3.1 As we have just seen, we get every Pythagorean triple \((a,b,c)\), \(b\) even, from:

\[(a,b,c) = (u^2-v^2, 2uv, u^2+v^2)\]

by substituting in different integers for \(u\) and \(v\). For example, \((u,v) = (2,1)\) gives the smallest triple \((3,4,5)\).

a) If \(u\) and \(v\) have a common factor, explain why \((a,b,c)\) will not be a primitive Pythagorean triple.

So if \(u\) and \(v\) have a common factor let:

\[u = xn, \quad n = \text{common factor} \quad \quad v = yn\]

Then:

\[
\begin{align*}
(a,b,c) &= (u^2-v^2, 2uv, u^2+v^2) \\
&= (xn^2 - yn^2, 2xyn, xn^2 + yn^2)
\end{align*}
\]

Multiplying out we get:

\[
\begin{align*}
(x^2n^2 - y^2n^2, 2xyn, x^2n^2 + y^2n^2)
\end{align*}
\]

You can pull out a factor of \(n^2\) leaving:

\[
\begin{align*}
(x^2 - y^2, 2xy, x^2 + y^2)
\end{align*}
\]

Thus if there is a common factor throughout the terms of a Pythagorean triple then these numbers are not primitive.
b) Find an example of integers $u > v > 0$ that do not have a common factor, yet the Pythagorean triple $(u^2 - v^2, 2uv, u^2 + v^2)$ is not primitive.

Basically, we did a trial and error method. Our first try was actually a success.

\[ \text{Ex: } u = 5 \quad v = 3 \]

Put into

\[ (u^2 - v^2, 2uv, u^2 + v^2) \]

\[ \rightarrow (5^2 - 3^2, 2 \cdot 5 \cdot 3, 5^2 + 3^2) \]

\[ = (25 - 9, 30, 25 + 9) \]

\[ = (16, 30, 34) \]

This Pythagorean triple works when $u$ and $v$ have no common factors, but the triple has a common factor of 2, and therefore is not primitive.

\[ \text{part c \rightarrow maple.} \]

\[ \text{part d \rightarrow on maple sheet} \]
Part C: Prove that your conditions in Part B really work.

So the basic stipulations we made were only that \( u \) and \( v \) must be opposite or even in terms of even or odd.

So if \( u \) is even \( \Rightarrow \) \( v \) must be odd or vice versa.

\( u = \text{even} = 2n \), and have common factor \( 2. \)

\( v = \text{odd} = 2n+1 \)

Then

\[
\left( (2n)^2 - (2n-1)^2, 2(2n)(2n-1), (2n)^2 + (2n-1)^2 \right)
\]

\[
\left( 4n^2 - 4n^2 + 4n - 1, 8n^2 - 8n + 4n^2 + 4n + 1 \right)
\]

\[
\left( 4n - 1, 8n(n-1), 8n^2 - 4n + 1 \right)
\]

This final computation shows that there are no common factors to be pulled out and thus we have a primitive Pythagorean triple.

\( \text{If } u = \text{even and } v = \text{odd or vice versa.} \)
continuance, as part e:

\[ u = 2m \]
\[ v = 2n + 1 \]

so when we plug this in we get:

\[ (4m^2 - 4n^2 - 4n - 1, 8mn + 4m, 4m^2 + 4n^2 + 4n + 1) \]

Show/prove they have no common factors.

so if we look at \( a = u^2 - v^2 \)

we can make \( u^2 = t \)
\[ v^2 = l \]

so we can have \( (t + l) + (t - l) = 2t \)
\[ (t + l) - (t - l) = 2l \]

We suppose that \( d \) is a common factor of \( t + l \) and \( t - l \)

So then for \( d \) to be a common factor it must be equal to one because that is the only number that they have in common.

So we now know \( t + l \) and \( t - l \) have no common factors.

According to the primitive Pythagorean triple definition we only have to show that two of them have no common factor which we showed for \( u^2 - v^2 \) and \( u^2 + v^2 \) because if there are only two numbers with a common factor it will not even turn out to be a triple.
3.1 part c

find pyth triples that arise when you substitute in all values of u and v with 1 < u < v < 10

For u^2 - v^2

> restart;

> TableSquares := proc(n, m)
local u, v, S, A;
S := {};
> for v from 1 to n do
for u from v+1 to m do
S := S union {(v, u) = (u^2 - v^2)};
end do;
end do;
A := Matrix(n, m, S);
end;

TableSquares := proc(n, m)
local u, v, S, A;
S := {};
for v to n do
for u from v + 1 to m do
S := union(S, {(v, u) = (u^2 - v^2)})
end do;
end do;
A := Matrix(n, m, S);
end proc;

