In Chapter 1 we saw that the $n^{th}$ triangular number $T_n$ is given by the formula:

$$T_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

The first few triangular numbers are 1, 3, 6, and 10. In the list of the first few Pythagorean triples $(a, b, c)$, we find $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, and $(9, 40, 41)$. Notice that in each case, the value of $b$ is four times a triangular number.

(a) Find a primitive Pythagorean triple $(a, b, c)$ with $b = 4T_n$. Do the same for $b = 4T_5$ and with $b = 4T_7$.

Data:

| $T_5 = 15$ | $b = 4T_5 = 60$ | $b = 4T_5 = 84$ | $b = 4T_7 = 112$ |
| $T_6 = 21$ | $b = 4T_6$ | $b = 4T_6 = 84$ | $b = 4T_7 = 112$ |
| $T_7 = 28$ | $b = 4T_7$ | $b = 4T_7$ | $b = 4T_7 = 112$ |

Observations: The first observation is in the primitive Pythagorean triple where $b = 4T_n$. As $n$ gets larger, $a$ gets larger. Also, for every $n+1$ number, $a$ is $2n+1$, where $n$ is always odd. The $a$ entry for $n+1$ will be the next odd number.

We also determined by starting with the primitive Pythagorean triple $(3, 4, 5)$ and $n=1$, then:

$$a = 2n+1, \quad b = 4T_n, \quad c = 4T_n + 1$$

Hypothesis: This works for all primitive Pythagorean triples where $b = 4T_n$. 
Examples:

let \( n = 4 \)

\[
\begin{align*}
\alpha &= 2(4) + 1 = 9 \\
\beta &= 4T_n = 4 \cdot 10 = 40 \\
\gamma &= 4T_n + 1 = 41
\end{align*}
\]

\( (9, 40, 41) \)

let \( n = 5 \)

\[
\begin{align*}
\alpha &= 2(5) + 1 = 11 \\
\beta &= 4T_n = 4 \cdot 15 = 60 \\
\gamma &= 4T_n + 1 = 61
\end{align*}
\]

\( (11, 60, 61) \)

(b) Do you think that for every triangular number \( T_n \), there is a primitive Pythagorean triple \( (\alpha, \beta, \gamma) \) with \( \beta = 4T_n \)? If you believe that this is true, then prove it.

Hypothesis: For every triangular number, \( T_n \), there is a primitive Pythagorean triple \( (\alpha, \beta, \gamma) \) with \( \beta = 4T_n \).

Data & Observations: In the textbook, on page 17, it says if we take \( t = 1 \), then we get a triple \( (s, \frac{s^2 - 1}{2}, \frac{s^2 + 1}{2}) \) whose \( b \) and \( c \) entries differ by 1.

So let \( a = s \), \( b = \frac{s^2 - 1}{2} \), \( c = \frac{s^2 + 1}{2} \).

We claim that if \( s = 2n + 1 \), then:

\[
\begin{align*}
\alpha &= 2n + 1 \\
\beta &= 4T_n \\
\gamma &= 4T_n + 1
\end{align*}
\]

Proof: (Our data from part (a) applies)

\[
\begin{align*}
\alpha &= s = 2n + 1 \\
\beta &= \frac{s^2 - 1}{2} = \frac{(2n+1)^2 - 1}{2} = \frac{4n^2 + 4n}{2} = 2n + \frac{2}{2} = 4T_n \\
\gamma &= \frac{s^2 + 1}{2} = \frac{(2n+1)^2 + 1}{2} = \frac{4n^2 + 4n + 2}{2} = 2n + \frac{2}{2} = 4T_n + 1
\end{align*}
\]
Conclusion: Our hypothesis is correct. For every triangular Pythagorean number \( T_n \) there is a primitive Pythagorean triple \((a, b, c)\) with \( b = 4T_n \).
Extension: Section 2 Problem 2.5:

For our extension, we noticed that the book was mistaken when they asked for the primitive Pythagorean triples with $b = T_n$. They meant to say $b = 4T_n$ (as we did when we re-voiced part (a)), so we decided to find primitive Pythagorean triples with $b = T_n$, $n = 5, 6, 7$.

**Data:**

<table>
<thead>
<tr>
<th>$b = T_n$</th>
<th>Primitive Pythagorean Triple $(a, b, c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = T_5$</td>
<td>$(8, 15, 17)$</td>
</tr>
<tr>
<td>$b = T_6$</td>
<td>$(20, 21, 29)$</td>
</tr>
<tr>
<td>$b = T_7$</td>
<td>$(45, 28, 53)$</td>
</tr>
</tbody>
</table>

However, watch what happens when $n = 4$:

$b = T_4 = 10 \Rightarrow$ there is no P.P.T. that exists with $b = 10$.

**Conclusion:** Although we were able to find primitive Pythagorean triples with $b = T_n$ for $n = 5, 6, 7$, we were able to prove with a counterexample that a primitive Pythagorean triple $(a, b, c)$ with $b = T_n$ does not exist for every triangular number $T_n$, but only for some.