Multivariable Tutte and Transition Polynomials

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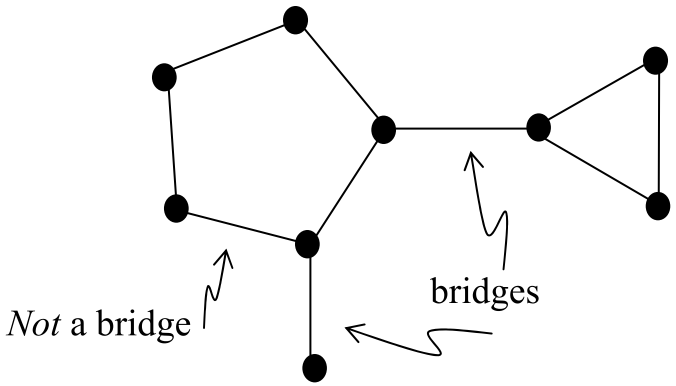
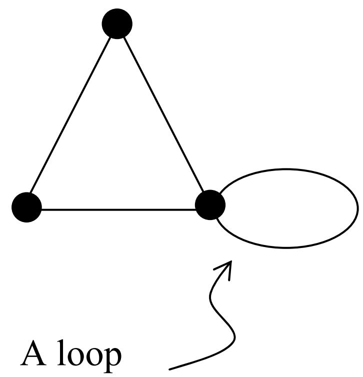
Bridges and Loops

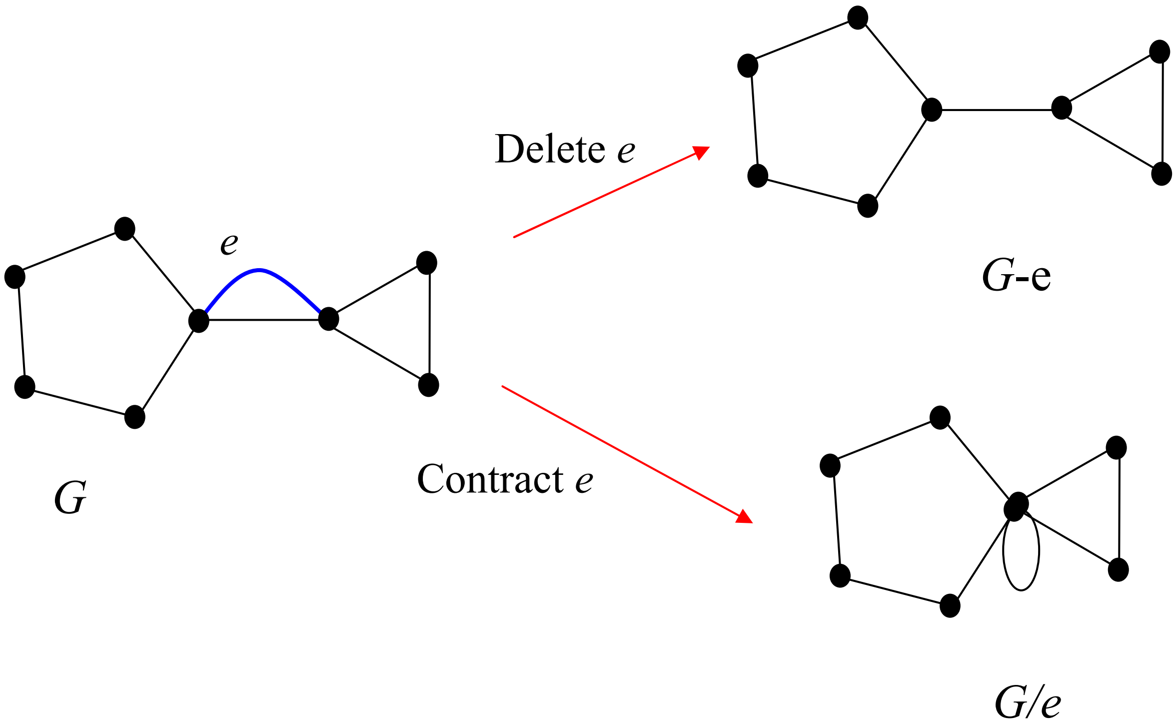




A *bridge* is an edge whose deletion separates the graph

A *loop* is an edge with both ends incident to the same vertex

Deletion and contraction



Circuit vs cycle…

Classical Dichromatic Polynomial a.k.a. Potts Model Partition Function

The classical dichromatic polynomial:

$$Z\left(G;u,v\right)=\sum_{A\subseteq E(G)} u^{k(A)}v^{|A|-n+k(A)}$$

Potts Model Partition Function with q spins, $v=\exp(J/\kappa T)-1$
and Hamiltonian $h(\omega)=-J\sum_{ij\in E(G)}\delta(\sigma_i,\sigma_j)$

$$Z\left(G;q,v\right)=\sum_{A\subseteq E(G)} q^{k(A)}v^{|A|-n+k(A)}=\sum_{states\ \omega}\exp\left(\frac{-h(\omega)}{\kappa T}\right)$$

