Chapter 3 Exercise 3.1

a) If \( u \) and \( v \) have a common factor, explain why \((a, b, c)\)
will not be a primitive Pythagorean triple.

Given: \((a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)\) and \(u = dm, \ v = dn\)

Substitute: \[\left((dm)^2 - (dn)^2, 2d^2mn, (dm)^2 + (dn)^2\right)\]

Then \((a, b, c) = (d^2(m^2 - n^2), 2d^2mn, d^2(m^2 + n^2))\)

Now, \((a, b, c)\) have a common factor of \(d^2\) and thus they
are not a primitive Pythagorean triple.

b) Need to find \(u > v > 0\), where \(u + v\) not common factor and
\((a, b, c)\) not primitive.

Need to find:

\[
\begin{align*}
\alpha &= u^2 - v^2 \\
\beta &= u^2 + v^2 \\
a + c &= 2u^2 &\Rightarrow u = \sqrt{\frac{a + c}{2}}
\end{align*}
\]

Take a primitive Pythagorean triple and multiply it by \(n\).

Thus take \((8, 15, 17)\) which is primitive,
then multiply \((8, 15, 17)n\). Now it's non-primitive.

Let \(n = 2 \rightarrow (16, 30, 34)\)

so \(u = \sqrt{\frac{16 + 34}{2}} = \sqrt{25} = 5\) and
\(c = u^2 + v^2 \Rightarrow c - u^2 = v^2; \ v^2 = 34 - 25 = 9 \rightarrow v = \sqrt{9} = 3\)

\(u = 5 > 3 = v \rightarrow 5 > 3 > 0. \checkmark\)
Chapter #3, Problem 3.1, Part C

> restart;

> TableSquares := proc(n, m)
> local u, v, S, A;
> S := {};
> for v from 1 to n do
> for u from v+1 to m do
> S := S union {(u, v) = u^2 + v^2};
> end do;
> end do;
> A := Matrix(n, m, S);
> end;

> TableSquares := proc(n, m)
> local u, v, S, A;
> S := {};
> for v to n do
> for u from v + 1 to m do
> S := union(S, {(u, v) = u^2 + v^2})
> end do;
> end do;
> A := Matrix(n, m, S);
> end proc;

> TableSquares(10, 10);

> TableSquares := proc(n, m)
> local u, v, S, A;
> S := {};
> for v from 1 to n do
> for u from v+1 to m do
> S := S union {(u, v) = u^2 + v^2};
> end do;
> end do;
> A := Matrix(n, m, S);
> end;

> TableSquares := proc(n, m)
> local u, v, S, A;
> S := {};
> for v from 1 to n do
> for u from v+1 to m do
> S := S union {(u, v) = u^2 + v^2};
> end do;
> end do;
> A := Matrix(n, m, S);
> end;

> TableSquares(10, 10);

0 0 0 0 0 0 0 0 0 0
5 0 0 0 0 0 0 0 0 0
10 13 0 0 0 0 0 0 0 0
17 20 25 0 0 0 0 0 0 0
26 29 34 41 0 0 0 0 0 0
37 40 45 52 61 0 0 0 0 0
50 53 58 65 74 85 0 0 0 0
65 68 73 80 89 100 113 0 0 0
82 85 90 97 106 117 130 145 0 0
101 104 109 116 125 136 149 164 181 0
> for v from 1 to n do
>   for u from v+1 to m do
>     S := S union {(u,v)=u^2-v^2};
>   end do;
> end do;
> A := Matrix(n,m,S);
> end;

> TableSquares := proc(n,m)
> local u, v, S, A;
> S := {};
> for v to n do for u from v+1 to m do S := union(S, {(u,v) = u^2 - v^2}) end do; end do;
> A := Matrix(n,m,S);
> end proc:

> TableSquares(10,10);

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 12 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 21 & 16 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 32 & 27 & 20 & 11 & 0 & 0 & 0 & 0 & 0 \\
48 & 45 & 40 & 33 & 24 & 13 & 0 & 0 & 0 & 0 \\
63 & 60 & 55 & 48 & 39 & 28 & 15 & 0 & 0 & 0 \\
80 & 77 & 72 & 65 & 56 & 45 & 32 & 17 & 0 & 0 \\
99 & 96 & 91 & 84 & 75 & 64 & 51 & 36 & 19 & 0
\end{bmatrix}
\]

> restart;

