I. Green's Theorem

Let \( C \) be a positively oriented, piecewise smooth, simple, closed curve around a region \( D \). If \( P(x,y) \) and \( Q(x,y) \) have continuous partial derivatives on an open region containing \( D \), then

\[
\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

or

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_C P \, dx + Q \, dy
\]
nice area application:

recall area of $D$ is

$$\int \int_D A$$

so we just need $P$ and $Q$

so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$
\[ \int \int_D dA = \int_y \int_{x_0}^{x_1} x \, dy = -\int_y \int_{x_0}^{x_1} y \, dx = \frac{1}{2} \int_x (x^2 y - y^2) \, dx \]

\[
\begin{align*}
\phi &= 0 \\
\phi &= x \\
\phi &= -y/2 \\
\phi &= x/2
\end{align*}
\]
Example:

\[
\oint_C (x^2 + y^2) \, dx + 2xy \, dy
\]

where \( C \) is \( y = x^2 \) from \((0, 0)\) to \((2, 4)\) then line from \((2, 4)\) to \((0, 4)\) then line from \((0, 4)\) to \((0, 0)\).

By Green's Theorem

\[
\oint_C (x^2 + y^2) \, dx + 2xy \, dy = \iint_D (2y - 2y) \, dA = 0
\]
\[ \begin{align*}
\oint & \int_{x^2 + y^2 = 1} x^2 \, dx - 3y^2 \, dy = 0 \\
\frac{\partial p}{\partial y} &= x^2 \\
\frac{\partial Q}{\partial x} &= 0 \\
\text{get } & \iint_D - \int_{-x^2} x^2 \, dA \\
& \text{go polar} \\
& \int_0^{2\pi} \int_{-r^2}^r (r \cos \theta)^2 \, r \, dr \, d\theta \\
& \int_0^{2\pi} \int_0^1 \cos^2 \theta \, d\theta \cdot \int_0^3 r^3 \, dr \\
& \text{(so } D \text{ is a unit circle)}
\end{align*} \]
Prove that the coordinates of the centroid of a region \( \mathcal{D} \)

\[(\bar{x}, \bar{y})\]

where

\[
\bar{x} = \frac{1}{m} \iint_{\mathcal{D}} x \, dA, \quad \bar{y} = \frac{1}{m} \iint_{\mathcal{D}} y \, dA
\]

can be given by

\[
\bar{x} = \frac{1}{2A} \int_{c}^{c} \int_{c}^{2A} x \, dy \, dx, \quad \bar{y} = \frac{1}{2A} \int_{c}^{2A} \int_{c}^{x^2} y \, dx \, dy
\]
Reason:
By Green's theorem
\[ \iint_D x \, dA = \oint_C P \, dx + Q \, dy \]
where \( \frac{dQ}{dx} - \frac{dP}{dy} = X \)

since we want the form given above, take \( Q = \frac{x^2}{2} \), \( P = 0 \)
then \( \frac{dQ}{dx} - \frac{dP}{dy} = X \), so apply Green's

to get
\[ \iint_D x \, dA = \iint_D 0 \, dx + \frac{x^2}{2} \, dy \]
\[ = \oint_C \frac{x^2}{2} \, dy \].