All edges have the *same* interaction energy in this model.

If q and v are indeterminates rather than physical values, these polynomials are the same (Fortuin & Kastelyn '72).

Essential Properties

- $Z(G;u,v) = Z(G-e;u,v) + Z\left(G/e;u,v\right)$ if e is not a loop.

- $Z\left(GH;u,v\right) = Z\left(G;u,v\right)Z\left(H;u,v\right)$ if GH is the disjoint union of G and H.

- $Z(G;u,v) = u(v+1)^n$ if G consists of a single vertex and n loops.

(Whitney) Tutte '67

These capture essential (local) physical properties, and are (defining) theoretical tools.

Some Multivariable Dichromatic Polynomials

Traldi '89:

$$Z\left(G;t,q,\mathbf{v}\right)=\sum_{A\subseteq E(G)} q^{k(A)}t^{|A|-n+k(A)}\prod_{e\in A} v_e \rightarrow Z\left(G;q,\mathbf{v}\right)=\sum_{A\subseteq E(G)} q^{k(A)}\prod_{e\in A} v_e$$

$$Z(G;u,\mathbf{v})=Z(G-e;u,\mathbf{v})+v_e Z\left(G/e;u,\mathbf{v}\right)$$

The interaction energy may depend on the edge

Fortuin & Kastelyn '72:

$$Z\left(G;q,\mathbf{v},\mathbf{w}\right)=\sum_{A\subseteq E(G)} q^{k(A)}\prod_{e\in A} v_e \prod_{e\in A^c} w_e$$

$$Z(G;u,\mathbf{v},\mathbf{w})=w_e Z(G-e;u,\mathbf{v},\mathbf{w})+v_e Z\left(G/e;u,\mathbf{v},\mathbf{w}\right)$$

Doubly weighted, but $w_e + v_e = 1$, so equivalent to above.

A slight shift…

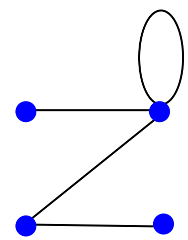
Classical Tutte polynomial:

Let e be an edge of G that is neither a bridge nor a loop. Then,

$$T(G;x,y) = T(G-e;x,y) + T(G \setminus e;x,y)$$

And if G consists of i bridges and j loops, then

$$T(G;x,y) = x^i y^j$$



The Tutte polynomial is a translation of the dichromatic/Potts

$$u^{k(G)} v^{|V|-k(G)} T\left(G;\frac{u+v}{v},v+1\right) = Z(G;u,v)$$

Fully 'multivariablized' Tutte polynomial

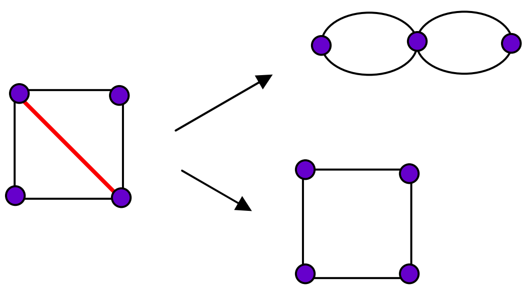
Zaslavsky '92, Bollobás & Riordan '99, (E-M &Traldi '06)

Let $W\left(E_n, c\right) = \alpha_n$, where E_n is the edgeless graph on n vertices.

