6.6 Group B
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Presentation: John Lucy

Sometimes we are only interested in solutions to $ax + by = c$ using nonnegative values for x and y.

(a) Explain why the equation $3x + 5y = 4$ has no solutions with x 0 and y 0.

It is impossible for a positive 3-value and a positive 5-value to be added together and equal 4. $3 + 5 = 8$, therefore x and y cannot both be 1. Anything larger than 1 for either x or y would make the sum greater than 8. If x is 0, and y is 1, the sum would be 5 and if x is 1 and y is 0, the sum would be 3. Therefore, $3x + 5y$ can never equal 4 by replacing x and y by integers.

(b) Make a list of some of the numbers of the form $3x + 5y$ with x 0 and y 0. Make a conjecture as to which values are not possible. Then prove that your conjecture is correct.

(See attached maple)

The greatest value not possible for x and y can be found by using the formula $x \cdot y - (x + y)$. For the formula $3x + 5y$, the values 7, 4, 2, and 1 are not possible values for the sum of $3x$ *and* $5y$ since there are no values that can manipulate 3 and 5 to be added together to equal 7, 4, 2, or 1. So, when we look at this example of $3x + 5y$, we see that $x \cdot y - (x + y)$ is simply $(3 \times 5) - (3 + 5) = 7$.

(c) For each of the following values of (a, b), find the largest number that is not of the form $ax + by$ with x 0 and y 0.

(i) (a, b) = (3, 7) Largest value not possible = 11
(ii) (a, b) = (5, 7) Largest value not possible = 23
(iii) (a, b) = (4, 11) Largest value not possible = 29

(See Maple)

(d) Let gcd $(a, b) = 1$. Using your results from (c), find a conjectural formula in terms of a and b for the largest number that is not of the form $ax + by$ with x 0 and y 0? Check your conjecture for at least two more values of (a, b).

Our conjectural formula is $a \cdot b - (a + b)$. By subtracting their sum from their product, you will get the greatest number not possible for that a and b.

This formula works for all pairs in part (c):

(3, 7) $3 \cdot 7 = 21$ $3+7 = 10$ $21-10 = 11$

(5, 7)	$5 \cdot 7 = 35$	$5+7 = 12$	$35-12 = 23$
(4, 11)	$4 \cdot 11 = 44$	$4+11 = 15$	$44-15 = 29$

A few more examples of this formula:

(5, 13)	$5 \cdot 13 = 65$	$5+13 = 15$	$65-18 = 47$
(4, 9)	$4 \cdot 9 = 36$	$4+9 = 13$	$36-13 = 23$

(e) Prove that your conjectural formula in (d) is correct.

(*See attached maple*).

(f) Try to generalize this problem to sums of three terms $ax + by + cz$ with x 0 and y 0. For example, what is the largest number that is not of the form $6x + 10y + 15z$ with nonnegative x, y, z?

We tried to do this on maple, but were unsuccessful. There were never any non-included numbers in the lists that we made. In our sequence list of possible terms, we were never able to see if 869 was actually skipped. 869 is $(x \cdot y \cdot z)$ - $(x+y+z)$, so Maple was unable to give us an exact answer. In between 22 and 877, there were 26980 terms that were not listed because that is an overload of terms. Unfortunately, 869 is in between 22 and 877, so we were unable to get it precisely to that number. It makes sense intuitively that this pattern, multiplying the coefficients of each variable and then subtracting their product, would be followed to obtain the largest number that is not of the form $6x + 10y + 15z$. Although we cannot prove this because of some technological difficulties, this is what we would assume is the correct answer.

