Comparison test

\[ a_n, b_n > 0 \text{ for all } n \]

If \( \sum b_n \) converges and \( a_n \leq b_n \text{ for all } n \), then \( \sum a_n \) converges.

If \( \sum b_n \) diverges and \( a_n \geq b_n \text{ for all } n \), then \( \sum a_n \) diverges.

(limit comparison)

If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \) with \( 0 < c < \infty \),

then \( \sum a_n \) converges

\( \iff \sum b_n \) converges.
Estimating

if $0 < a_n, b_n$ and $a_n \leq b_n$

and $\sum b_n$ converge (hence $\sum a_n$ converge by comparison test)

Then observe

$\sum b_n = S = S_n + R_n = S_n + b_{n+1} + b_{n+2} + \ldots$

$\sum a_n = T = T_n + T_n = T_n + a_{n+1} + a_{n+2} + \ldots$

so $T_n \leq R_n$

usually $\sum b_n$ is a "known" series eg

eg geometric or power series

so you can either compute $R_n$ (eas.) or bound it with an integral (p-series)

(then this gives a bound for the error $T_n$ from estimating $T = \sum a_n$ by $T_n$.)
\[ \sum_{n=1}^{\infty} \frac{4+3^n}{2^n} \]

\[ \frac{4+3^n}{2^n} \geq \frac{3^n}{2^n} = \left( \frac{3}{2} \right)^n \]

but \[ \sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n \]

geometric series

with \[ r = \frac{3}{2} > 1 \]

hence diverges.

\[ \sum_{n=1}^{\infty} \frac{4+3^n}{2^n} \]

diverges by comparison test.
\[ \sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1} \leq \frac{n^2}{3n^4} = \frac{1}{3n^2} \]

Compare to

\[ \sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2} \]

Since the series converges, by the \( p \)-series test, with \( p = 2 > 1 \).
34. \[ \sum_{n=1}^{10} \frac{1 + \cos(n)}{n^5} = 1.55972 \]

(note that \( 0 < \frac{1 + \cos(n)}{n^5} < \frac{n+1}{n^5} < \frac{2}{n^5} \))

Thus our error will be less than the error in \[ \sum_{n=1}^{\infty} \frac{2}{n^5} \]

\[ \text{i.e.} \quad R_{10} \]
But this is a p-series, so from last class

\[ R_{10} \leq \sum_{n=1}^{\infty} \frac{2^n}{x^5} \text{ d}x = \lim_{t \to \infty} \int_{10}^{t} \frac{2x-5}{x^5} \text{ d}x \]

\[ = \frac{1}{2} \cdot \frac{1}{104} = .00005 \]