Show that

\[ x^3 - 15x + k = 0 \]

has at most one root in \( \mathbb{Z}_2 \).
3WOC, suppose there are two - say 
\[ f(x) = x^3 - 15 + k \]
has \( f(a) = 0 \) and \( f(b) = 0 \) with \( a, b \in [-2, 2] \)
Then Rolle's theorem says

\[ f(c) \in (a, b) \)

\( -2 < c < 2 \)

So \( f'(c) = 0 \)
but

\[ f'(x) = 3x^2 - 15 \]

so \( f'(c) = 0 \) \implies

\[ 3c^2 - 15 = 0 \]

\implies \[ 3c^2 = 15 \]

\[ c^2 = 5 \]

\[ \implies c = \pm \sqrt{5} \approx 2.23... \]

but we said \(-2 < c < 2\)

so this can't happen.
4.3

**Theorem**

\[ f'(x) > 0 \text{ on an interval} \]

\[ \Rightarrow f(x) \text{ is increasing on the interval}. \]
Proof:

Let \( x_1, x_2 \in I \) with \( x_1 < x_2 \).

By MVT, \( \exists c \in (x_1, x_2) \) such that

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)
\]

positive, positive since \( x_2 > x_1 \),

so \( f(x_2) - f(x_1) > 0 \),

so \( f(x_2) > f(x_1) \) so \( f \) is increasing.
First Derivative Test

Let $c$ be a critical number for a continuous function $f$.

\[
\begin{array}{c|ccc}
\text{sign of } f' & - & + \\
\text{behavior of } f & \text{dec} & \text{inc} & f(c) \text{ is a rel min}
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{sign of } f' & + & - \\
\text{behavior of } f & \text{inc} & \text{dec} & f(c) \text{ is a rel max}
\end{array}
\]

If the sign of $f'(x)$ doesn't change from the left to the right of $c$, then $f(c)$ is neither a rel max, nor rel min.
The second derivative determines concavity

eg. \( f'' > 0 \) it means that \( f' \) is increasing
The function is concave up.

- Slopes increase.
\[
\begin{array}{cc}
\text{if } f' > 0, f'' > 0 & \text{if } f' > 0, f'' < 0 \\
\text{f is inc + cc } \uparrow & \text{f is inc + cc } \downarrow \\
\text{if } f' < 0, f'' > 0 & \text{if } f' < 0, f'' < 0 \\
\text{f is dec + cc } \uparrow & \text{f is dec + cc } \downarrow \\
\end{array}
\]
2nd derivative test:

if \( f'(c) = 0 \)

and

\[ f''(c) < 0 \] then \( f(c) \) is local max

\[ f''(c) > 0 \] then \( f(c) \) is local min

\[ f''(c) = 0 \] No conclusion
Inflection point: if 
\[ f''(c) = 0 \quad \text{and} \quad \]
either 
\[ f'' < 0 \quad \frac{1}{c} \quad + \quad \]
or 
\[ f'' > 0 \quad \frac{1}{c} \quad - \quad \]
then say \( f(c) \) is an inflection pt.
\[ f(x) = \frac{x^2}{x^2 + 3} \]

\[ f'(x) = \frac{2x(x^2 + 3) - x^2 (2x)}{(x^2 + 3)^2} \]

\[ = \frac{6x}{(x^2 + 3)^2} \]
\[ f''(x) = \frac{6 (x^2 + 3)^2 - 6x \cdot 2(x^2 + 3)^1 \cdot 2x}{(x^2 + 3)^4} \]

\[ f''(x) = \frac{6 (x^2 + 3) \left[ x^2 + 3 - 4x^2 \right]}{(x^2 + 3)^4} \]

\[ f''(x) = \frac{18 (1 - x^2)}{(x^2 + 3)^3} = \frac{18 (1-x)(1+x)}{(x^2 + 3)^2} \]