Extension

In these problems that we had to solve for *6.6,* the *a* and *b* values have no common factors. This leads one to ask the question of whether it would be possible to have the *a* and *b* values with a common factor? When we look at this, we see that these two values cannot have common factors because there will be a way to get that largest possible value that should not be able to be achieved by our conjecture in *part b*. This stated that if one is given an *a* and a *b* value, and if they do not have any common factors besides 1, then to find the largest possible value that cannot be achieved by inserting integers for *a* and *b* is given by $(x \cdot y \cdot z) - (x+y+z)$. We have already demonstrated how this works with some examples where *a* and *b* do not have common factors, but if we use this for two numbers that have common factors, we see that it does not hold true. For example, if we look at (6, 9) for (*a, b*) respectively, these two numbers have a common factor of 3. If we use our conjecture from above, we find that we get 39 as the largest possible value that shouldn't be able to be achieved. Although, we find that 39 is indeed attainable with the given *a* and *b* values. If 6 is multiplied by 5 and 9 multiplied by 1, we get 30 + 9, which does equal 39. Another example of this could be (5, 10). 5 multiplied by 10 minus 15 (5+10) is 35. If we multiply 5 by 3, this gives us 15, and if we multiply 10 by 2, we get 20. 20 + 15 equals 35, so our conjecture still does not work for *a* and *b* values that have a common factor. When we investigate why *a* and *b* values with the same factors do not work, we find that when we perform our conjecture upon them, we are multiplying the two *a* and *b* values together, and we are also subtracting a value that has that same common factor in it. When we add *a* and *b*, that is still going to have a common factor of whatever the two *a* and *b* values had as their common factor. For example, when we looked at (5, 10), we see that when we add them, we get 15, and if we multiply them, we are obviously going to get a factor of 5, so when you subtract a factor of 5 from a factor of 5, the end result will also be a factor of 5. So, in closing, this conjecture does not work for an *a* and *b* value with a common factor because the resulting answer will also have that same common factor.

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> restart;
> with(LinearAlgebra):
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B)

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> M:=Matrix(7,7, [seq([seq(5*x + 3*y, x=0..6)], y=0..6)]);
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$$M := \begin{bmatrix} 0 & 5 & 10 & 15 & 20 & 25 & 30 \\ 3 & 8 & 13 & 18 & 23 & 28 & 33 \\ 6 & 11 & 16 & 21 & 26 & 31 & 36 \\ 9 & 14 & 19 & 24 & 29 & 34 & 39 \\ 12 & 17 & 22 & 27 & 32 & 37 & 42 \\ 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 18 & 23 & 28 & 33 & 38 & 43 & 48 \end{bmatrix}$$

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> sort([seq(seq(3*x+5*y, x=0...10), y=0...10)]);
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[0, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 15, 16, 17, 18, 18, 19, 20, 20, 21, 21, 22, 23, 23, 24, 24, 25, 25, 26, 26, 27, 27, 28, 28, 29, 29, 30, 30, 30, 31, 31, 32, 32, 33, 33, 34, 34, 35, 35, 35, 36, 36, 37, 37, 38, 38, 39, 39, 40, 40, 40, 41, 41, 42, 42, 43, 43, 44, 44, 45, 45, 45, 46, 46, 47, 47, 48, 48, 49, 49, 50, 50, 50, 51, 51, 52, 52, 53, 53, 54, 54, 55, 55, 56, 56, 57, 57, 58, 59, 59, 60, 60, 61, 62, 62, 63, 64, 65, 65, 66, 67, 68, 69, 70, 71, 72, 74, 75, 77, 80]

C)

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> sort([seq(seq(3*x+7*y, x=0...10), y=0...10)]);
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[0, 3, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 21, 22, 23, 24, 24, 25, 26, 27, 27, 28, 28, 29, 30, 30, 31, 31, 32, 33, 34, 34, 35, 35, 36, 37, 37, 38, 38, 39, 40, 41, 41, 42, 42, 43, 44, 44, 45, 45, 46, 47, 48, 48, 49, 49, 50, 51, 51, 52, 52, 53, 54, 55, 55, 56, 56, 57, 58, 58, 59, 59, 60, 61, 62, 62, 63, 63, 64, 65, 65, 66, 66, 67, 68, 69, 69, 70, 70, 71, 72, 72, 73, 73, 74, 75, 76, 76, 77, 78, 79, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 90, 91, 93, 94, 97, 100]

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> sort([seq(seq(5*x+7*y, x=0...10), y=0...10)]);
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[0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 35, 36, 37, 38, 39, 40, 40, 41, 42, 42, 43, 44, 45, 45, 46, 47, 47, 48, 49, 49, 50, 50, 51, 52, 52, 53, 54, 54, 55, 56, 56, 57, 57, 58, 59, 59, 60, 61, 61, 62, 63, 63, 64, 64, 65, 66, 66, 67, 68, 68, 69, 70, 70, 71, 71, 72, 73, 73, 74, 75, 75, 76, 77, 78, 78, 79, 80, 80, 81, 82, 83, 84, 85, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100, 101, 103, 105, 106, 108, 110, 113, 115, 120]

