\[ F = \frac{mgR^2}{(x+R)^2}, \quad F_{net} = ma = m \frac{dv}{dt} \]

\( x(t) \) is position (height) above earth, so \( \frac{dx}{dt} = v \)

Thus \( m \frac{dv}{dt} = -\frac{mgR^2}{(x+R)^2} \) (\( -\) since gravity acts downward)

\[ \Rightarrow \quad \frac{dv}{dt} = \frac{-gR^2}{(x+R)^2} \]

We would like to "separate this, but have \( dt \) on the left, and \( x \), not \( t \), on the right. Need to switch one or the other. Note that we can think of velocity as a function of \( x \) (i.e., what is velocity at a certain height rather than time), i.e., as \( V(x(t)) \)

So by the chain rule,

\[ \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}, \quad \text{but} \quad \frac{dx}{dt} = v \]

Thus \( \frac{dv}{dt} = \frac{dv}{dx} \cdot v \)

So \( v \frac{dv}{dx} = \frac{-gR^2}{(x+R)^2} \) so now we can think of this as a separable diff. eq.
\[ S_v dv = -gR^2 \int \frac{1}{(x+R)^2} \, dx \]

\[ \frac{v^2}{a} = \frac{gR^2}{(x+R)} + C \]

but here \( v \) is a function of \( x \)

when \( t=0, \ x=0, \ \) so \( v(0) = v_0 \) (initial velocity)

and we get that

\[ \frac{v_0^2}{a} = \frac{gR^2}{(0+R)} + C, \ \text{so} \ \frac{v_0^2}{a} = gR = C \]

so \( \frac{v^2}{a} = \frac{gR^2}{(x+R)} + \frac{v_0^2}{a} - gR \).

Now, at the top of the flight, \( x=h \) (given)

and \( v=0 \), so get

\[ 0 = \frac{gR^2}{h+R} + \frac{v_0^2}{a} - gR \]

\[ gR - \frac{gR^2}{h+R} = \frac{v_0^2}{a} \]

\[ \frac{gR (h+R) - gR^2}{h+R} = \frac{v_0^2}{a} \Rightarrow a \sqrt{\frac{2gR \cdot h}{h+R}} = v_0 \]
b. \[ V_e = \lim_{h \to \infty} \sqrt{\frac{2gR}{h+R}} = \sqrt{2gR} \cdot \lim_{h \to \infty} \frac{\sqrt{h}}{\sqrt{h+R}} \]

\[ = \sqrt{2gR} \cdot 1 = \sqrt{2gR} \]

\[ V_e = \sqrt{20.32 \cdot 3960 \text{ ft/mile}} \]

\[ \text{Suppose} \quad \lim_{h \to \infty} \frac{\sqrt{h}}{\sqrt{h+R}} = L \quad \text{this is} \quad \frac{\infty}{\infty} \quad \text{form} \]

so can use L'Hopital

\[ L = \lim_{h \to \infty} \frac{\sqrt{h}}{\sqrt{h+R}} = \lim_{h \to \infty} \frac{\frac{1}{2} h^{-\frac{1}{2}}}{\frac{1}{2} (h+R)^{-\frac{1}{2}}} = \]

\[ = \lim_{h \to \infty} \frac{\sqrt{h}^{\frac{1}{2}}}{\sqrt{h+R}^{\frac{1}{2}}} = \lim_{h \to \infty} \frac{\sqrt{h}}{\sqrt{h+R}} = L \]

\[ \implies L = \frac{1}{L} \implies L^2 = 1 \implies L = \pm 1 \]

but \[ \frac{\sqrt{h}}{\sqrt{h+R}} \] is always non-negative, so \[ L = +1 \]

\[ \lim_{h \to \infty} \frac{\sqrt{h}}{\sqrt{h+R}} = 1 \]