Chaos

Understanding Chaotic Behavior Through Differential Equations

The Simple Three

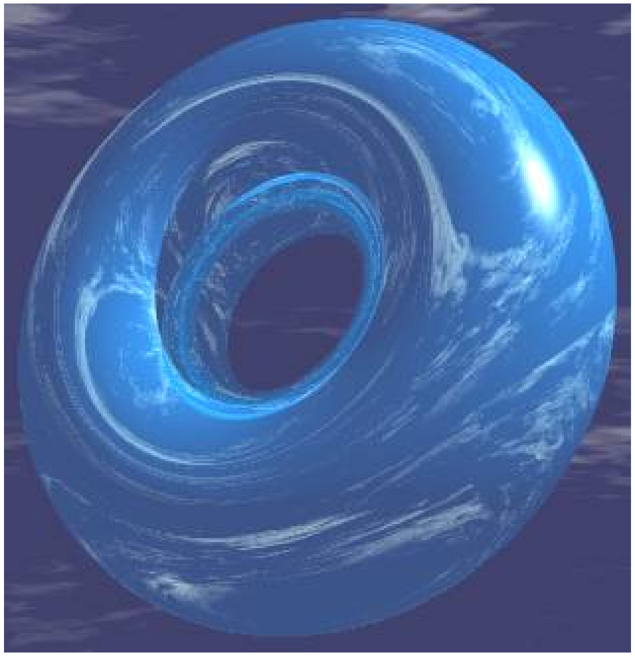
$$dx/dt = \sigma(y - x)$$

$$dy/dt = rx - y - xz$$

$$dz/dt = xy - bz$$

Quasi-periodicity on a Torus

- Consider the quotient group R/Z (the reals mod the integers). There is a map on the torus (looks like a bagel or a doughnut- show picture of maple torus) which assigns to each real number x its equivalence class [x].



Quasi-periodicity on a Torus

- When the surface area is peeled off and flattened out it looks like a 1 by 1 square that can be viewed in the 2 dimensional phase plane.
- Thus we now have an area on a graph that we can apply 2 differential equations, called $dx/dt=f(x, y)$ and $dy/dt=g(x, y)$ to move along the flattened out torus. Both of these equations have period one that is they are evaluated mod 1
  - i.e. $f(x+1, y+1) = f(x,y)$ and $g(x+1, y+1) = g(x,y)$

Quasi-periodicity on a Torus

Looking at the simple flow of two linear and constant trajectories on the torus:

- dx/dt= w_1
- dy/dt= w_2

- This leaves us with the resulting solutions of the DEs:

$$x(t) = w1t+ c1$$

$$y(t)=w2t+c2$$

- Each has slope $\frac{w_2}{w_1}$ and passes through (c1, c2)

Quasi-periodicity on a Torus

- 2 Things can now happen:
  - The slope $\frac{w_2}{w_1}$ is Rational or it is irrational.
  - If it is Rational I claim that we will get closed trajectories on the Torus

Quasi-periodicity on a Torus

- So assume $\frac{w_2}{w_1}$ is some p/q where p and q are integers in lowest terms.
- Now consider that WLOG our trajectory passes through the origin (c1,= c2=0) to simplify our work. Thus we have the resulting equation.

$$x(t) = w_1 t$$

$$y(t) = (p/q) w_1 t$$

Quasi-periodicity on a Torus

- Now consider then time t/w1 has elapsed. This is just an arbitrary time but it allows us to simplify and better analyze our equations. So if q/w1 time has passed our equations now look like:

$$x(t)= q$$

$$y(t)= p$$

And thus we have a closed curve on the 1 by 1 square and also the torus.

Quasi-periodicity on a Torus

- The Slope is irrational: Claim:

  a) trajectories are not closed

  b) the trajectory fills-in the torus;
  i.e. is dense in itself.

  c) will come arbitrarily close to any point on the torus

Quasi-periodicity on a Torus

- First consider the first circle represented by the y axis. We are going to study the flow using a Poincare map obtained by taking a pt (0,y) and taking the point (1, F(y)) where the orbit through (0, y) hits the line x = 1
- Now by computing the function F and considering the orbit through (0,y) as

