Some Remarks on Domination

by

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We give two results on domination in graphs, including a proof of a conjecture of Favaron, Henning, Mynhart and Puech [2]. Corollary 2 was found by four separate subsets of the authors. We decided to give this joint presentation of our results. We first offer a result about bipartite graphs.

Lemma 1 *Let G be a bipartite graph with partite sets (X, Y) whose vertices in Y are of minimum degree at least 3. Then there exists a set $A \subset X$ of size at most $|X \cup Y|/4$ such that every vertex in Y is adjacent to a vertex in A.*

Proof: The proof is by induction on $|V(G)| + |E(G)|$. The smallest graph as described in the lemma is $K_{1,3}$, for which the statement holds. This gives the start of our induction. Let $x = |X|$ and $y = |Y|$. If there exists a vertex v in Y of degree at least 4, then delete any edge e incident to v. The subset A of $G - e$ guaranteed by the inductive hypothesis is adjacent in G to every vertex in Y as desired. So we may assume that the vertices in Y are all of

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degree exactly 3. If there exists a vertex v in X of degree at least 3, then delete that vertex and all of its neighbors. Adding v into the subset A from this smaller graph yields our desired subset for G. So we may assume that every vertex in X is of degree at most 2. If there exists an isolated vertex $v \in X$, then again the set A in $G - v$ is adjacent to every vertex in Y. So we may assume that every vertex in X is of degree at least 1.

We now know enough about G to prove the existence of the desired subset A directly. Let X_i denote the vertices in X of degree i, $i = 1, 2$, and let $x_i = |X_i|$. Under the conclusions of the previous paragraph, $x = x_1 + x_2$, and since all vertices in Y are of degree three, $3y = x_1 + 2x_2$. The desired result is a set $A \subset X$ of cardinality at most $(x + y)/4 = x_1/3 + 5x_2/12$.

Form the graph G' from G where $V(G') = X_2$, and $uv \in E(G')$ if and only if u, v are adjacent with a common vertex in G. Since the maximum degree in G of a vertex $x \in X$ is 2, G' is of maximum degree 4. Hence, by Brooks' Theorem, G' has an independent set of size at least $x_2/4$ (the possible exceptional case $G' = K_5$ cannot arise by this construction). The corresponding vertices in G have disjoint neighborhoods, hence they are adjacent to at least $x_2/2$ different vertices in Y. The set A uses these vertices, and for each remaining vertex in Y an adjacent vertex in X. Now $|A| \leq x_2/4 + y - 2x_2/4 = x_1/3 + 5x_2/12$ as desired. ■

A *total dominating set* in a graph H is a subset A of vertices such that every vertex in H is adjacent to a vertex in A.

Corollary 2 *Every graph H of order n and of minimum degree at least 3 has a total dominating set of size at most $n/2$.*

Proof: Construct a bipartite graph G from H as follows. Each vertex v_i in H gives two vertices x_i, y_i in G. Each edge $v_i v_j$ in H gives two edges $x_i y_j$ and $x_j y_i$ in G. The bipartition is $(X, Y) = (\{x_i\}, \{y_i\})$. A total dominating set A in H corresponds to an $A \subset X$ in G adjacent to every vertex in Y. The result now follows by the previous lemma. ■

This corollary settles a conjecture of Favoron et al. [2], which also contains some history of the problem. As shown in [2] the statement in the corollary is tight. For every n divisible by four there are 3-regular graphs of order n having no dominating set of size less than $n/2$. In fact, using a somewhat more complicated argument, it is possible to show [6] that any extremal graph must be 3-regular and n must be divisible by 4. Reference [4] extends

Corollary 2 to the set of graphs of minimum degree at least 2 where no degree-2 vertex is adjacent to two other degree-2 vertices. Reference [3] shows that every connected graph on n vertices and $e \geq 2$ edges and maximum degree at most 3 is totally dominated by a set of $n - e/3$ vertices, from which Collorary 2 follows. They also examine when the bound is tight.

The result can also be phrased in terms of transversals of rank 3 hypergaphs. In this context Lemma 1 is related to work by Chvátal and McDiarmid [1]. This relation and Corollary 2 was noted by Thomasse and Yeo [5].

The advantage to our approach is studying total domination through the corresponding bipartite graph. This allows more subtle inductive steps. These techniques may have other applications to total domination.

References

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