> TableSquares(9, 10);

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td>8</td>
<td>15</td>
<td>24</td>
<td>35</td>
<td>48</td>
<td>63</td>
<td>80</td>
<td>99</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>12</td>
<td>21</td>
<td>32</td>
<td>45</td>
<td>60</td>
<td>77</td>
<td>96</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>16</td>
<td>27</td>
<td>40</td>
<td>55</td>
<td>72</td>
<td>91</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9</td>
<td>20</td>
<td>33</td>
<td>48</td>
<td>65</td>
<td>84</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>11</td>
<td>24</td>
<td>39</td>
<td>56</td>
<td>75</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>13</td>
<td>28</td>
<td>45</td>
<td>64</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>32</td>
<td>51</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>17</td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>19</td>
</tr>
</tbody>
</table>

From the table above, we found that when we substitute in all values of u, v for 1 < v < u < 10
for 2uv

> restart;

> TableSquares:=proc(n, m)
local u, v, S, A;
S:={};

> for v from 1 to n do
  for u from v+1 to m do
    S:=S union {(v,u)=(2*u*v)};
  end do;
end do;

> A:=Matrix(n,m,S);
end:

TableSquares := proc(n, m)
local u, v, S, A;
S:={};
for v to n do for u from v + 1 to m do S := union(S, {(v, u) = 2*u*v}) end do; end do;
A := Matrix(n, m, S);
end proc:

> TableSquares(9,10);

5
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>32</td>
<td>36</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>42</td>
<td>48</td>
<td>54</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>40</td>
<td>48</td>
<td>56</td>
<td>64</td>
<td>72</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
<td>100</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>84</td>
<td>96</td>
<td>108</td>
<td>120</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>112</td>
<td>126</td>
<td>140</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>144</td>
<td>160</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>180</td>
</tr>
</tbody>
</table>

For u^2+v^2

> restart;

> TableSquares:=proc(n, m)
local u, v, S, A;
S:={};

> for v from 1 to n do
  for u from v+1 to m do
    S:=S union {(v,u)=(u^2+v^2)};
  end do;
end do;
\begin{itemize}
\item end do;
\item end do;
\end{itemize}
\begin{verbatim}
> A := Matrix(n, m, S);
end;
TableSquares := proc(n, m)
local u, v, S, A;
S := {};
for v to n do for u from v + 1 to m do S := union(S, \{(v, u) = u^2 + v^2\}) end do; end do;
A := Matrix(n, m, S);
end proc;
\end{verbatim}

\begin{verbatim}
> TableSquares(9, 10);
\end{verbatim}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\hline
1 & 0 & 5 & 10 & 12 & 17 & 26 & 37 & 50 & 65 & 82 & 101 \\
\hline
2 & 0 & 0 & 8 & 13 & 20 & 29 & 40 & 53 & 68 & 85 & 104 \\
\hline
3 & 0 & 0 & 0 & 12 & 25 & 34 & 45 & 58 & 73 & 90 & 109 \\
\hline
4 & 0 & 0 & 0 & 0 & 16 & 24 & 32 & 40 & 58 & 76 & 94 \\
\hline
5 & 0 & 0 & 0 & 0 & 0 & 20 & 24 & 36 & 56 & 76 & 96 \\
\hline
6 & 0 & 0 & 0 & 0 & 0 & 0 & 41 & 61 & 81 & 101 & 121 \\
\hline
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 85 & 109 & 133 & 157 \\
\hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & 130 & 160 \\
\hline
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 145 & 164 \\
\hline
\end{tabular}
\caption{TableSquares(9, 10)}
\end{table}

\begin{itemize}
\item From our tables we find that primitive Pythagorean triples are generated by \(u\) being even and \(v\) being odd, or vice versa.
\item This is represented in our tables by diagonal lines.
\end{itemize}

\textbf{Where is part e?}
Extended:

We weren't sure what to do for an extension. So we went back and looked over our observations once more times we found that on our tables, there was a pattern.

For finding the c term of the PPT starting at 5 and increasing by 4n starting at n = 2 will always give you a prime. In c = 1.

Once we discovered this we tried to find the a, b, is that would correlate to the c.

For b we find that starting from 4 the numbers increase by a factor of 4. So b correlating to c before would have to be 4n where $n = 2$ as it starts at 4.

Finally a was observed from the table to merely be increasing by a factor of 2 starting at the number 3 so $a = 2n$ starting at $n = 1$ and $a = 3$.

So from the above observations I think we could generate an infinite amount of PPT with the same simple generalization.