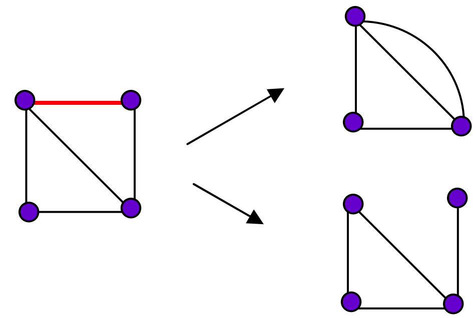
$$W\left(G,c\right) = \begin{cases} X_e W\left(G/e,c\right) \text{ if } e \text{ is a bridge} \\ Y_e W\left(G-e,c\right) \text{ if } e \text{ is a loop} \\ x_e W\left(G/e,c\right) + y_e W\left(G-e,c\right) \text{ else} \end{cases}$$

Each edge has four parameters associated with it: two 'interaction energies', a loop value, and a bridge value.

Need to be careful about order…



vs.

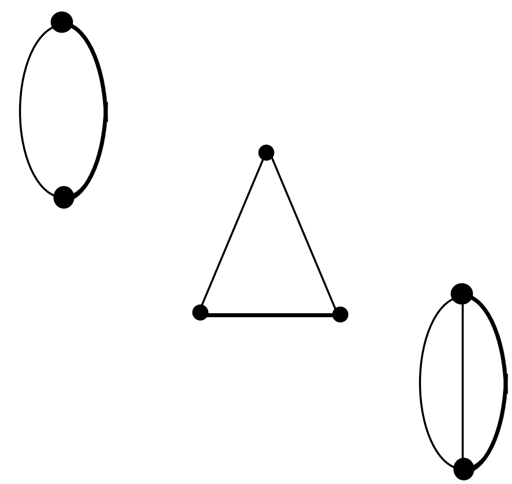


Need to have:

$$X_\lambda y_\mu - y_\lambda X_\mu - x_\lambda Y_\mu + Y_\lambda x_\mu = 0$$

$$Y_\nu \left(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu \right) = 0$$

$$X_\nu \left(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu \right) = 0$$



Necessary and sufficient to assure the function is well-defined, i.e. independent of the order of deletion and contraction.

Dichromatic analog

Don't always get something analogous to the dichromatic, but do in all 'reasonable' situations, e.g. parameters in a field and not all zero, domain = all graphs.

In this case, with $X_e = x_e + qy_e,\ \ Y_e = x_e + ry_e$

$$Z(G;q,r,\mathbf{x},\mathbf{y}) = \sum_{S\subseteq E(G)} q^{k(S)-k(G)} r^{|S|-n+k(S)} \prod_{e\in S} x_e \prod_{e\in S^c} y_e$$

$$\to \sum_{S\subseteq E(G)} q^{k(S)} \prod_{e\in S} x_e \prod_{e\in S^c} y_e$$

This gives a model with two degrees of freedom in the weights.

The Martin, or circuit partition polynomial

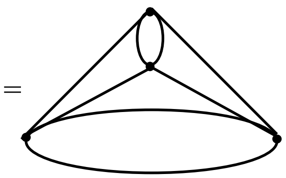
Martin '77, cf Las Vergnas

Let G be an Eulerian graph (all vertices of even degree).

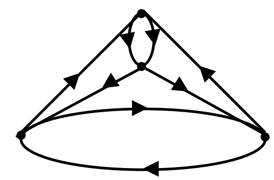
Let $\vec{G}$ be an Eulerian digraph (directed edges and Kirchhoff laws).

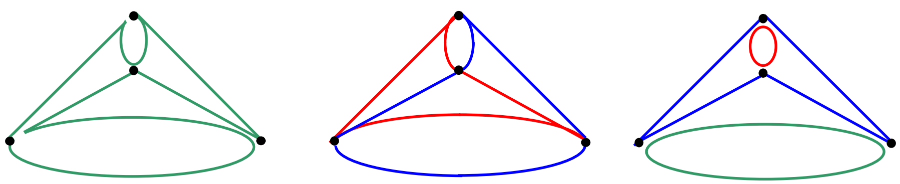
$$J(G;x) = \sum_{k=1} f_k(G)x^k, \quad j(\vec{G};x) = \sum_{k=1} f_k(\vec{G})x^k$$

$G =$



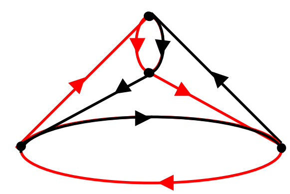
An Eulerian digraph G :





1-partition 2-Partition 3-Partition

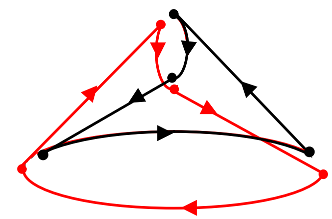
An Eulerian 2-partition of G :



These polynomials are generating functions for Eulerian k-partitions in a graph.

