EMBEDDING GRAPHS

Let's start with the plane:
Let $G$ be a graph with $v$ vertices, $e$ edges, and $f$ faces.
Assume $G$ is simple and connected.
Suppose $G$ is embedded in the plane.

Euler's Theorem: $v - e + f = 2$. 
Regular graphs:
A graph is regular of degree $r$ if every vertex has degree $r$.

Platonic Solids are graphs that are regular of degree $r$ and for which every face is bordered by a cycle of length $s$. 
Example:

- **Tetrahedron**
  - Vertices: 4
  - Edges: 6
  - Faces: 4
  - Vertices: 3
  - Edges: 3
  - Faces: 3

Example:

- **Hexahedron**
  - Vertices: 8
  - Edges: 12
  - Faces: 6
  - Vertices: 3
  - Edges: 4
  - Faces: 3

(tetrahedron)

(hexahedron)
To find all such graphs, we have five variables:
\( (r, s, v, e, f) \)
Satisfying these three equations:

1. \( v - e + f = 2 \)
2. \( \sum \deg v = 2e \) so \( vr = 2e \)
3. \( sf = 2e \)
Write \( v, e \), and \( f \) in terms of \( r \) and \( s \):

\[
\begin{align*}
v &= \frac{2e}{r} \\
f &= \frac{2e}{s} \quad \text{such that} \quad \frac{2e}{r} - e + \frac{2e}{s} &= 2 \\
e \left( \frac{2}{r} - 1 + \frac{2}{s} \right) &= 2 \\
e &= \frac{2}{\frac{2}{r} - 1 + \frac{2}{s}} \\
e &= \frac{2rs}{2s + 2r - rs}
\end{align*}
\]
Use $c$ to write $v$ and $f$:

\[ v = \frac{2c}{f} = \frac{2(2s)}{f(2r+2s-\sqrt{s})} \]

So:

\[ v = \frac{4s}{2r+2s-\sqrt{s}} \]

Similarly:

\[ f = \frac{2c}{s} = \frac{2(2s)}{8(2r+2s-\sqrt{s})} \]

\[ f = \frac{4s}{2r+2s-\sqrt{s}} \]
Denominator is
\[ 2(5\sigma^r) - rs \]

in order for \( v, e \) and \( t \) to make sense,
\[ 2(5\sigma^r) - rs \geq 0 \]

restricts to small numbers.
Consider $r \geq 3$ and $s \geq 3$.

\[
\begin{align*}
\text{if } r &= 3 \quad \frac{4}{\sqrt{2(3+s)-3s}} \\
\text{if } s &= 2 \quad \text{too small to make a solid} \\
\text{if } s &= 3 \quad v = 4 \\
\text{if } s &= 4 \quad v = 8 \\
\text{if } s &= 5 \quad v = 20 \\
\text{if } s &= 6 \quad \text{divided by 0} \\
\text{if } s &> 6 \quad v < 0
\end{align*}
\]
So: we're in the case where
\( r = 3, \ s \in \{3, 4, 5\} \).
What are \( v, e, f \)?
(What are the solids we generate?)

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- Tetrahedron
- Cube
- Dodecahedron
cloakedhedron:
What if $r = 4$?

Then \[ V = \frac{4s}{2(4+s)-4s} = \frac{4s}{8-2s} \]

So \[ V = \frac{2s}{4-s} \]

So $s < 4$ and $4-s | 2s$.
(Consider only $s \geq 3$.)

$r = 4$  \[ v = 6 \quad e = 12 \quad f = 8 \quad \text{Octahedron} \]
$s = 3$
Octahedron:

(Vocab note: When \( s = 3 \), the graph is a triangulation.)
Another case:

\[ r = 5 \quad \frac{4s}{10 + 3s - 5s} = \frac{4s}{10 - 3s} \]

\[ 10 - 3s > 0 \quad \text{and} \quad 10 - 3s \mid 4s \quad \text{and} \quad s \geq 3 \]

so \( s = 3 \).

\[ r = 5 \quad v = 12 \quad e = 30 \quad f = 20 \]

icosahedron
icosahedron:
If \( r = 6 \)?

\[
V = \frac{4s}{2(6+5) - 6s} = \frac{4s}{12 - 4s} = \frac{s}{3 - s}
\]

Crisis: \( s > 3 \) and \( s < 3 \).

We can show that these equations are not satisfied if \( s > 5 \).
Duality of Platonic Solids:

- Tetrahedron is self-dual

- Octahedron and cube are dual
- Dodecahedron and icosahedron are dual
NOTE: If you allow the denominator of \( v/p, f \) to be 0, you get infinite tessellations.

- Self-dual
- Dual
These calculations are based on Euler's formula: \( V - E + F = 2 \), which only holds in the plane (also the sphere).

Let's consider:
1) the torus
2) the projective plane.
ORIENTABLE SURFACES:

The donut is a surface called the torus.

It is equivalent to taking the sphere and adding a handle.
It's nice to draw graphs on spheres with handles to eliminate edge crossings.

*Ex.:* $K_5$ isn't planar but it embeds with 1 handle.

It is too hard to draw on spheres.

apply handle here.
Take your donut:

Cut it, and unfold:

This is a polygon that represents the torus.

Identification Polygon

abc
Let's draw $K_5 \subset T$

($K_5$ embedded on the torus):

$v = 5$
$e = 10$
$f = 5$

$v - e + f = 0$ ?
We can also do $K_7$:

$K_7$ is triangulating the torus!

$v = 7$, $e = 21$, $f = 14$

$v - e + f = 0$?
Euler's formula on the torus is
\[ v - e + f = 0. \]
We can make other surfaces by adding handles.

double torus

(also sphere w/two handles)

\(v - e + f = -2\)
A sphere with $n$ handles has formula:

$$v - e + f = 2 - 2n$$

Eq: $n = 0$

Sphere

$v - e + f = 2$

$n = 1$

Torus

$v - e + f = 0$

$n = 2$

Double torus

$v - e + f = -2 \ldots$
Homework:
do calculations for regular solids
on the torus
(use \( v - E + f = 0 \) instead
of \( v - e + f = 2 \)).

(Google platonic solids
for good pictures, too.)
If you have ?s:

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