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Final Draft: Section 2 Problem 2.5:

In Chapter 1 we saw that the n^{TH} triangular number T_n is given by the formula:

$$T_n = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

The first few triangular numbers are 1, 3, 6, and 10. In the list of the first few Pythagorean triples (a, b, c), we find (3, 4, 5), (5, 12, 13), (7, 24, 25), and (9, 40, 41). Notice that in each case, the value of b is four times a triangular number.

(a) Find a primitive Pythagorean triple (a, b, c) with $b = 4T_5$. Do the same for $b = 4T_6$ and with $b = 4T_7$.

Data:

	$b = 4T_n$	Primitive Pythagorean Triple (a, b, c)
$T_5 = 15$	$b = 4T_5 = 60$	(11, 60, 61)
$T_6 = 21$	$b = 4T_6 = 84$	(13, 84, 85)
$T_7 = 28$	$b = 4T_7 = 112$	(15, 112, 113)

Observations: The first observation is in the primitive Pythagorean triple where $b = 4T_n$. As n gets larger a gets larger. Also, for every n+1 number, a is 2n+1, where a is always odd. The a entry for n+1 will be the next odd number.

We also determined by starting with the primitive Pythagorean triple (3, 4, 5) and n = 1, then:

$$a = 2n+1, \quad b = 4T_n, \quad c = 4T_n + 1$$

Hypothesis: This works for all primitive Pythagorean triples where $b = 4T_n$.

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Examples :

let $n=4$

$$\left.\begin{aligned} a &= 2(4)+1=9 \\ b &= 4T_4 = 4\cdot 10 = 40 \\ c &= 4T_4+1 = 41 \end{aligned}\right\} (9,40,41)$$

let $n=5$

$$\left.\begin{aligned} a &= 2(5)+1=11 \\ b &= 4T_5 = 4\cdot 15 = 60 \\ c &= 4T_5+1 = 61 \end{aligned}\right\} (11,60,61)$$

(b) Do you think that for every triangular number T_n, there is a primitive Pythagorean triple (a,b,c) with $b=4T_n$? If you believe that this is true, then prove it.

Hypothesis : For every triangular number, T_n, there is a primitive Pythagorean triple (a,b,c) with $b=4T_n$.

Data & Observations: In the textbook, on page 17, it says if we take $t=1$, then we get a triple $(s, \frac{s^2-1}{2}, \frac{s^2+1}{2})$ whose b and c entries differ by 1.

So let $a=s$, $b=\frac{s^2-1}{2}$, $c=\frac{s^2+1}{2}$

We claim that if $s=2n+1$, then:

$$a=2n+1,\ b=4T_n,\ c=4T_n+1$$

Proof : (Our data from part (a) applies)

$$a=s=2n+1$$

$$b=\frac{s^2-1}{2}=\frac{(2n+1)^2-1}{2}=\frac{4n^2+4n}{2}=\frac{4(n^2+n)}{2}=4T_n \quad \checkmark$$

$$c=\frac{s^2+1}{2}=\frac{(2n+1)^2+1}{2}=\frac{4n^2+4n+2}{2}=\frac{4(n^2+n)}{2}+\frac{2}{2}=4T_n+1 \quad \checkmark$$

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Conclusion: Our hypothesis is correct. For every triangular Pythagorean number, T_n, there is a primitive Pythagorean triple (a, b, c) with $b = 4T_n$

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Extension: Section 2 Problem 2.5:

For our extension we noticed that the book was mistaken when they asked for the primitive Pythagorean triples with $b = T_n$. They meant to say $b = 4T_n$ (as we did when we re-wrote part (a)), so we decided to find primitive Pythagorean triples with $b = T_n$, $n = 5, 6, 7$.

Data:

$b = T_n$	Primitive Pythagorean Triple (a, b, c)
$b = T_5 = 15$	$(8, 15, 17)$
$b = T_6 = 21$	$(20, 21, 29)$
$b = T_7 = 28$	$(45, 28, 53)$

However, watch what happens when $n = 4$:

$b = T_4 = 10 \Rightarrow$ There is no P.P.T. that exists with $b = 10$.

Conclusion: Although we were able to find primitive Pythagorean triples with $b = T_n$ for $n = 5, 6, 7$, we were able to prove with a counterexample that a primitive Pythagorean triple (a, b, c) with $b = T_n$ does not exist for every triangular number, T_n, but only for some.