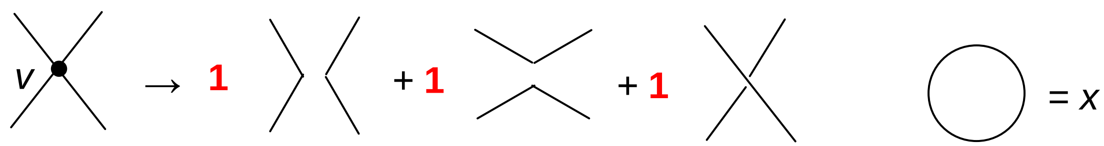
State Model and Recursive Forms

$$J\left(G;x\right)=\sum_{S\in states(G)} x^{c(S)}, \quad j\left(\vec{G};x\right)=\sum_{S\in states(\vec{G})} x^{c(S)}$$

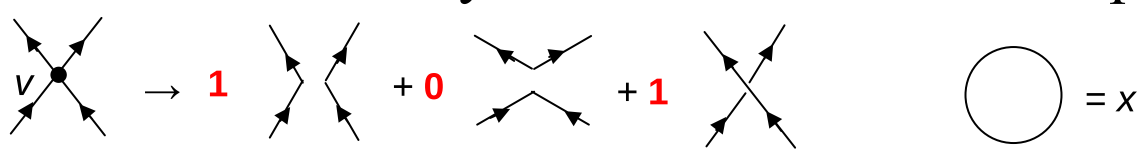


A graph state with 2 components

The Circuit Partition Polynomial



The Circuit Partition Polynomial for Oriented Graphs:



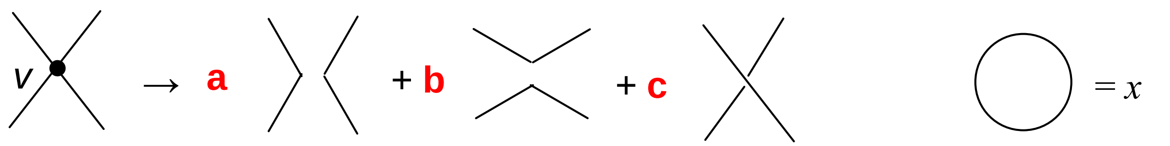
(The weight is 1 for *coherent* states, 0 else)

Multiple formulations facilitate theoretical techniques, e.g. induction.

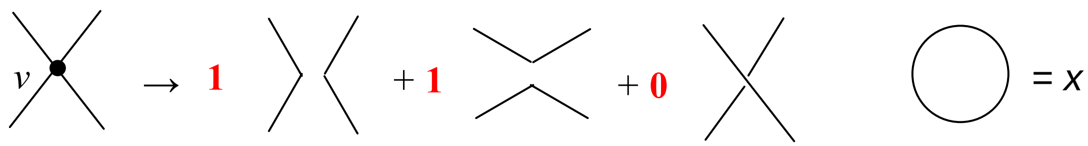
A multivariable, or weighted, version

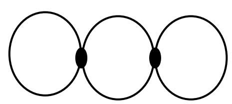
"On Transition Polynomials of 4-Regular Graphs" --F. Jaeger, 1987

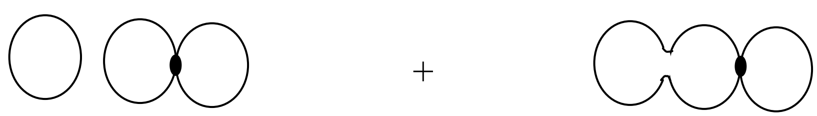
The general form:

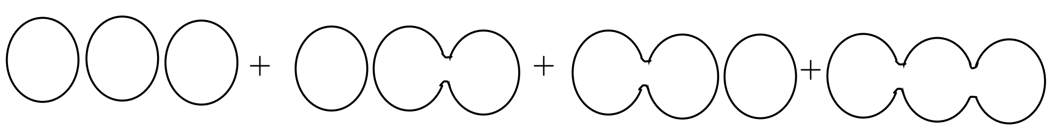


For example, if the weight system is:



