Exercise 8.5
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Find all incongruent solutions to each of the following linear congruences.

First we need to remember that in the formula $ax = c$ (mod m) the number of solutions is determined by the gcd of (a,m). We will let g = gcd (a,m). If g does not divide c, then the congruence $ax \equiv c$ (mod m) has no solutions. If g does divide c, then there are exactly g incongruent solutions to $ax \equiv c$ (mod m).

(a) $8x \equiv 6$ (mod 14)

First we find gcd (8,14)
$14 = 8 \cdot 1 + 6$
$8 = 6 \cdot 1 + 2$
$6 = 2 \cdot 3 + 0$
gcd (8,14) = 2

Since 2 does divide 6, there are 2 solutions.

To find the solutions we use $8u$-$14v$=2. In this case, we found $(u,v) = (2,1)$. Multiplying by 6/2=3 gives the solution $(x,y) = (6,3)$ to the equation $8x$-$14y$=6. Finally the complete set of solutions to $8x \equiv 6$ (mod 14) is obtained by starting with x=6(mod 14) and adding multiples of the quantity 14/2=7. The value of the solutions can not go above 14 therefore starting with 6, 6+7=13 (the next solution) stopping there because 20>14.
The two solutions are x=6 and y=13.

(b) $66x \equiv 100$ (mod 121)

gcd (121,66)
$121 = 66 \cdot 1 + 55$
$66 = 55 \cdot 1 + 11$
$55 = 11 \cdot 5 + 0$
gcd (121,66) = 11

Since 11 does not divide 100, there are no solutions to this problem.

(c) $21\,x \equiv 14$ (mod 91)

gcd (91,21)
$91 = 21 \cdot 4 + 7$
$21 = 7 \cdot 3 + 0$
gcd(91,21) = 7

Since 7 divides 14, there are 7 solutions to $21\,x \equiv 14 \pmod{91}$.

Now to find the solutions, solve $21u - 91v = 7$. $(u, v) = (9,2)$, so multiplying by $14/7 = 2$ gives the solution $(x,y) = (18,4)$ to the equation $21x - 91y=14$. The rest of the solutions to $21x \equiv 14 \pmod{91}$ are obtained by starting with $x \equiv 18 \pmod{91}$ and adding multiples of the quantity $91/7 = 13$.

The 7 solutions are $x = \{18, 31, 44, 57, 60, 73, 86\}$ which all prove to be true.

EXTENSION

$12x \equiv 34 \pmod{56}$
gcd (56, 12)
$56 = 4 \cdot 12 + 8$
$12 = 1 \cdot 8 + 4$
$8 = 2 \cdot 4 + 0$

gcd (56,12) = 4

4 does not divide 34, therefore there are no solutions to $12x \equiv 34 \pmod{56}$

$65x \equiv 43 \pmod{21}$
gcd (65, 21)
$65 = 3 \cdot 21 + 2$
$21 = 10 \cdot 2 + 1$
$2 = 2 \cdot 1 + 0$

gcd (65, 21) = 1

1 does divide 43, therefore there is one answer

Now to find the solutions, solve $65u - 21v = 1$. $(u, v) = (11, 34)$, so multiplying by 43/1 = 43 gives the solution $(x,y) = (473, 1462)$ to the equation $65x - 21y = 43$. The rest of the solutions to $65x \equiv 43 \pmod{21}$ are obtained by starting with $x \equiv 473 \pmod{21}$ and adding multiples of the quantity 21/1=21.

473 is greater than 21, so I'm not sure these values worked. It could be the values I picked, or something went wrong in the calculations, but either way this example doesn't work.