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> sort([seq(seq(4*x+11*y, x=0...12), y=0...12)]);
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[0, 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 44, 45, 46, 47, 48, 48, 49, 50, 51, 52, 53, 54, 55, 55, 56, 57, 58, 59, 59, 60, 61, 62, 63, 64, 65, 66, 66, 67, 68, 69, 70, 70, 71, 72, 73, 74, 75, 76, 77, 77, 78, 79, 80, 81, 81, 82, 83, 84, 85, 86, 87, 88, 88, 89, 90, 91, 92, 92, 93, 94, 95, 96, 97, 98, 99, 99, 100, 101, 102, 103, 103, 104, 105, 106, 107, 108, 109, 110, 110, 111, 112, 113, 114, 114, 115, 116, 117, 118, 119, 120, 121, 121, 122, 123, 124, 125, 125, 126, 127, 128, 129, 130, 131, 132, 132, 133, 134, 135, 136, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 152, 153, 154, 156, 157, 158, 160, 161, 164, 165, 168, 169, 172, 176, 180]

D)

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> sort([seq(seq(5*x+13*y, x=0...12), y=0...12)]);
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[0, 5, 10, 13, 15, 18, 20, 23, 25, 26, 28, 30, 31, 33, 35, 36, 38, 39, 40, 41, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 170, 171, 172, 173, 175, 176, 177, 178, 180, 181, 183, 185, 186, 188, 190, 191, 193, 196, 198, 201, 203, 206, 211, 216]

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> sort([seq(seq(4*x+9*y, x=0...12), y=0...12)]);
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[0, 4, 8, 9, 12, 13, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 36, 37, 38, 39, 40, 40, 41, 42, 43, 44, 44, 45, 45, 46, 47, 48, 48, 49, 49, 50, 51, 52, 53, 53, 54, 54, 55, 56, 57, 57, 58, 58, 59, 60, 61, 62, 62, 63, 63, 64, 65, 66, 66, 67, 67, 68, 69, 70, 71, 71, 72, 72, 73, 74, 75, 75, 76, 76, 77, 78, 79, 80, 80, 81, 81, 82, 83, 84, 84, 85, 85, 86, 87, 88, 89, 89, 90, 90, 91, 92, 93, 93, 94, 94, 95, 96, 97, 98, 98, 99, 99, 100, 101, 102, 102, 103, 103, 104, 105, 106, 107, 107, 108, 108, 109, 110, 111, 111, 112, 112, 113, 114, 115, 116, 116, 117, 118, 119, 120, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 134, 135, 136, 138, 139, 140, 143, 144, 147, 148, 152, 156]

E)

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> sort([seq(seq(5*x+8*y, x=0...12), y=0...12)]);
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[0, 5, 8, 10, 13, 15, 16, 18, 20, 21, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 40, 41, 42, 43, 44, 45, 45, 46, 47, 48, 48, 49, 50, 50, 51, 52, 53, 53, 54, 55, 55, 56, 56, 57, 58, 58, 59, 60, 60, 61, 61, 62, 63, 63, 64, 64, 65, 66, 66, 67, 68, 68, 69, 69, 70, 71, 71, 72, 72, 73, 74, 74, 75, 76, 76, 77, 77, 78, 79, 79, 80, 80, 81, 82, 82, 83, 84, 84, 85, 85, 86, 87, 87, 88, 88, 89, 90, 90, 91, 92, 92, 93, 93, 94, 95, 95, 96, 96, 97, 98, 98, 99, 100, 100, 101, 101, 102, 103, 103, 104, 105, 106, 106, 107, 108, 108, 109, 110, 111, 111, 112, 113, 114, 115, 116, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 130, 131, 132, 133, 135, 136, 138, 140, 141, 143, 146, 148, 151, 156]

