16.3

\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} \text{ is independent of path in } D \iff \int_{C} \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for every closed path in } D \]
Reasoning

\[ \int \mathbf{F} \cdot d\mathbf{r} = \int \mathbf{F} \cdot d\mathbf{r} \]
\[ c_1 \quad c_2 \]

\[ \Rightarrow \int \mathbf{F} \cdot d\mathbf{r} - \int \mathbf{F} \cdot d\mathbf{r} = 0 \]
\[ c_1 \quad c_2 \]

\[ \Rightarrow \int \mathbf{F} \cdot d\mathbf{r} + \int -\mathbf{F} \cdot d\mathbf{r} = 0 \]
\[ c_1 \quad -c_2 \]

\[ \Rightarrow \int \mathbf{F} \cdot d\mathbf{r} = 0 \]
\[ c_1 - c_2 \]

\[ \Rightarrow \int \mathbf{F} \cdot d\mathbf{r} = 0 \]
\[ c \]
Another thing...

If \( F \) is a continuous vector field on an open connected region \( D \) and \( \oint_C F \cdot dr \) is independent of path on \( \partial D \) then \( F \) is conservative.

ie \( \exists f \) so that \( F = \nabla f \).
Example: find the work done by
\[
\vec{F}(x,y) = \frac{y^2}{x^2} \mathbf{i} - \frac{2y}{x} \mathbf{j}
\]
in moving an object from \((1,1)\) to \((4,-2)\).
1st check if $F$ is conservative.

Sure: \[ \frac{2y}{x^2} = \frac{2y}{x} = \frac{\partial}{\partial x} \]

Thus $F = Df$ for some $f$ and \( \int_C F \cdot dr = f(4,2) - f(1,1) \).
So all we have to do is find $f$

$$f_x(x,y) = \frac{y^2}{x^2}$$

$$f(x,y) = \int \frac{y^2}{x^2} \, dx = -\frac{y^2}{x} + g(y)$$

Then $f_y(x,y) = -2yx^{-1} + g'(y)$

From also $f_y(x,y) = -2yx^{-1} + \ln A$

$F = \langle f_x, f_y \rangle$

So $g'(y) = 0$

$g(y) = \text{constant}$
So \( f(x, y) = -\frac{y^2}{x} + C \)

So \( \int_{C} F \cdot dr = f(4, -2) - f(1, 1) \)

\[
C = \frac{-\frac{9}{4} - \frac{1}{1}}{1} = 0
\]
Proof that if $\mathbf{F}$ is a cont. vector field on an open connected $D$, and $\int_c \mathbf{F} \cdot d\mathbf{r}$ is indep. of path, then $\mathbf{F}$ is conservative, i.e. $\int_c \mathbf{F} \cdot d\mathbf{r} = \nabla f$. 