  {(w1t, w2t + y), $t \in R$ }

Quasi-periodicity on a Torus

- Consider now that x(t) = w_1 t = 1 and

  F(y) = w_2 t + y
- This then gives us t = 1/ w_1 and when we plug this into F(y) we get
- F(y) = $\frac{w_2}{w_1}$ + y (on the torus this equation will be mod 1)
- If this equation is repeated we get iterates of F(y) which can be represented by the set
- A = {n$\frac{w_2}{w_1}$+ m: n$\geq$0 $\in$Z, m $\in$ Z}

Quasi-periodicity on a Torus

- E.g. let $\frac{w_2}{w_1}$ be the $\sqrt{2}$ Then our set A (after a few computations) would yield…

Quasi-periodicity on a Torus

- Note that the resulting numbers are all contained within the interval [0, 1])
- Thus because all irrational numbers are unique and thus so are there multiples it suffices to say that A is dense in R and on the torus as well. A is dense on the torus because:
- **A is clearly a subgroup**
- It is closed under the operation of addition (a sum of two members in A will give another element of the same form: n $\frac{w_2}{w_1}$ + m; we also have additive inverses and an additive identity:0)
- **0 is an accumulation point of A:**
- Since A $\subseteq$ [0, 1] and we are in the Reals we know that our set A is bounded. Thus there must also be a subsequence which will converge to 0. i.e. for any ε>0 there are trajectories that come within this ε of 0 (come arbitrarily close to 0). Thus irrational trajectories will fill in the torus and hence is dense on the Torus.

Chaos

- Defined:
  - Is “aperiodic long term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.”

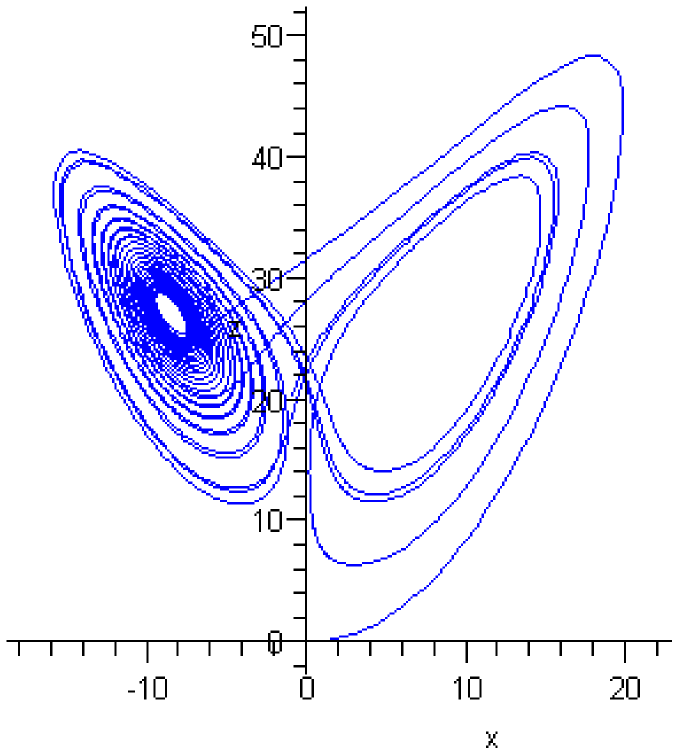
Chaos

- The Lorenz Attractors:

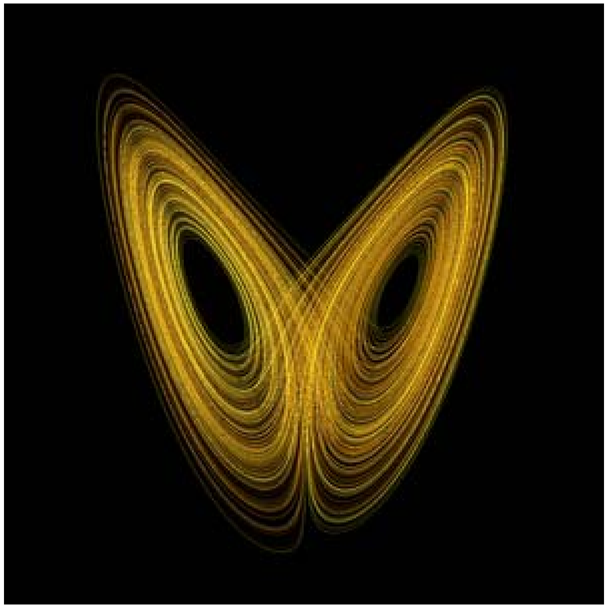
$$dx / dt = \sigma(y - x)$$

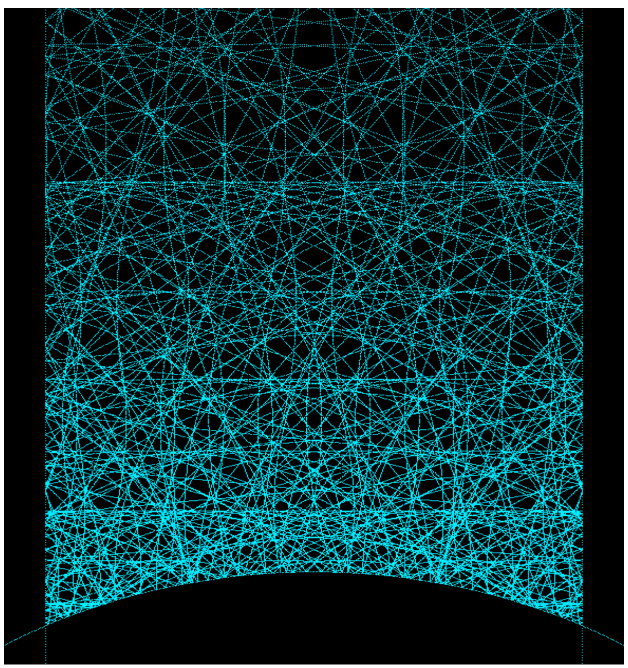
$$dy / dt = rx - y - xz$$

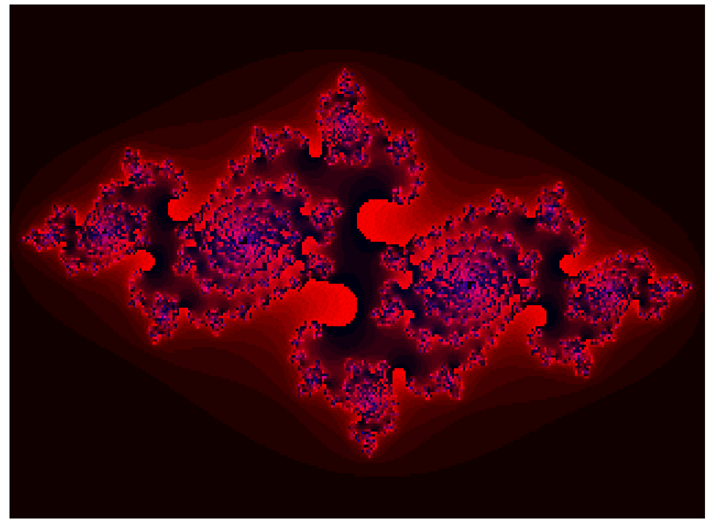
$$dz / dt = xy - bz$$

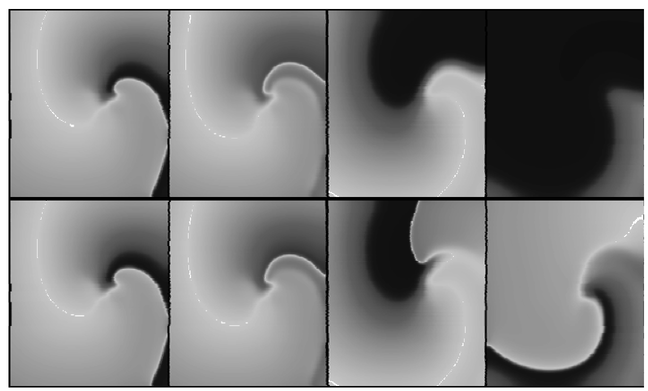


Chaos









Chaos: *More Than Just Pretty Pictures and Maple Plots*

- In biology, chaos is used in the identification of new evolutionary processes leading to understanding the genetic algorithim, artificial life simulations, better understanding of learning processes in systems including the brain
- consciousness and the mind
- thermodynamics in particular, chaos is applied in the study of turbulence leading to the understanding of self-organizing systems and system states
- answering previously unsolvable problems in quantum mechanics and cosmology.
- heart arrhthmias and brain function
- encryption schemese needed for computer networking and telecommunications
- for studying data that was previously thought of as random