Group A
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Problem 2.1 part A:
Show that either a or b of a Primitive Pythagorean Triple (a,b,c) must be a multiple of 3.

We start with the equation:

$$a^2 + b^2 = c^2$$

We can rewrite the above equation as:

$b^2 = c^2 - a^2$ and we can factor this into:

$b^2 = (c-a)(c+a)$

From earlier proofs we know that no two of a, b and c have a common factor greater than one. We also know that that c is always odd and one of a or b is even and the other is odd. Let's assume that a is odd which makes b even.

Since odd + odd = even and odd – odd = even, we know that both c+a and c-a will give even answers. This means that we can divide both sides of the equation by 2 and still get positive integers.

$$(b/2)^2 = [(c-a)/2] \times [(c+a)/2]$$

Now we have the product of two integers whose greatest common divisor is 2 and whose product is a square.
The two integers have a greatest common divisor of 2 because:

$(c+b) + (c-b) = 2c$
$(c+b) - (c-b) = 2b$
$d|c+b$ and $d|c-b$
so, $d|2b$ and $d|2c$
Thus, the greatest common divisor is 2 and only 2.

For a product to be a square it means that the two factors, since they have a greatest common divisor of 2, have to each be squares too. For example: (4)(9) = 36

$$(2^2)(3^2) = (6^2)$$

This means that (c-a)/2 and (c+a)/2 are both squares.

To make our computations easier we create two new variables r and s.

Let $r^2 = (c+a)/2$
Let $s^2 = (c-a)/2$
(we use r^2 and s^2 because (c-a)/2 and (c+a)/2 are both squares)

We can now change the equation $(b/2)^2 = [(c-a)/2] \times [(c+a)/2]$ to:

$(b/2)^2 = (r^2)(s^2)$ which simplifies to:

$b/2 = (r)(s)$ or

$b = 2rs$

We also know that $r^2 = s^2 + a$

Proof: $r^2 = s^2 + a$

$(c+a)/2 = (c-a)/2 + a$

$(c+a)/2 = (c-a)/2 + 2a/2$

$(c+a)/2 = (c-a+2a)/2$

$(c+a)/2 = (c+a)/2$

And we know that that r is greater than s because $r^2 = s^2 + a$ and $s^2 + a > s^2$.

We can rewrite $r^2 = s^2 + a$ as:

$a = r^2 - s^2$

And since we know that a is odd, this means that one of r^2 or s^2 is odd and one is even.

(odd = odd – even *or* odd = even – odd)

Finally we can conclude that if either r or s is a multiple of 3 then b is a multiple of 3 because b = 2rs. And if neither r or s is a multiple of 3, then we can split the equation

$a = r^2 - s^2$ into

$a = (r^2 - 1) - (s^2 - 1)$ and this can be split into

$a = (r+1)(r-1) - (s+1)(s-1)$

So if r is not divisible by 3 then either r+1 or r-1 will be divisible by 3 and if s is not divisible by 3 then either s+1 or s-1 will be divisible by 3 and this makes a divisible by 3.

Therefore either a or b will always be divisible by 3.

Part B: By examining the above list of primitive Pythagorean triples, make a guess about when a, b or c is a multiple of 5. Try to show that your guess is correct.

After examining the list of primitive Pythagorean triples, we noticed that one of a, b or c will always be divisible by 5.

We attempted to prove this using a similar method as we used in part A:
So if r or s is a multiple of 5 then b is a multiple of 5 since b = 2rs (similar to part A).
Then we looked at doing a similar method to see when a was divisible by 5:

$a = r^2 - s^2$ split into

$a = (r^2 - 4) - (s^2 - 4)$ and this can be split into

$a = (r+2)(r-2) - (s+2)(s-2)$

However, this method didn't really seem to prove anything.

Extension:

Now we will make a guess about when a,b, or c is a multiple of 4 and try to prove.

We know that $a = r^2 - s^2$ and is odd
And b = 2rs and is even

We also know that one of r or s is even and one is odd.

When we multiply r and s, the product will always be an even number.
(even)(odd) = even

We then are multiplying this product of r x s by 2. When any even number is multiplied by 2 it will always be divisible by 4.

Ex: 2 x 2 = 4
2 x 4 = 8
2 x 6 = 12
2 x 18 = 36
*all products are divisible by 4

Thus, b will always be divisible by 4.