Then for $G =$ , get





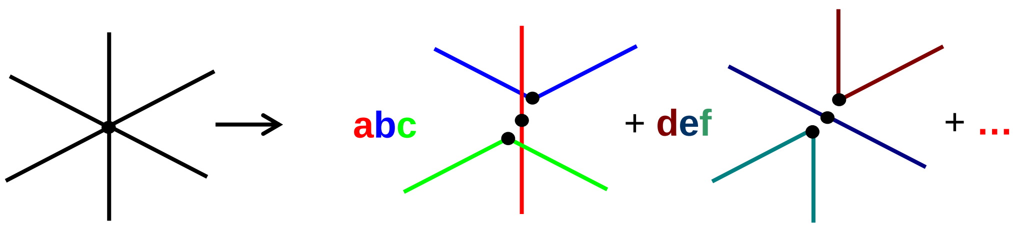
So $Q(G; A, x) = x^3 + 2x^2 + x$

A is the weight system specifying the coefficient for each transition.

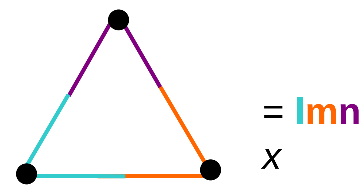
The generalized transition polynomial

E-M 98, E-M & Sarmiento 02

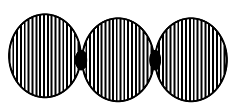
$N(G;W,x)$, where G is an arbitrary Eulerian graph, and W is a weight system which assigns a value in R (a ring with unit) to every pair of adjacent half edges in G. Then,

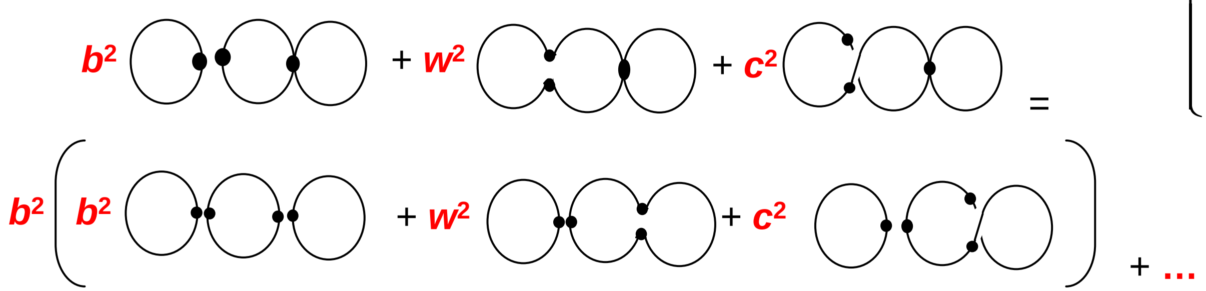


(New edge pairs have weight 1)



$$W = \begin{cases} b \text{ if pair bounds the same black face} \\ w \text{ if pair bounds the same white face} \\ c \text{ else} \end{cases}$$

E.g. $G =$ 



$$= b^4x^3 + b^2w^2x^2 + b^2c^2x^2 + \ldots$$

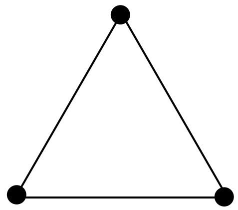
Overview of relationships

The Circuit Partition Polynomial → The Generalized Transition Polynomial

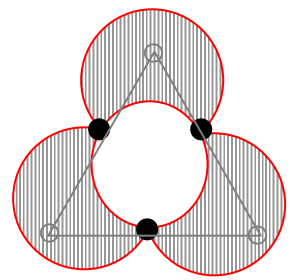
For planar graphs via the medial graph

The Classical Tutte Polynomial → Parameterized Tutte Polynomials

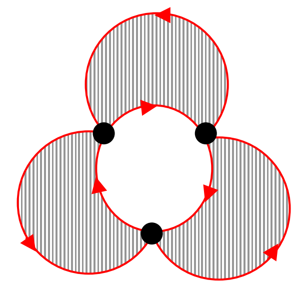
The Tutte-transition connection



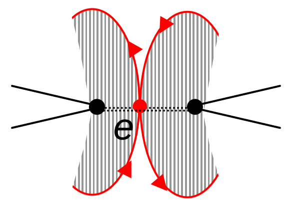
A Planar graph *G*



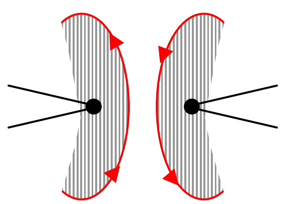
G_m with the vertex faces colored black

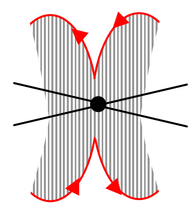


Orient G_m so that black faces are to the left of each edge.