> TableSquares := proc(n,m)
> local u, v, S, A;
> S := {};
> for v from 1 to n do
>   for u from v+1 to m do
>     S := S union {(u,v)=2*u*v};
>   end do;
> end do;
> A := Matrix(n,m,S);
> end;

> TableSquares := proc(n,m)
local $u, v, S, A$
$S := \{\}$
for $v$ to $n$ do
  for $u$ from $v + 1$ to $m$ do
    $S := \text{union}(S, \{(u, v) = 2^u v^v\})$
  end do
end do
$A := \text{Matrix}(n, m, S)$
end proc

$\text{TableSquares}(10, 10)$

$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 16 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 20 & 30 & 40 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 24 & 36 & 48 & 60 & 0 & 0 & 0 & 0 & 0 \\
14 & 28 & 42 & 56 & 70 & 84 & 0 & 0 & 0 & 0 \\
16 & 32 & 48 & 64 & 80 & 96 & 112 & 0 & 0 & 0 \\
18 & 36 & 54 & 72 & 90 & 108 & 126 & 144 & 0 & 0 \\
20 & 40 & 60 & 80 & 100 & 120 & 140 & 160 & 180 & 0
\end{bmatrix}$
> restart;
> for v from 1 to 9 do
>     for u from v+1 to 10 do
>         print((u^2-v^2), (2*u*v), (u^2+v^2));
>     end do;
> end do;

\[
\begin{align*}
\text{primitive} & \quad u \\
2 & 3, 4, 5 \\
& 8, 6, 10 \\
4 & 15, 8, 17 \\
& 24, 10, 26 \\
6 & 35, 12, 37 \\
& 48, 14, 50 \\
8 & 65, 16, 65 \\
& 80, 18, 82 \\
10 & 99, 20, 101 \\
3 & 5, 12, 13 \\
& 12, 16, 20 \\
5 & 21, 20, 29 \\
& 32, 24, 40 \\
7 & 45, 28, 53 \\
& 60, 32, 68 \\
9 & 77, 36, 85 \\
& 96, 40, 104 \\
4 & 24, 25 \\
& 16, 30, 34 \\
& 27, 36, 45 \\
& 40, 42, 58 \\
8 & 55, 48, 73 \\
& 72, 54, 90 \\
10 & 91, 60, 109 \\
5 & 9, 40, 41 \\
& 20, 48, 52 \\
7 & 33, 56, 65 \\
& 48, 64, 80 \\
9 & 65, 72, 97 \\
& 84, 80, 116
\end{align*}
\]
The Pythagorean triple \((u^2-v^2, 2uv, u^2+v^2)\) is primitive whenever \(v\) is odd and \(u\) is even or vice versa and also when \(u \neq v\) have no common factors.

\[
\begin{align*}
\text{For } v = 6: & \\
\text{For } v = 9: & \\
\text{For } v = 10: & \\
\end{align*}
\]
e.) Prove that your conditions in (d) really work.

Proof:

Case 1: Let \( u = \text{even} \), \( u > v \), \( u = 2m \), \( m \leq n \)

\[ v = \text{odd} \quad v = 2n + 1 \]

\[ a = u^2 - v^2 \]
\[ = 4m^2 - (4n^2 + 4n + 1) \]
\[ = 4m^2 - 4n^2 - 4n - 1 \]

\[ b = 2uv \]
\[ = 2(2n)(2n + 1) \]
\[ = 8mn + 4n \]

\[ c = u^2 + v^2 \]
\[ = 4m^2 + (4n^2 + 4n + 1) \]
\[ = 4m^2 + 4n^2 + 4n + 1 \]

Now let \( m = u^2 \) and \( n = v^2 \), so \( a = m - n \) and \( c = m + n \). We know from part (a) that \( u \) and \( v \) have no common factors, thus it also follows to say that \( u^2 \) and \( v^2 \) have no common factors. So in setting \( m = u^2 \) and \( n = v^2 \), we know that \( m \) and \( n \) have no common factors. So check that \( a = m - n \) and \( c = m + n \) have no common factors.

Assume that \( a \) and \( c \) have a common factor of \( d \), then:

\[ d | m + n \] and \( d | m - n \)

if they both have common factor \( d \) then when we add or subtract them the resulting number will also have a factor of \( d \). So

\[ (m + n) + (m - n) = 2m \]
\[ (m + n) - (m - n) = 2n \]

and since \( m \) and \( n \) don't have a common factor, \( d \in \{1, 2, 3\} \)

So if we multiply the two numbers \( a \) and \( c \) together the resulting number will be divisible by \( d \).
So,

$$(m+n)(m-n) = m^2 - n^2 = u^4 - v^4$$

and we know that $u^4$ will always be even, since $u$ is even, and we know that $v^4$ will always be odd, since $v$ is odd, and we know that an even minus an odd will always be odd and thus never divisible by 2.

$$: d$$ must equal 1 and $a$ and $c$ have no common factors.

