1.1

6.3. \( a_1 = 1 \), \( a_{n+1} = 3 - \frac{1}{a_n} \)

\[ a_1 = 1 \]
\[ a_2 = 3 - \frac{1}{a_1} = 3 - \frac{1}{1} = 2 \]
\[ a_3 = 3 - \frac{1}{a_2} = 3 - \frac{1}{2} = \frac{5}{2} = 2.5 \]
\[ a_4 = 3 - \frac{1}{a_3} = 3 - \frac{1}{\frac{5}{2}} = \frac{15}{5} - \frac{2}{5} = \frac{13}{5} = 2.6 \]
\[ a_5 = 3 - \frac{1}{a_4} = 3 - \frac{1}{\frac{13}{5}} = \frac{15}{13} - \frac{5}{13} = \frac{10}{13} = 2.615 \]
\[ a_6 = 3 - \frac{1}{a_5} = 3 - \frac{1}{\frac{10}{13}} = \frac{39}{13} - \frac{13}{13} = \frac{26}{13} = 2.617 \]
see if $3 - \frac{1}{a_n}$ is monotonic

i.e. $a_{n+1} > a_n$

is $3 - \frac{1}{a_n} > 3 - \frac{1}{a_{n-1}}$

is $\frac{-1}{a_n} > \frac{-1}{a_{n-1}}$

is $\frac{1}{a_n} < \frac{1}{a_{n-1}}$

is $a_{n-1} < a_n$
Proof by induction:

show \( a_{n+1} > a_n \)

Base case:

know \( a_2 > a_1 \), since \( 2 > 1 \)

Now assume

\( a_n > a_{n-1} \)

and show that \( a_{n+1} > a_n \)
given \( a_n > a_{n-1} \)

then \( \frac{1}{a_{n-1}} > \frac{1}{a_n} \)

so \( \frac{-1}{a_{n-1}} < \frac{-1}{a_n} \)

\( 3 - \frac{1}{a_{n-1}} < 3 - \frac{1}{a_n} \)

\( a_n < a_{n+1} \)

Wahoo!
Show bounded above.
Try bounded by 3.\ (i.e. show $a_n < 3 \forall n$)

By induction again:

Base case: show true for $n = 1$

i.e show $a_1 < 3$

Now assume $a_{n-1} < 3$ and prove $a_n < 3$

\[ a_n = 3 - \frac{1}{a_{n-1}} \]

Is \[ 3 - \frac{1}{a_{n-1}} \leq 3 \] ?

Is \[ \frac{-1}{a_{n-1}} \leq 0 \] yes.
$\exists a_n \geq 3$ converges since increasing and bounded above by 3.

Suppose
$$\lim_{n \to \infty} a_n = L,$$
then we want to know what $L$ is.
\[ L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} 3 - \frac{1}{a_{n-1}} \]

\[ = 3 - \frac{1}{\lim_{n \to \infty} a_{n-1}} \]

Thus

\[ L = 3 - \frac{1}{L} \]

and now we can solve for \( L \)
we get
\[ L(L) = (3 - \frac{1}{L}) L \]
\[ L^2 = 3L - 1 \]
\[ L^2 - 3L + 1 = 0 \]
\[ L = \frac{3 \pm \sqrt{5}}{2} \]
What does this mean.

If \( \sum a_n \) is defined by

\[
a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}
\]

Then \( \lim_{n \to \infty} a_n = \frac{3 + \sqrt{5}}{2} \)