$ is the number of Eulerian orientations of G with k anticircuits. An Eulerian orientation of an Eulerian graph is an assignment of a direction to each edge so that at each vertex the number of incoming edges is equal to the number of outgoing edges. An anticircuit of an oriented Eulerian graph G is a closed path where the direction of the edges changes at every vertex of degree 4.

(J, 1) $J(G;1) = \prod_v (\deg v - 1)!!$, where $n!! = (n-1)(n-3)\ldots 3\cdot 1$, for n even. By convention, $(\deg v - 1)!! = 1$ if $\deg v = 0$.

THE TRANSFORMED POLYNOMIAL:

The following transformation of the Martin polynomials was originally motivated by a Hopf algebraic structure. Although this structure is not used explicitly here, the combinatorial manipulations which generate the results of this paper are intimately related to the fact that the transformed polynomial below is a Hopf map, and in fact a valuation of a more general universal graph polynomial. The details of this relationship can be found in [E-M.N].

DEFINITION 2.3 (GENERATING FUNCTION FORM). The polynomial $J(G;x)$ of an Eulerian graph G is given by $J(G;x) = \sum_{k \geq 0} f_k(G)x^k$, where $f_k(G)$ is the number of Eulerian k-partitions of G. As in the previous example, since the graph G above has 44 1-partitions, 32 2-partitions, and 5 3-partitions, its polynomial is $J(G;x) = 44x + 32x^2 + 5x^3$.

DEFINITION 2.4 (STATE MODEL FORM). Since $f_k(G)$ is the number of graph states with k components, $J(G)$ can be defined equivalently as $J(G) = \sum_S x^{k(S)}$, where the sum is over all graph states S of G, and $k(S)$ is the number of components in the state S.

PROPOSITION 2.5 (THE TRANSFORMATION IDENTITY). $J(G;x) = xM(G;x+2)$, and $M(G;x) = \frac{1}{x-2} J(G;x-2)$.

It is understood that any division above is performed before any evaluation. Thus, except for the Martin polynomial on the empty graph E, which is now defined to be $\frac{1}{x-2}$, the expressions above are all polynomial.

Some of the results that follow were derived using the original form of the Martin polynomial, and some were derived using the transformed polynomial. Since one form or the other may be easier to use in any given application, this transformation identity will be used without comment hereafter to give results in terms of either form of the polynomial.

KNOWN RELATIONSHIPS AND EVALUATIONS:

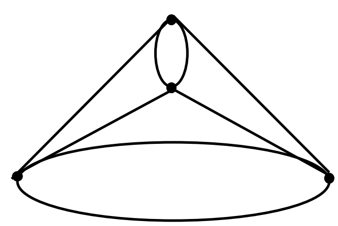
The labeling of results below is inspired by the very useful mnemonic system in [B.T]. Results (J_2,-4) through (J,2), in terms of the original Martinpolynomial, can be found in [LV.P]. The remainder of the results appear in [E-M.N] or [E-M.M]. When there is no danger of confusion A will be written for $G|_A$, the restriction of G to the edges of the subset A of the edge set $E(G)$.

2. THE MARTIN POLYNOMIAL, WITH EVALUATIONS AND IDENTITIES

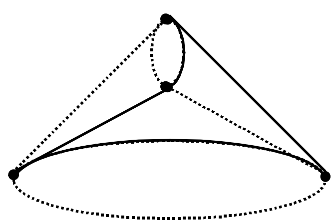
THE MARTIN POLYNOMIAL:

The definitions below generally follow the notation given in [LV.P].

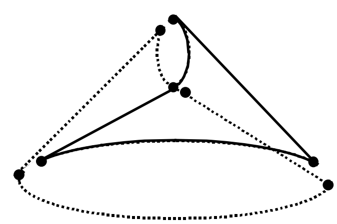
DEFINITION 2.1. An Eulerian k-partition of a graph G is a partition of the edge set of G into $k \geq 0$ closed paths. Some care must be taken in counting loops with this definition. Eulerian k-partitions can be defined equivalently as *graph states* with k components, where a graph state is the result of choosing pairings of all of the edges at each vertex. This alternative definition clarifies the loop counting and accounts for some slight simplification in one or two formulas below. Details of this can be found in [E-M.M]. Let $f_0(G) = 1$ if $G = E$, and 0 otherwise.



An Eulerian graph G



An Eulerian 2-partition of G



The equivalent graph state with 2 components

DEFINITION 2.2. The Martin polynomial of an Eulerian graph G is given by $M(G;x) = \sum_{k \geq 0} f_{k+1}(G)(x-2)^k$, where $f_k(G)$ is the number of Eulerian k-partitions of G (or equivalently, the number of graph states with k components). For example, the graph G above has 44 1-partitions, 32 2-partitions, and 5 3-partitions, so its Martin polynomial is $M(G;x) = 5x^2 + 12x$.

Abstract

The Martin polynomial of a graph, introduced by Martin in his 1977 thesis [M], encodes information about the families of closed paths in Eulerian graphs. A translation of the Martin polynomial, $J(G;x)$, is used in this paper to obtain differentiation formulas for both the original form of the Martin polynomial and for $J(G;x)$. These formulas are then used to establish various combinatorial identities for Eulerian graphs.

Key words and phrases: Martin polynomial, graph invariants, Eulerian graphs, Eulerian orientations, graph polynomials.

Mathematics subject classification: primary-- 05C38, secondary-- 16W30.

1. INTRODUCTION

The Martin polynomial of a graph, introduced by Martin in his 1977 thesis [M], encodes information about the families of closed paths in an Eulerian graph. This polynomial and its properties were further refined and developed by Las Vergnas in *Le Polynome de Martin d'un Graphe Eulerien* in 1983 [LV], and then generalized to isotropic systems by Bouchet (see [B.I], [B.T]). A reduction identity for the Martin polynomial was derived using Hopf-algebraic techniques in [E-M.U] and combinatorially in [E-M.M]. The computations in these papers involved translating the Martin polynomial to a more easily manipulated form, $J(G;x)$. This translation is used in the current paper to obtain differentiation formulas for the Martin polynomial. Since the Martin polynomial encodes information about Eulerian graphs, these differentiation formulas can then be used to establish combinatorial interpretations for various evaluations of the derivatives and to construct combinatorial relationships for the properties of Eulerian graphs.

The following conventions are used through this paper. Unless otherwise indicated, graphs may have loops and multiple edges. A graph is called Eulerian if all its vertices have even degrees, but connectedness is not required. Unless otherwise indicated, all graphs in this paper are assumed to be Eulerian. A cycle is a graph isomorphic to a polygon, and is denoted c_n when it has n vertices.

DIFFERENTIATING THE MARTIN POLYNOMIAL

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