Note: It is sufficient to prove that any 2 factors of a triple (in this case we chose $a$ and $c$) do not have a common factor, to see if they are primitive. Since in a non-primitive triple, all three terms will have a common factor of $d$. Note also that if $b$ were to have a common factor $d$ with a but not $c$, or vice versa, then $(a,b,c)$ would not fulfill the requirements to be a Pythagorean triple.

Thus $$(d_4, d_3, b, c)$$

$$d_4^2 + d_3^2 b^2 = c^2$$

$$d_4^2 (d_3^2 + b^2) = c^2$$

But $c^2$ doesn’t have a factor of $d_2$ and you will not end up with an integer for the triple.
e) (continued)

**CASE 3:** Let \( u = \text{even}, \ u = 2n \quad \forall n > m \) since \( u > v \).

\[
\begin{align*}
a &= u^2 - v^2 \\
&= (2n)^2 - (2m)^2 \\
&= 4n^2 - 4m^2 \\
&= 2(2n^2 - 2m^2)
\end{align*}
\]

\[
\begin{align*}
b &= 2uv \\
&= 2(2n)(2m) \\
&= 8nm \\
&= 2(4nm)
\end{align*}
\]

\[
\begin{align*}
c &= u^2 + v^2 \\
&= (2n)^2 + (2m)^2 \\
&= 4n^2 + 4m^2 \\
&= 2(2n^2 + 2m^2)
\end{align*}
\]

*: when \( u = \text{even}, v = \text{even} \) we don't get primitive Pythagorean triples since \( a, b \) & \( c \) have a common factor of 2.

**CASE 4:** Let \( u = \text{odd}, \ u = 2n+1 \quad \forall n > m \) since \( u > v \).

\[
\begin{align*}
a &= u^2 - v^2 \\
&= (2n+1)^2 - (2m+1)^2 \\
&= 4n^2 + 4n + 1 - 4m^2 - 4m - 1 \\
&= 4(n^2 - m^2) + 4(n - m) \\
&= 4(2(n^2 - m^2) + (n - m)) \\
&= 2(2((n^2 - m^2) + (n - m)))
\end{align*}
\]

\[
\begin{align*}
b &= 2uv \\
&= 2(2n+1)(2m+1) \\
&= 2(4nm + 2n + 2m + 1) \\
&= 2(4nm + 2n + 2m + 1)
\end{align*}
\]

\[
\begin{align*}
c &= u^2 + v^2 \\
&= (2n+1)^2 + (2m+1)^2 \\
&= 4n^2 + 4n + 1 + 4m^2 + 4m + 1 \\
&= 4(n^2 + m^2) + 4(nm) + 2 \\
&= 2(2(n^2 + m^2) + 2(nm) + 1)
\end{align*}
\]

*: when \( u = \text{odd}, v = \text{odd} \) we don't get primitive Pythagorean triples since \( a, b \) & \( c \) have a common factor of 2.

Therefore, our conditions in (a) really work. √
Extension:

**Question:** Can we look at this problem from an abstract algebra perspective?

→ Yes! We searched online and found a proof of primitive Pythagorean triples using abstract algebra. It uses the properties of parity and gcd. (http://planetmath.org/encyclopedia/ProofofPythagoreanTriples.html)

**Proof:** Let $a, b, c$ be primitive positive integers such that $a^2 + b^2 = c^2$.

Let: $\frac{m}{n} = \frac{a+c}{b}$ and reduce so that $\gcd(m, n) = 1$.

Then, $\frac{m}{n} \cdot b - a = c$. Now, square both sides and multiply by $n^2$:

$$\Rightarrow m^2 b^2 - 2mnab + n^2 a^2 = n^2 a^2 + n^2 b^2$$

$$\Rightarrow b(m^2 - n^2) = a(2mn).$$

From this we can see that there are two cases where $m$ & $n$ are opposite parity, or they are both odd. We know since $\gcd(m, n) = 1$, they both cannot be even.

Then:
If both cases are looked at, it can be proved that to be a primitive Pythagorean triple $u > v$, $\gcd(u, v) = 1$ and $u$ & $v$ are of opposite parity.