

Problem 3.1

GROUP A

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A) If u and v have a common factor greater than 1, (a,b,c) will not be a primitive Pythagorean triple.

We started by rewriting u and v as:

$$u = kx$$

$$v = ky$$

so that u and v both have a common factor of k , that is greater than 1.

Our original equations:

$$a = u^2 - v^2$$

$$b = 2uv$$

$$c = u^2 + v^2$$

We can then substitute kx for u and ky for v and get:

$$a = (kx)^2 - (ky)^2$$

$$b = 2(kx)(ky)$$

$$c = (kx)^2 + (ky)^2$$

We can then factor a k^2 out of each term:

$$a = k^2(x^2 - y^2)$$

$$b = k^2(2xy)$$

$$c = k^2(x^2 + y^2)$$

Since a , b , and c each have a common factor of k^2 , (a,b,c) can not be a primitive Pythagorean triple.

B) An example of integers that do not have a common factor but give a Pythagorean triple that is not primitive is $u=3$, $v=1$.

Proof:

$$a: u^2 - v^2 = (3)^2 - (1)^2 = 8$$

$$b: 2uv = 2(3)(1) = 6$$

$$c: u^2 + v^2 = (3)^2 + (1)^2 = 10$$

This gives the Pythagorean triple $(6,8,10)$ which is not primitive because a , b , and c all have a common factor of 2.

C) See attached Maple Calculations

D) By analyzing the Pythagorean triplets we found with Maple, we have noticed when u was odd and v was even, or when u was even and v was odd, the triplets are primitive. However, the only limitation to this is that u and v must have no common factors. For example, when u equals 10 and v equals 5, they do not result in a primitive triplet.

E) The conditions necessary to have a primitive triple are that both a and c are odd, and b is even, and that there is no common factor greater than one in all three values. Below considers all of the possible conditions of u and v and demonstrates how the limitations set in part D are the only that result in primitive triplets.

1. When u and v are both even, they share a common factor, k . Also, as previously stated, it is possible for u and v to be different based on odd and even, but still have a common factor, k . As demonstrated in Part A, we will observe this common factor, k , in the triplet, and determine that this set of a , b , and c , cannot be a primitive Pythagorean triple because a common factor of k^2 can be found in all three values.

$$u = kx \quad ; \quad v = ky$$

$$a = (kx)^2 - (ky)^2$$

$$b = 2(kx)(ky)$$

$$c = (kx)^2 + (ky)^2$$

$$a = k^2(x^2 - y^2)$$

$$b = k^2(2xy)$$

$$c = k^2(x^2 + y^2)$$

2. Additionally, when u and v are both odd, it is possible to express them in terms of $2m+1$ and $2n+1$, and prove that they will not produce a primitive Pythagorean triple. This is so based on the fact that the values of a and c must be odd, and the value of b must be even for a triple to be primitive, and when u and v are both odd, this condition is not met.

$$u = 2m + 1 \quad ; \quad v = 2n + 1$$

$$a = (2m+1)^2 - (2n+1)^2$$

$$= 4(m^2 + m - n^2 - n)$$

$$= \text{EVEN!}$$

$$b = 2(2m+1)(2n+1)$$

$$= \text{EVEN!}$$

$$c = (2m+1)^2 + (2n+1)^2$$

$$= 2(2m^2 + 2m + 2n^2 + 2n + 1)$$

$$= \text{EVEN!}$$

3. Next, we will observe the relationship of a, b, and c, where u is even and v is odd, thus having values of 2m and 2n+1. Notice how the resulting values of a and c will be odd, and b will be even, which proves partly that the Pythagorean triple will be primitive.

$$\begin{aligned} a &= (2m)^2 - (2n+1)^2 \\ &= 4(m^2 - n^2 - n) - 1 \\ &= (\text{EVEN}) - 1 \\ &= \text{ODD!} \end{aligned}$$

$$\begin{aligned} b &= 2(2m)(2n+1) \\ &= 4(2mn + m) \\ &= \text{EVEN!} \end{aligned}$$

$$\begin{aligned} c &= (2m)^2 + (2n+1)^2 \\ &= 4(m^2 + n^2 + n) + 1 \\ &= (\text{EVEN}) + 1 \\ &= \text{ODD!} \end{aligned}$$

4. Finally, in order to prove that the a, b, and c values above are, in fact, a set of primitive Pythagorean triples when u is even and v is odd, we must prove that u and v, and a and c, have no common factors. This is done by substituting u for m^2 and v for n^2 , as squaring an even number results in an even, and squaring an odd will also result in an odd. Next, we compare the values of a and c with a common divisor d, and as they are both odd and should not share a common factor greater than 1. This is done by multiplying the two values together, and proving that 1 is their only common factor.

$$u = m^2 \quad ; \quad v = n^2$$

$$a = (u^2 - v^2) = (m - n) \quad ; \quad c = (u^2 + v^2) = (m + n)$$

So,

$$d \mid m + n \quad \text{and} \quad d \mid m - n$$

And,

$$\begin{aligned} (m + n) + (m - n) &= 2m \\ (m + n) - (m - n) &= 2n \end{aligned}$$

Therefore,

$$d \mid 2m \quad \text{and} \quad d \mid 2n \quad ; \quad \text{possible common factors} = \{1, 2\}$$

However, 2 cannot be a factor because

$$\begin{aligned} (m + n)(m - n) &= m^2 - n^2 \\ &= u^4 - v^4 \\ &= \text{EVEN} - \text{ODD} \\ &= \text{ODD} \end{aligned}$$

And an odd number is not divisible by 2.

