5.2 Fundamental Theorem of Calc:

If $f$ is cont. on $[a, b]$

then...

\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

where $F(x) = f(x)$
Proof of I.\[
\frac{d}{dx} \int_{a}^{x} \! f(t) \, dt = \lim_{h \to 0} \frac{\int_{a}^{x+h} \! f(t) \, dt - \int_{a}^{x} \! f(t) \, dt}{h}\]

Graph of the function $f(t)$ with points $a$, $x$, and $x+h$.
\[ \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt \times b(t) \]

Recall

\[ m \cdot h \leq \int_{x}^{x+h} f(t) \, dt \leq M \cdot h \]

so \( m \leq \frac{1}{h} \int_{x}^{x+h} f(t) \, dt \leq M \)
Thus

\[ \lim_{h \to 0} m \leq \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt \leq \lim_{h \to 0} M \]

\[ \lim_{h \to 0} x \leq \lim_{h \to 0} x \]

\[ f(x) \leq \lim_{h \to 0} \int_{x}^{x+h} f(t) \, dt \leq f(x) \]

\[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \]

\[ \text{by pinching theorem} \]

\[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \]
Proof of F

That \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \)

where \( F'(x) = f(x) \)

i.e. \( F(x) \) is any antiderivative of \( f(x) \)
Proof:

First recall that if

\[ F'(x) = f(x) \quad \text{and} \quad \]
\[ G'(x) = f(x) \quad \text{then} \]

\[ F(x) - G(x) = C \quad \text{(a constant)} \]
Now let \( F(x) \) be any antiderivative of \( f(x) \), and let 
\[ G(x) = \int_{a}^{x} f(t) \, dt \]

So by part I, \( G'(x) = \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \)

\[ \therefore G(x) \text{ is also an antiderivative of } f(x). \]
Thus

\[ F(x) - G(x) = c \quad \text{(a constant)} \]

In particular

\[ F(a) - G(a) = c, \quad \text{but } G(a) = 0 \]

so \[ F(a) = c \]
Therefore
\[ f(x) - g(x) = F(a) \text{ for all } x \]

In particular if \( x = b \) get
\[ F(b) - g(b) = F(a) \]
so \[ F(b) - F(a) = \frac{b}{a} \int_a^b f(x) \, dx \]

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]
Let $F$ be an indefinite integral of $f$.
Let $A(x) = \int_a^x f(t) \, dt$; by part 2 of the Fundamental Theorem, $A$ is also an indefinite integral of $f$.

What we called $G(x)$
A different proof of Part II

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \]

where \( x_i^* \) is any point in \([x_{i-1}, x_i]\)
Let $F(x)$ be an antiderivative of $f(x)$, i.e. $F'(x) = f(x)$.

Recall MVT says that if $g(x)$ is continuous on $[a, b]$ and $g'(x)$ exists on $(a, b)$, then for $c \in (a, b)$ such

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$
So we know (by part I) that $f(x)$ is cont on $[x_{i-1}, x_i]$ and $F'(x)$ exist on $(x_{i-1}, x_i)$ (in fact $F'(x) = f(x)$)

\[ F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \]

So let's choose $x_i^*$ to be $c_i$
So,
\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*) \Delta x \]

\[ \geq \lim_{n \to \infty} \sum_{i=1}^n f(c_i) \cdot (x_i - x_{i-1}) \]

\[ = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) - f(x_{i-1}) \left( \frac{x_i - x_{i-1}}{x_i - x_{i-1}} \right) \]
\[
\lim_{n \to \infty} \sum_{i=1}^{n} F(x_i) - F(x_{i-1})
\]

\[
= \lim_{n \to \infty} \left( F(x_1) - F(x_0) + F(x_2) - F(x_1) + \ldots + F(x_n) - F(x_{n-1}) \right)
\]

\[
= \lim_{n \to \infty} -F(x_0) + F(x_n) = -F(a) + F(b)
\]

\[
= F(b) - F(a)
\]