Weak Tutte Functions of Matroids – w/ T. Zaslavsky.

Barcellona 10/05 ①
Talk

1. Recall the classical Tutte polynomial:

1. $T(M; x, y) = T(M/e; x, y) + T(M\setminus e; x, y)$
2. $T(M_1 \oplus M_2) = T(M_1)\, T(M_2)$
3. $T(D) = x^i y^j$ if D is a discrete matroid with i coloops + j loops.

And the universality:

a if $f(\emptyset) = 1$
b $f(M) = a f(M/e) + b f(M\setminus e)$ if e is non-sep.
c $f(M_1 \oplus M_2) = f(M_1) f(M_2)$

Then $f(M) \xrightarrow{\text{essentially}} T(M; f(\text{coloop}), f(\text{loop}))$
(unique extension)

Question: Where is the action really? I.e. how much can we throw away and still get a universal contraction deletion invariant?

Builds on essential ideas in Zas92, BR92, unified/reconciled in E-MT

Ans Every thing except the contract-deletion! For example β invariant coef of x
(and in fact can replace a by a_e, b by b_e and throw out invariance too.)

1^{st} need some mechanism to keep track of the contraction/deletion parameters:

Let U be a set. Let γ, δ be parameter assignments γ & $\delta : U \to R$ (a ring)
and let $\mathcal{M}$ be a minor-closed class of matroids on U

A weak Tutte function with parameter assignments γ & γ is a function $F : \mathcal{M} \to L$ (an R module)

(PDC) s.th $F(M) = \gamma_e F(M/e) + \delta_e F(M\setminus e)$

Note: 1. function, ie well-defined, ie indep't of order of del/cont
2. F must be specified for all discrete matroids in $\mathcal{M}$
3. Without specified relations among the γ_e's & δ_e's F is not a matroid invariant

especially 4, γ and δ make it possible to always view F as a module Hom: Let $P = \{ c_e, d_e \mid e \in U \}$ and let $\mathbb{Z}[P]$ be the polynomial ring and $\mathbb{Z}[P]\mathcal{M}$ the free module over $\mathbb{Z}[P]$ generated by the set $\mathcal{M}$.

Then sending $c_e \mapsto \gamma_e$, $d_e \mapsto \delta_e$ makes

$$F: \mathbb{Z}[P]\mathcal{M} \longrightarrow L \quad \text{a } \mathbb{Z}[P] \text{ module hom.}$$

Now what about indep't of order?

If $\Gamma = \langle M - c_e M/e - d_e M \backslash e \rangle$
Then F is a weak Tutte function $\Leftrightarrow$ $\Gamma \subseteq \ker F$
ie F factors through $\mathbb{Z}[P]\mathcal{M}/\Gamma$
Remarkable is that this information can be encoded entirely in terms of discrete matroids:

$$\begin{tikzcd}
\Delta \arrow[r, hook] \arrow[d, hook] & \mathbb{Z}[P]\mathcal{D} \arrow[r, two heads, "\pi_{\mathcal{D}}"] \arrow[d, hook, shift left] & \mathbb{Z}[P]\mathcal{D}/\Delta \arrow[d, hook, shift left, "i"] \\
\Gamma \arrow[r, hook] & \mathbb{Z}[P]\mathcal{M} \arrow[r, two heads, "\pi_{\mathcal{M}}"'] \arrow[u, shift left, "t_\sigma"] \arrow[ur, dashed, "t"] & \mathbb{Z}[P]\mathcal{M}/\Gamma \arrow[u, shift left, "\tau"]
\end{tikzcd}$$

So what is Δ? t_σ, t, τ.

As in the case (Zas 92, BR99), All the action takes place with the little guys:  although here w/ discretes attached say what this is

Δ is the submodule $\mathbb{Z}[P]\mathcal{D}$ generated by the relations making the little guys order indept — $\oplus D$ for any discrete matroid.

eg, for ${}^{e}\bigcirc^{f} \oplus D \in \mathcal{M}$

need $c_e \, \bigcirc^{f} \oplus D + d_e \, |^{f} \oplus D - (c_f \, \overset{e}{\circ} \oplus D + d_f \, {}^{e}| + D) \in \Delta \subseteq \mathbb{Z}[P]\mathcal{D}$

$t_\sigma(M)$ = weighted sum of discrete matroids from

$M \to c_e M/e + d_e M\backslash e$ if e is a non-sep

in the order given by σ

$t = \pi_{\mathcal{D}} \circ t_\sigma$

$\tau = t \circ \pi_{\mathcal{M}}^{-1}$

Δ is where the action is - in fact:

Thrm 1 If I is any submodule of $\mathbb{Z}[P]\mathcal{D}$, and

proj: $\mathbb{Z}[P]\mathcal{D} \to \mathbb{Z}[P]\mathcal{D}/I$

then proj $\circ\, t_\sigma$ is well defined, ie indep't of order

$\Leftrightarrow \Delta \subseteq I$

In particular, t is well defined

The proof is much the same flavor, up to technical details, as Zas 92, E-M Traldi, (BR 99)

Another way to see that Δ is where the action is

is that $\mathbb{Z}[P]\mathcal{D}/\Delta \cong \mathbb{Z}[P]\mathcal{M}/\Gamma = \langle M - c_e M/e - d_e M\backslash e \rangle$

optional Proof that $\mathbb{Z}[P]\mathcal{D}/\Delta \cong \mathbb{Z}[P]\mathcal{M}/\Gamma$

ι is cannonical inclusion

$$D+\Delta \longmapsto D+\Gamma$$

$\tau = t \cdot \pi_m^{-1}$ Note τ is well defined since

$t = \pi_{\mathcal{D}}\, t_\sigma$ $\Gamma \subseteq \ker t$

Then $\tau\ \iota(D+\Delta) = t\ \pi_m^{-1}(D+\Gamma) = \pi_{\mathcal{D}}\, t_\sigma(D) = \pi_{\mathcal{D}}(D) = D+\Delta$ ✓

and $\iota\ \tau(M+\Gamma) = \iota\ t\ \pi_m^{-1}(M+\Gamma) = \iota\ \pi_{\mathcal{D}}\, t_\sigma(m) =$

$\iota(t_\sigma(m)+\Delta) = t_\sigma(m)+\Gamma = M-M+t_\sigma(m)+\Gamma =$

$M+\Gamma$ since $M - t_\sigma(m) \in \Gamma$ ✓

---

t, the tutte form, has the desired universalitiy & unique extension properties:

rel papers:
Fortuin & Kasteleyn 72
Traldi 89
Bollobás, Pebody, Riordan 2000

Thrm: If F is any weak Tutte function $F: \mathbb{Z}[P]\mathcal{M} \to L$ then $F(M) = \hat{F}(t(M))$ where $\hat{F}: \mathbb{Z}[P]\mathcal{D}/\Delta \to L$ by $\hat{F}(\bar{D}) = F(D)$ and conversely, if $G: \mathbb{Z}[P]\mathcal{D}/\Delta \to L$ is a module hom. Then $F = G \circ t$ is a weak Tutte function.

What does this really buy us?

1. If F is any weak Tutte function, then F can be given as an evaluation of t - just replace all the (classes of) discrete matroids in the expression of t by F(discrete)
2. If $f: \mathbb{Z}[P]\mathcal{D} \to L$ with $\Delta \subseteq \ker f$, then we can extend f uniquely to a weak Tutte function, loosely
$f(M) =$ "delete/contract, in any order, down to discretes, then evaluate" *
+ this will be indept of order