EXTENSION: Looking at the calculations derived from Maple, we noticed that the a term was decreasing each time by $2v+1$, the b term was increasing by $2u$, and the c term was increasing each time by $2v+1$.

We tried to prove this by using the Pythagorean Theorem: $a^2 + b^2 = c^2$

We also know that

$$a = u^2 - v^2$$

$$b = 2uv$$

$$c = u^2 + v^2$$

So if we subtract $2v-1$ from a, and add $2u$ to b, and add $2v+1$ to c, we should get the next terms in the sequence and these terms will fit the theorem $a^2 + b^2 = c^2$. We tried to plug in the values of u equals 4 and v equals 3, hoping to find some equality in the final calculation.

$$[(u^2 - v^2) - (2v+1)]^2 + [2uv + 2u]^2 = [(u^2 + v^2) + (2v+1)]^2$$

$$[(7) - (7)]^2 + [32]^2 = [(25) + (7)]^2$$

$$0 + 1024 = 1024$$

$$1024 = 1024$$

As shown above, this assumption does seem to work, as the two sides are equal.

Group A

```
> TableSquares := proc (n, m)
  local u, v, S, C;
  S := {};
  > for u from 2 to m do
    for v from 1 to (u-1) do
      S := S union {(u, v) = u^2 + v^2};
    end do;
  end do;
  > C := Matrix(n, m, S);
  > end;


TableSquares := proc(n, m)
local u, v, S, C;
  S := {};
  for u from 2 to m do for v to u - 1 do S := union(S, {(u, v) = u^2 + v^2}) end do; end do;
  C := Matrix(n, m, S);
end proc;
```

```
> TableSquares(10, 10);
```

0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
10	13	0	0	0	0	0	0	0	0
17	20	25	0	0	0	0	0	0	0
26	29	34	41	0	0	0	0	0	0
37	40	45	52	61	0	0	0	0	0
50	53	58	65	74	85	0	0	0	0
65	68	73	80	89	100	113	0	0	0
82	85	90	97	106	117	130	145	0	0
101	104	109	116	125	136	149	164	181	0

```
> TableSquares := proc (n, m)
  local u, v, S, B;
  S := {};
  > for u from 2 to m do
    for v from 1 to (u-1) do
      S := S union {(u, v) = 2*u*v};
    end do;
  end do;

  B := Matrix(n, m, S);
  end;
```



```
TableSquares := proc(n, m)
```

```
local u, v, S, B;
```

```
S := {};
```

```
for u from 2 to m do for v to u - 1 do S := union(S, {(u, v) = 2*u*v}) end do; end do;
```

```
B := Matrix(n, m, S);
```

```
end proc;
```

```
> TableSquares(10, 10);
```

0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
6	12	0	0	0	0	0	0	0	0
8	16	24	0	0	0	0	0	0	0
10	20	30	40	0	0	0	0	0	0
12	24	36	48	60	0	0	0	0	0
14	28	42	56	70	84	0	0	0	0
16	32	48	64	80	96	112	0	0	0
18	36	54	72	90	108	126	144	0	0
20	40	60	80	100	120	140	160	180	0

```
> restart;
```

```
> TableSquares:=proc(n,m)
```

```
local u, v, S, A;
```

```
S:={};
```

```
> for u from 2 to m do
```

```
for v from 1 to (u-1) do
```

```
S:=S union {(u,v)=[u^2-v^2, 2*u*v, u^2+v^2]}
```

```
end do;
```

```
end do;
```

```
A:=Matrix(n,m,S);
```

```
end;
```

```
>
```

```
TableSquares := proc(n, m)
```

```
local u, v, S, A;
```

```
S := {};
```

```
for u from 2 to m do for v to u - 1 do S := union(S, {(u, v) = [u^2 - v^2, 2*u*v, u^2 + v^2]})
```

```
end do;
```

```
end do;
```

```
A := Matrix(n, m, S);
```

```
end proc;
```

> TableSquares(10,10);

[[0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [[3, 4, 5], 0, 0, 0, 0, 0, 0, 0, 0, 0], [[8, 6, 10], [5, 12, 13], 0, 0, 0, 0, 0, 0, 0, 0],
[[15, 8, 17], [12, 16, 20], [7, 24, 25], 0, 0, 0, 0, 0, 0, 0],
[[24, 10, 26], [21, 20, 29], [16, 30, 34], [9, 40, 41], 0, 0, 0, 0, 0, 0],
[[35, 12, 37], [32, 24, 40], [27, 36, 45], [20, 48, 52], [11, 60, 61], 0, 0, 0, 0, 0],
[[48, 14, 50], [45, 28, 53], [40, 42, 58], [33, 56, 65], [24, 70, 74], [13, 84, 85], 0, 0, 0, 0],
[[63, 16, 65], [60, 32, 68], [55, 48, 73], [48, 64, 80], [39, 80, 89], [28, 96, 100], [15, 112, 113], 0, 0, 0],
[[80, 18, 82], [77, 36, 85], [72, 54, 90], [65, 72, 97], [56, 90, 106], [45, 108, 117], [32, 126, 130],
[17, 144, 145], 0, 0], [[99, 20, 101], [96, 40, 104], [91, 60, 109], [84, 80, 116], [75, 100, 125],
[64, 120, 136], [51, 140, 149], [36, 160, 164], [19, 180, 181], 0]]

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