1. Let \( \mathbf{u} \) be any unit vector \( \langle a, b \rangle \) and let \( \langle x_0, y_0, z_0 \rangle \) be a point on the surface. Slice through the surface in the \( \mathbf{u} \) and find the slope of the tangent line. This is \( D_{\mathbf{u}} f(x_0, y_0) \), the directional derivative of \( f \) at \( (x_0, y_0) \) in the direction \( \mathbf{u} \).
\[ D_u f(x, y) = f_x(x, y) a + f_y(x, y) b \]
\[ = \langle f_x, f_y \rangle \cdot \langle a, b \rangle \]

\[ \nabla f = \langle f_x, f_y \rangle \quad (\text{gradient}) \]

\[ \nabla \text{grad } f, \text{ or } \text{del } f \]

Thus,

\[ D_u f(x, y) = \nabla f \cdot \hat{u} \]
Theorem

The maximum of $D_u f(x, y)$ is $|\nabla f| \hat{u}$.

Proof:

$D_u f = \nabla f \cdot \hat{u} = |\nabla f| |\hat{u}| \cos \theta$

$= |\nabla f| \cos \theta$

biggest when $\theta = 0$, i.e. when $\nabla f$ and $\hat{u}$ are $\parallel$, i.e. $\hat{u}$ points in the $\nabla f$ direction.
eg. find the rate of change and direction it occurs in \( \nabla f \) at \((4, 3, -1)\) if \( f(x, y, z) = x + \frac{y}{z} \).

\[ \nabla f = \langle f_x, f_y, f_z \rangle \]
\[ \nabla f = \langle 1, \frac{1}{z}, -\frac{y}{z^2} \rangle \]

At \((4, 3, -1)\), get

\[ \nabla f = \langle 1, -\frac{1}{3}, -3 \rangle \]
The max rate of change is
Thus \[ |\langle 1, -1, -3 \rangle| = \sqrt{1 + 1 + 9} = \sqrt{11} \]
and occurs in the direction
\[
\frac{\langle 1, -1, -3 \rangle}{\sqrt{11}} = \left\langle \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-3}{\sqrt{11}} \right\rangle
\]
Tangent plane to a surface given by an equation in $x, y, z$:

Write all junk on left hand side and call this $F(x, y, z)$.

Then tangent plane at $(a, b, c)$ is given by

$$F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0$$

or $\nabla F \cdot (\langle x, y, z \rangle - \langle a, b, c \rangle) = 0$

Thus $\nabla F$ is the normal to the tangent plane.
\[ xe^{yz} = 1 \]

\[ F_x = e^{yz} \rightarrow 1 \]
\[ F_y = xe^{yz} \rightarrow 5 \]
\[ F_z = xe^{yz} \rightarrow 0 \]

\[ \nabla F = \langle 1, 5, 0 \rangle \]

\[ (x-1) + 5(y-0) + 0(z-5) = 0 \]

\[ x-1 + 5y = 0 \quad \Rightarrow \quad x + 5y = 1 \]
Show $D$ follows the product rule:

\[ D(uv) = u Dv + v Du \]

\[
\nabla uv = \left< (uv)_x, (uv)_y \right>
\]
\[
= \left< u_x v + u v_x, v_y v + v v_y \right>
\]
\[
= u \left< v_x, v_y \right> + v \left< u_x, u_y \right>
\]
\[
= u \nabla v + v \nabla u
\]