we want

\[ h(x) = \frac{f(x) - f(a)}{(b - a)(x - a)} + f(a) \]

blue-red = function-line

slope is \( f'(c) \)

line is

\[ y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a) \]

or

\[ f(x) = \frac{f(b) - f(a)}{b - a} (x - a) + f(a) \]
Apply Rolle's theorem to
\[ h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a) \]

1st need to check that \( h(x) \) satisfies the hypotheses:

1. Is \( h(x) \) continuous on \([a, b]\)?
   - Yes because both \( f(x) \) and the line are.

2. Does \( h'(x) \) exist on \((a, b)\)?
   - Yes, because \( f'(x) \) exists on \((a, b)\) and a line is always differentiable.
3. does $h(a) = h(b)$?

\[ h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a) \]

so

\[ h(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a-a) - f(a) = f(a) - f(a) = 0 \]

\[ h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a) \]

\[ = f(b) - f(b) + f(a) - f(a) \]

\[ = 0. \]

so yes $h(a) = h(b)$
If $c \in (a, b)$ then
\[ h'(c) = 0 \]
So, what is $h'(x)$?
\[ h(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a) - f(a) \]
so
\[ h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \cdot 1 - 0 \]
So there is some $c \in (a, b)$ such that
\[ 0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \]
I.e.,
\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]
260

dist

100

slope is 80
Cor. If \( f'(x) = 0 \ \forall \ x \in (a,b) \)
then \( f(x) \equiv k \ \text{on} \ (a,b) \)

Proof: by MVT

take \( r \in (a,b) \)

and apply MVT on \([a,r]\)

\( f(x) \) is cont on \([a,r]\)

\( f'(x) \) exist (in fact = 0) on \((a,r)\)

So \( \exists \ C \in (a,r) \) s.t.

\[
f'(c) = \frac{f(r) - f(a)}{r - a}
\]

but \( f'(c) = 0 \) since \( f'(x) = 0 \ \forall \ x \in (a,b) \) \( (\text{given}) \)

so \( 0 = \frac{f(r) - f(a)}{r - a} \)

\[
\Rightarrow \ f(r) = f(a) \ \forall \ r \in (a,b)
\]
Cor. 

\[ f'(x) = g'(x) \quad \forall x \in (a, b) \]

Then \[ f(x) - g(x) = k \quad \forall x \in (a, b) \]

i.e. \[ f(x) = g(x) + k \]

Proof: Apply MVT to \[ f(x) - g(x) \]

get that \[ f'(x) - g'(x) = 0 \quad \forall x \in (a, b) \]

So by previous cor., \[ f(x) - g(x) = k \].
Show

$2x - 1 - \sin x = 0$ has exactly one real root.

By intermediate value theorem:

$f(0) = 0 - 1 - 0 = -1 < 0$

$f(\pi) = 2\pi - 1 - \sin \pi = 2\pi - 1 > 0$

So there must be at least one zero between 0 and $\pi$. 
Suppose there were 2 zeros.

Say \( f(a) = f(b) = 0 \)

Then MVT says if \( c \in (a, b) \)

so that \( f'(c) = \frac{f(b) - f(a)}{b - a} = 0 \)

but... \( f'(x) = 2 - \cos x \geq 0 \)

so no such \( c \)

\( \therefore \) can't have two roots.