delete





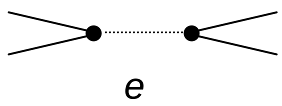
contract

Then, with this orientation of G_m, $j\left(\vec{G}_m; x\right) = x^{k(G)} t(G; x+1, x+1)$

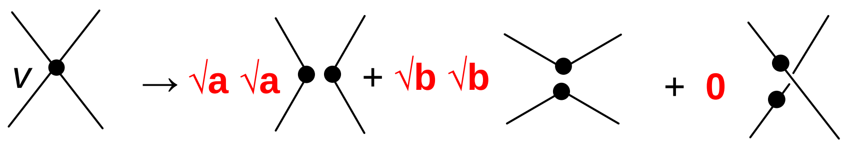
Martin 77

Same construction for multivariable case

G a planar graphs with oriented medial graph $\vec{G}_m$



with weights **a** and **b**



The edge weights of G induce a weight system on $\vec{G}_m$

Then $$N\left(\vec{G}_m; W, x\right) = x^{k(G)} Z(G; x, x, \mathbf{v}, \mathbf{w})$$

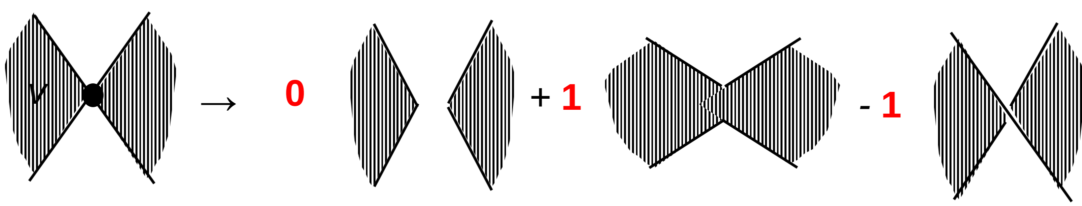
The multivariable Tutte & transition polynomials are also related for planar graphs via the medial construction.

Beyond Tutte: The Penrose Polynomial

"Applications of Negative Dimensional Tensors"—R. Penrose, 1969

Defined for planar graphs and computed via the medial graph. Motivated by diagrams from tensor analysis, and surprisingly, for 3-regular planar graphs,

$$P(G;3) = \left(\frac{-1}{4}\right)^{\frac{|V|}{2}} P(G;-2) = \text{\# 3-edge colorings} \leftrightarrow \text{4 color theorem}$$

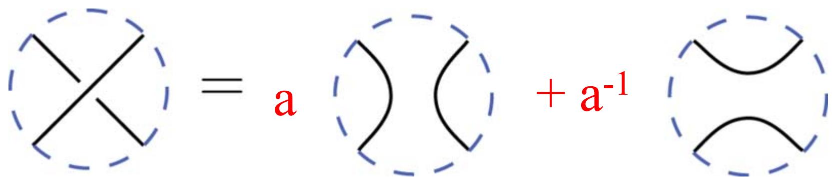


$$N(G_m; A, x) = P(G; x)$$

N assimilates polynomials the Tutte polynomial doesn't.

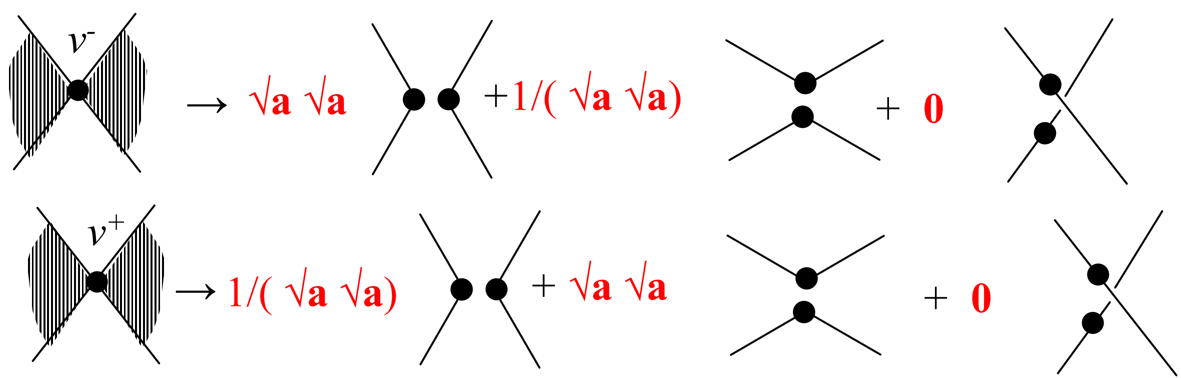
The Kauffman bracket of knot theory

The Kauffman bracket of a link *L*:



$K[\bigcirc \cup L] = 1 \qquad K[\bigcirc] = 1$

Let G_L be the signed, 2 face colored universe of a link *L*:



$$N\left(G_L; W, a^2 + a^{-2}\right) = (a^2 + a^{-2})K[L]$$

Hopf Algebras—definition by example

The Binomial Bialgebra B

B is an infinite dimensional vector space over **C** with basis $\{1, x, x^2 \ldots\}$

B is an algebra, with (associative) multiplication given by:

$$m : B \otimes B \to B \quad \text{by} \quad m(x^r \otimes x^s) = x^{r+s}$$

B is a coalgebra, with (coassociative) comultiplication given by:

$$\Delta x^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} \otimes x^r$$

B is a bialgebra, since the multiplication and the comultiplication are compatible, i.e. the comultiplication is an algebra map:

$$\Delta\left(x^r \cdot x^s\right) = \Delta x^r \cdot \Delta x^s, \text{ where } \left(a \otimes b\right) \cdot \left(c \otimes d\right) = ac \otimes bd$$

B is a Hopf algebra, with antipode: $S\left(x\right) = -x$

Underlying Algebraic Structure

$$\Gamma = \{ \ (G,W) \ \}$$

Γ is a locally finite partially ordered set with order relation

$$(A,W(A)) \leq (G,W(G))$$

if A is an Eulerian subgraph of G, and if A has a weight system

$W(A)$ that is *inherited* from $W(G)$

Γ is also closed under disjoint union (direct product)

Γ is a hereditary family

Incidence Hopf Algebra

$\Gamma = span_R\{\ (G,W)\ \}$ is an incidence Hopf algebra (Schmitt, 94).

Multiplication: $m:\Gamma\otimes\Gamma\to\Gamma$ by

$$m\big((G,W(G))\otimes(H,W(H))\big)=(GH,W(GH))$$

Unit: $\mu:R\to\Gamma$ by $\mu(r)=r(\mathcal{E},W)$

Comultiplication: $\Delta:\Gamma\to\Gamma\otimes\Gamma$ by

$$\Delta(G,W)=\sum(A_1,W(A_1))\otimes(A_2,W(A_2))$$

Counit: $\varepsilon:\Gamma\to R$ by $\varepsilon(G,W)=\begin{cases}1 \text{ if } G=\mathcal{E}\\ 0 \text{ else}\end{cases}$

Antipode:

$$\zeta(G,W)=\sum(-1)^{|P|}\big(A_1\dots A_{|P|},W(A_1\dots A_{|P|})\big)$$

sum over all ordered partitions P of G into $|P|$ edge-disjoint Eulerian subgraphs.

N is a Hopf algebra map

Theorem: If we give $R[x]$ the structure of a binomial bialgebra, then $N : \Gamma \to R[x]$ is a Hopf algebra map.

The proof is straightforward combinatorics, but requires hereditary properties, and hence the pair weights and not just state weights.

Structural identities

I. (From the comultiplication)

$$N\left(G;W,x+y\right)=\sum N\left(A_1;W\left(A_1\right),x\right)N\left(A_2;W\left(A_2\right),y\right)$$

II. (From the antipode)

$$N\left(G;W,-x\right)=N\left(\zeta\left(G,W\right);x\right)$$

These are powerful theoretic tools, particularly for inductive arguments.

Some applications

Identity I:

Combinatorial interpretations for the Martin polynomials for all integers (previously only known for –2, -1, 0, 1 in the oriented case, and –2, 0, 2 in the unoriented case) –E-M, also Bollobas.

Combinatorial interpretations for the diagonal Tutte polynomial (also derivatives) of a planar graph for all integers (previously –1, 3 were the only known non-trivial values) – E-M.

Identity II:

Used to determine combinatorial interpretations for the Penrose polynomial for all negative integers (previously only known for positive integers)—Sarmiento, E-M&Sarmiento.

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