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> sort([seq(seq(6*x+5*y, x=0...12), y=0...12)]);
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[0, 5, 6, 10, 11, 12, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 30, 31, 32, 33, 34, 35, 35, 36, 36, 37, 38, 39, 40, 40, 41, 41, 42, 42, 43, 44, 45, 45, 46, 46, 47, 47, 48, 48, 49, 50, 50, 51, 51, 52, 52, 53, 53, 54, 54, 55, 55, 56, 56, 57, 57, 58, 58, 59, 59, 60, 60, 60, 61, 61, 62, 62, 63, 63, 64, 64, 65, 65, 66, 66, 66, 67, 67, 68, 68, 69, 69, 70, 70, 71, 71, 72, 72, 72, 73, 73, 74, 74, 75, 75, 76, 76, 77, 77, 78, 78, 79, 79, 80, 80, 81, 81, 82, 82, 83, 84, 84, 85, 85, 86, 86, 87, 87, 88, 89, 90, 90, 91, 91, 92, 92, 93, 94, 95, 96, 96, 97, 97, 98, 99, 100, 101, 102, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 114, 115, 116, 117, 120, 121, 122, 126, 127, 132]

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F)

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> sort([seq(seq(seq(2*x+3*y+5*z, x=0...8), y=0...8), z=0...8)]);
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[0, 2, 3, 4, 5, 5, 6, 6, 7, 7, 8, 8, 8, 9, 9, 9, 10, 10, 10, 10, 11, 11, 11, 11, 12, 12, 12, 12, 12, 13, 13, 13, 13, 13, 14, 14, 14, 14, 14, 14, 15, 15, 15, 15, 15, 15, 15, 16, 16, 16, 16, 16, 16, 17, 17, 17, 17, 17, 17, 17, 18, 18, 18, 18, 18, 18, 18, 18, 19, 19, 19, 19, 19, 19, 19, 19, 19, 20, 20, 20, 20, 20, 20, 20, 20, 20, 20, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 22, 22, 22, 22, 22, 22, 22, 22, 22, 22, 22, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 26, 26, 26, 26, 26, 26, 26, 26, 26, 26, 26, 26, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 28, 28, 28, 28, 28, 28, 28, 28, 28, 28, 28, 28, 28, 28, 29, 29, 29, 29, 29, 29, 29, 29, 29, 29, 29, 29, 29, 29, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 31, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 32, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 33, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 34, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 35, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 37, 37, 37, 37, 37, 37, 37, 37, 37, 37, 37, 37, 37, 37, 37, 37, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 39, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 43, 43, 43, 43, 43, 43, 43, 43, 43, 43, 43, 43, 43, 43, 43, 43, 44, 44, 44, 44, 44, 44, 44, 44, 44, 44, 44, 44, 44, 44, 44, 44, 45, 45, 45, 45, 45, 45, 45, 45, 45, 45, 45, 45, 45, 45, 45, 45, 46, 46, 46, 46, 46, 46, 46, 46, 46, 46, 46, 46, 46, 46, 46, 46, 47, 47, 47, 47, 47, 47, 47, 47, 47, 47, 47, 47, 47, 47, 47, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 51, 51, 51, 51, 51, 51, 51, 51, 51, 51, 51, 51, 51, 51, 52, 52, 52, 52, 52, 52, 52, 52, 52, 52, 52, 52, 52, 53, 53, 53, 53, 53, 53, 53, 53, 53, 53, 53, 53, 53, 54, 54, 54, 54, 54, 54, 54, 54, 54, 54, 54, 54, 54, 55, 55, 55, 55, 55, 55, 55, 55, 55, 55, 55, 55, 55, 56, 56, 56, 56, 56, 56, 56, 56, 56, 56, 56, 56, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 59, 59, 59, 59, 59, 59, 59, 59, 59, 59, 60, 60, 60, 60, 60, 60, 60, 60, 60, 60, 61, 61, 61, 61, 61, 61, 61, 61, 61, 62, 62, 62, 62, 62, 62, 62, 62, 63, 63, 63, 63, 63, 63, 63, 63, 64, 64, 64, 64, 64, 64, 64, 65, 65, 65, 65, 65, 65, 65, 66, 66, 66, 66, 66, 66, 67, 67, 67, 67, 67, 68, 68, 68, 68, 68, 69, 69, 69, 69, 70, 70, 70, 70, 71, 71, 71, 72, 72, 72, 73, 73, 74, 74, 75, 75, 76, 77, 78, 80]

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> sort([seq(seq(seq(6*x+10*y+15*z, x=0...29), y=0...29), z=0...29)]);
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[0, 6, 10, 12, 15, 16, 18, 20, 21, 22, *[...26980 terms...]*, 877, 878, 879, 881, 883, 884, 887, 889, 893, 899]

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