Problem 2.1 part A:
Show that either a or b of a Primitive Pythagorean Triple (a,b,c) must be a multiple of 3.

We start with the equation:
\[ a^2 + b^2 = c^2 \]

We can rewrite the above equation as:
\[ b^2 = c^2 - a^2 \]
and we can factor this into:
\[ b^2 = (c-a)(c+a) \]

From earlier proofs we know that no two of a, b and c have a common factor greater than one. We also know that that c is always odd and one of a or b is even and the other is odd. Let’s assume that a is odd which makes b even.

Since odd + odd = even and odd – odd = even, we know that both c+a and c-a will give even answers. This means that we can divide both sides of the equation by 2 and still get positive integers.
\[ \frac{(b/2)^2}{2} = \frac{[(c-a)/2] \times [(c+a)/2]}{2} \]

Now we have the product of two integers whose greatest common divisor is 2 and whose product is a square.
The two integers have a greatest common divisor of 2 because:
\[ (c+b) + (c-b) = 2c \]
\[ (c+b) - (c-b) = 2b \]
d\[c+b\] and d\[c-b\]
so, d\[2b\] and d\[2c\]
Thus, the greatest common divisor is 2 and only 2.

For a product to be a square it means that the two factors, since they have a greatest common divisor of 2, have to each be squares too. For example: (4)(9) = 36
\[ (2^2)(3^2) = (6^2) \]
This means that (c-a)/2 and (c+a)/2 are both squares.

To make our computations easier we create two new variables r and s.
Let \[ r^2 = \frac{(c+a)}{2} \]
Let \[ s^2 = \frac{(c-a)}{2} \]
(we use \[ r^2 \] and \[ s^2 \] because (c-a)/2 and (c+a)/2 are both squares)
We can now change the equation \((b/2)^2 = [(c-a)/2] \times [(c+a)/2]\) to:
\[(b/2)^2 = (r^2)(s^2)\] which simplifies to:
\[b/2 = r(s)\] or
\[b = 2rs\]

We also know that \(r^2 = s^2 + a\)

**Proof:**
\[r^2 = s^2 + a\]
\[(c+a)/2 = (c-a)/2 + a\]
\[(c+a)/2 = (c-a)/2 + 2a/2\]
\[(c+a)/2 = (c-a+2a)/2\]
\[(c+a)/2 = (c+a)/2\]

And we know that that \(r\) is greater than \(s\) because \(r^2 = s^2 + a\) and \(s^2 + a > s^2\).

We can rewrite \(r^2 = s^2 + a\) as:
\[a = r^2 - s^2\]

And since we know that \(a\) is odd, this means that one of \(r^2\) or \(s^2\) is odd and one is even.

\((\text{odd} = \text{odd} - \text{even or odd} = \text{even} - \text{odd})\)

Finally we can conclude that if either \(r\) or \(s\) is a multiple of 3 then \(b\) is a multiple of 3 because \(b = 2rs\). And if neither \(r\) or \(s\) is a multiple of 3, then we can split the equation
\[a = r^2 - s^2\] into
\[a = (r^2 - 1) - (s^2 - 1)\] and this can be split into
\[a = (r+1)(r-1) - (s+1)(s-1)\]

So if \(r\) is not divisible by 3 then either \(r+1\) or \(r-1\) will be divisible by 3 and if \(s\) is not divisible by 3 then either \(s+1\) or \(s-1\) will be divisible by 3 and this makes \(a\) divisible by 3.

Therefore either \(a\) or \(b\) will always be divisible by 3.

**Part B:** By examining the above list of primitive Pythagorean triples, make a guess about when \(a\), \(b\) or \(c\) is a multiple of 5. Try to show that your guess is correct.

After examining the list of primitive Pythagorean triples, we noticed that one of \(a\), \(b\) or \(c\) will always be divisible by 5.

We attempted to prove this using a similar method as we used in part A:

So if \(r\) or \(s\) is a multiple of 5 then \(b\) is a multiple of 5 since \(b = 2rs\) (similar to part A).

Then we looked at doing a similar method to see when \(a\) was divisible by 5:

\[a = r^2 - s^2\] split into
\[a = (r^2 - 4) - (s^2 - 4)\] and this can be split into
\[a = (r+2)(r-2) - (s+2)(s-2)\]

However, this method didn't really seem to prove anything.
Extension:

Now we will make a guess about when a, b, or c is a multiple of 4 and try to prove.

We know that \( a = r^2 - s^2 \) and is odd
And \( b = 2rs \) and is even

We also know that one of \( r \) or \( s \) is even and one is odd.

When we multiply \( r \) and \( s \), the product will always be an even number.
\[ (\text{even})(\text{odd}) = \text{even} \]

We then are multiplying this product of \( r \times s \) by 2. When any even number is multiplied by 2 it will always be divisible by 4.

Ex:  
\[
\begin{align*}
2 \times 2 &= 4 \\
2 \times 4 &= 8 \\
2 \times 6 &= 12 \\
2 \times 18 &= 36
\end{align*}
\]

*all products are divisible by 4

Thus, \( b \) will always be divisible by 4.